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Operational formulae for certain classical polynomials

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OPERATIONAL FORMULAE FOR CERTAIN CLASSICAL POLYNOMIALS

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1. Recently Srivastava [7, p. 43] has defined a set of polynomials $A_n^{(a)}(x)$ related to the Laguerre polynomials by means of the relations

\[ \sum_{r=0}^{n} A_r^{(a)}(x) L_{n-r}^{(a+r)}(x) = 0, \quad n \geq 1, \]

\[ A_0^{(a)}(x) = 1. \]

He also gave the generating function, hypergeometric representation and the Rodrigues’ formula for these polynomials [7, pp. 44-45] in the forms:

\[ (1 + t)^{-1-a} e^{xt} = \sum_{r=0}^{\infty} t^r A_r^{(a)}(x), \]

\[ A_n^{(a)}(x) = \frac{1}{(n)!} \sum_{r=0}^{\infty} \frac{(-n)_r (1 + \alpha)_r}{(r)!} x^{n-r}, \]

and

\[ A_n^{(a)}(x) = \frac{x^{n+a+1}}{(n)!} D^n \left[ x^{-a-1} e^x \right], \quad \left( D = \frac{d}{dx} \right), \]

respectively.

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In this paper we give some operational formulae for these as well as Laguerre polynomials and employ them to derive many interesting results.

The first operational formula to be proved is

\[(1.5) \quad \prod_{j=1}^{n} (xD + x - \alpha - j) = (n)! \sum_{r=0}^{n} \frac{x^r}{(r)!} A_{n-r}^{(a)} (x) D^r.\]

Note that the formula (1.5) corresponds to the one given by Carlitz [3, p. 219] in the case of Laguerre polynomials [8, p. 428].

To prove (1.5) we observe that if

\[\Omega_n = \prod_{j=1}^{n} (xD + x - \alpha - j), \quad \Omega_0 = 1,\]

it can be proved very easily by the method of induction that

\[\Omega_n (y) = \frac{x^{n+a+1}}{e^{-x} D^n [e^x x^{-a-1} y]},\]

where \(y\) is some differentiable function of \(x\).

Next since

\[D^n [e^x x^{-a-1} y] = \sum_{r=0}^{n} \binom{n}{r} D^{n-r} [e^x x^{-a-1}] D^r y = \sum_{r=0}^{n} x^{-n-a-1} e^x (n)! \frac{x^r}{(r)!} A_{n-r}^{(a)} (x) D^r y,\]

(1.5) follows immediately.

In (1.5) if we take \(y = 1\), we obtain

\[(1.6) \quad \prod_{j=1}^{n} (xD + x - \alpha - j) \cdot 1 = (n)! A_n^{(a)} (x).\]

As an application of (1.5) and (1.6), let us consider

\[(m + n)! A_{m+n}^{(a)} (x) = \prod_{j=1}^{m} (xD + x - \alpha - n - j) \prod_{j=1}^{n} (xD + x - \alpha - j) \cdot 1\]

\[= (n)! \prod_{j=1}^{m} (xD + x - \alpha - n - j) A_n^{(a)} (x)\]

\[= (m)! \sum_{r=0}^{n} \frac{x^r}{(r)!} A_{m-r}^{(a+n)} (x) D^r A_n^{(a)} (x).\]
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But since

$D^r A_n^{(a)}(x) = A_{n-r}^{(a)}(x)$,

we readily get

(1.7) \( \binom{m + n}{n} A_{m+n}^{(a)}(x) = \sum_{r=0}^{\min(m,n)} \frac{x^r}{r!} A_{m-r}^{(a+n)}(x) A_{n-r}^{(a)}(x) \).

Further, from (1.7) we have

\[
\sum_{m=0}^{\infty} \binom{m + n}{n} t^m A_{m+n}^{(a)}(x) = \sum_{r=0}^{\infty} \frac{(xt)^r}{r!} A_{m-r}^{(a)}(x) \sum_{m=0}^{\infty} t^m A_{m}^{(a+n)}(x)
\]

\[= \sum_{r=0}^{\infty} \frac{(xt)^r}{r!} A_{m-r}^{(a)}(x) \cdot e^{xt} (1 + t)^{-(a+n+1)},\]

and making use of the relation [7, p. 45]

\[A_n^{(a)}(x + y) = \sum_{r=0}^{n} \frac{y^r}{r!} A_{n-r}^{(a)}(x),\]

we get the known formula [6, p. 7]

(1.8) \( \sum_{m=0}^{\infty} \binom{m + n}{n} t^m A_{m+n}^{(a)}(x) = e^{xt} (1 + t)^{-(a+n+1)} A_n^{(a)}(x(1 + t)) \).

Another operational formula for the polynomials $A_n^{(a)}(x)$ is

(1.9) \( (1 + D)^{-1-a} x^n = (n)! A_n^{(a)}(x), \quad \left( D = \frac{d}{dx} \right) \).

To prove it we note that

\[(1 + D)^{-1-a} x^n = \sum_{r=0}^{n} (-1)^r \frac{(1 + \alpha)^r}{r!} D^r x^n = \sum_{r=0}^{n} \frac{(1 + \alpha)^r}{r!} (\frac{n}{r}) x^{n-r},\]

which evidently yields the formula (1.9).
From (1.9) we have

$$ (n!) \ A_n^{(a+\beta)} (x) = (1 + D)^{-1-a-\beta} \ x^n, $$

(1.10)

the last formula gives us

$$ A_n^{(a+\beta)} (x) = (1 + D)^{-\beta} \ A_n^{(a)} (x); $$

Further let us operate on the identity

$$ A_n^{(a+\beta)} (x) = \sum_{r=0}^{n} (-1)^r (\beta)^r \ A_{n-r}^{(a)} (x). $$

(1.11)

Further let us operate on the identity

$$ e^{xt} = \sum_{n=0}^{\infty} \frac{t^n}{(n)!} \ x^n, $$

by $(1 + D)^{-1-a}$; the familiar shift rule then gives us

$$ (1 + D)^{-1-a} \ e^{xt} = e^{xt} (1 + t)^{-1-a} \left\{ 1 + \frac{D}{1 + t} \right\}^{-1-a} \cdot 1 = (1 + t)^{-1-a} \ e^{xt}. $$

On the other hand the second member yields

$$ \sum_{n=0}^{\infty} t^n \ A_n^{(a)} (x) $$

with the help of (1.9).

We thus arrive at the familiar generating function (1.2).

Now replace by $tD_y$, \( D_y = \frac{d}{dy} \) in (1.2) and operate on both sides by $(1 + D_y)^{-1-\beta}$. The left-hand side gives us

$$ (1 + D_y)^{-1-\beta} \ e^{xty} (1 + ty) = e^{xty} \left\{ 1 + xt + D_y \right\}^{-1-\beta} (1 + yt)^{-1-a} $$

$$ = e^{xty} (1 + xt)^{-1-\beta} \sum_{r=0}^{\infty} (-1)^r \frac{(1 + \beta)^r}{(r)!} \cdot \frac{D_y^r}{(1 + xt)^r} (1 + yt)^{-1-a} $$

$$ = e^{xty} (1 + xt)^{-1-\beta} (1 + yt)^{-1-a} \ {}_2F_0 \left[ 1 + \alpha, 1 + \beta; -; \frac{t}{(1 + xt)(1 + yt)} \right] $$
and the right-hand side yields

\[ \sum_{n=0}^{\infty} (n!) \, t^n \, A_n^{(a)} (x) \, A_n^{(b)} (y). \]

Combining these two sides we finally get

\[ \sum_{n=0}^{\infty} (n!) \, t^n \, A_n^{(a)} (x) \, A_n^{(b)} (y) \]

\[ \cong e^{yt} (1 + xt)^{-1-\beta} (1 + yt)^{-1-\alpha} \, _2F_0 \left[ 1 + \alpha, 1 + \beta; -; \frac{t}{(1 + xt)(1 + yt)} \right]. \]

From the relation (1.12) we have

\[ (1 + xt)^{-1-\beta} (1 + yt)^{-1-\alpha} \, _2F_0 \left[ 1 + \alpha, 1 + \beta; -; \frac{t}{(1 + xt)(1 + yt)} \right] \]

\[ \sum_{r, s, k = 0}^{\infty} \frac{r! \, (1 + \alpha)_r \, (1 + \beta)_s}{(r) ! \, (s) ! \, (k) !} \, (-1)^{s+k} \, x^k \, y^s \cdot t^{r+s+k} \]

\[ \sum_{n=0}^{\infty} (-t)^n \sum_{k=0}^{n} \frac{k! \, (1 + \alpha)_{n+r-k}(1 + \beta)_r}{(k) ! \, (n-k) !} \, x^{k-r} \, y^{n-k} \]

\[ \sum_{n=0}^{\infty} (-t)^n \sum_{k=0}^{n} \frac{(1 + \beta)_k \, (1 + \alpha)_{n-k}}{(n-k) !} \, y^{n-k} \sum_{r=0}^{k} (-1)^{r} \frac{(1 + \alpha + n - r)_r}{(n-k)!} \, x^{k-r} \]

\[ \sum_{n=0}^{\infty} (-1)^{n} \sum_{k=0}^{n} \frac{(1 + \beta)_k (1 + \alpha)_{n-k}}{(n-k) !} \, A_k^{(a+n-k)} (x). \]

Thus (1.12) is equivalent to

\[ \sum_{r=0}^{n} \frac{(r)!}{(n-r)!} \, (-1)^{r} \, (xy)^{n-r} \, A_n^{(a)} (x) \, A_n^{(b)} (y) \]

\[ = \sum_{k=0}^{n} \frac{(1 + \beta)_k (1 + \alpha)_{n-k}}{(n-k) !} \, y^{n-k} \, A_k^{(a+n-k)} (x). \]
2. Making use of the relation \([7, \text{p. 45}]\)

\[ L_{n}^{-(a+n+1)} (-x) = \Delta_n^{(a)} (x) \]

and our formula (1.8), we get

\[
(1 - D)^{a+n} (-x)^n = (n)! L_n^{(a)} (x),
\]

which yields the following interesting result

\[
L_n^{(a+\beta)} (x) = (1 - D)^{\beta} L_n^{(a)} (x).
\]

Now consider

\[
L_n^{(a+\beta+1)} (x) = (1 - D)^{\beta+1} L_n^{(a)} (x) = \left[ 1 + \frac{D}{1 - D} \right]^{-\beta-1} L_n^{(a)} (x)
\]

\[
= \sum_{r=0}^{n} (-1)^r \frac{(\beta + 1)_r}{(r)!} (1 - D)^{-r} D^r L_n^{(a)} (x),
\]

and it follows that

\[
L_n^{(a+\beta+1)} (x) = \sum_{r=0}^{n} \frac{(\beta + 1)_r}{(r)!} L_n^{(a-r)} (x).
\]

Formula (2.3) was proved earlier in a different way by Al-Salam \([1, \text{p. 131}]\), and our proof differs markedly with that of Rainville \([5, \text{p. 209}]\).

Next let us consider the expression,

\[
(-1)^n e^x D^n [e^{-x} L_n^{(a)} (x)]
\]

which by the usual shift rule gives us

\[
(-1)^n e^x D^n [e^{-x} L_n^{(a)} (x)] = (1 - D)^n L_n^{(a)} (x).
\]

On making use of the relation (2.2) we obtain

\[
(-1)^n e^x D^n [e^{-x} L_n^{(a)} (x)] = L_n^{(a+n)} (x),
\]
which may be put in the form

\[(2.4) \quad R_n(1 + \alpha, x) = (-1)^n e^x D^n [e^{-x} L_n^{(\alpha)}(x)],\]

where \(R_n(\alpha, x)\) is the pseudo Laguerre set defined by Shively [5, p. 298] as

\[R_n(\alpha, x) = \binom{a_{2n}}{(n)! (a_n)} F_1 \left( -n ; a + n ; x \right) .\]

The formula (2.4) has been proved recently by Khandekar in a different way (see [4], p. 2).

Further, from (2.1) we have

\[\frac{(-x)^n}{(n)!} = (1 - D)^{-a-n} L_n^{(\alpha)}(x) = \sum_{r=0}^{n} \frac{(\alpha + n)_r}{(r)!} D^r L_n^{(\alpha)}(x)\]

which gives

\[(2.5) \quad \frac{(-x)^n}{(n)!} = \sum_{r=0}^{n} (-1)^r \frac{(\alpha + n)_r}{(r)!} L^{(\alpha+r)}_{n-r}(x) .\]

Next consider the identity

\[e^{-xt} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n)!} t^n ,\]

operate on both sides by \((1 - D)^{\alpha}\), make use of (2.1) and proceed as in the cases of (1.12) and (1.13). We then get the generating function

\[(2.6) \quad (1 + t)^{\alpha} e^{-xt} = \sum_{n=0}^{\infty} t^n L_n^{(\alpha-n)}(x) ,\]

due to Erdélyi, and the known formula [2, p. 151]

\[(2.7) \quad \sum_{n=0}^{\infty} (n)! t^n L_n^{(\alpha-n)}(x) L_n^{(\beta-n)}(y) \]

\[= \begin{cases} e^{yt} (1 - yt)^{\alpha-\beta} t^\beta (\beta) ! L^{(\alpha-\beta)}_{\beta} \left( \frac{(1 - xt)(1 - yt)}{t} \right) \\ e^{yt} (1 - xt)^{\beta-\alpha} t^\alpha (\alpha) ! L^{(\beta-\alpha)}_{\alpha} \left( \frac{(1 - xt)(1 - yt)}{t} \right) \end{cases} .\]
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