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ON THE ASYMPTOTIC BEHAVIOR OF THE ONE-SIDED GREEN'S
FUNCTION FOR A DIFFERENTIAL OPERATOR
NEAR A SINGULARITY

STEVEN BANK *)

1. Introduction.

In this paper we consider nth order linear differential operators \( \Omega \), whose coefficients are complex functions defined and analytic in unbounded sectorial regions, and have asymptotic expansions, as the complex variable \( x \to \infty \) in such regions, in terms of real (but not necessarily integral) powers of \( x \) and/or functions which are of smaller rate of growth (\(<\)) than all powers of \( x \) as \( x \to \infty \). (We are using here the concept of asymptotic equivalence (\( \sim \)) as \( x \to \infty \), and the order relation «\( < \)» introduced in [8; § 13]. (A summary of the necessary definitions from [8] appears in § 2 below.) However, it should be noted (see [8; § 128 (g)]) that the class of operators treated here includes, as a special case, those operators whose coefficients are analytic and possess asymptotic expansions (in the customary sense) of the form \( \Sigma c_j x^{-\lambda_j} \) with \( \lambda_j \) real and \( \lambda_j \to +\infty \) as \( j \to \infty \). More specifically, we are concerned here with the asymptotic behavior of the one-sided Green's function \( H(x, \zeta) \) for the operator \( \Omega \) (see [7; p. 33] or § 3 below), near the singular point at \( \infty \). This function plays a major role in determining the asymptotic behavior near \( \infty \) of solutions of the non-homogeneous equation \( \Omega(y) = f \) (for functions \( f \) analytic in a sectoral region \( D \)), since

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the function \( y(x) = \int_{x_0}^{x} H(x, \zeta)j(\zeta)d\zeta \) is a solution of \( \Omega(y) = f \) satisfying zero initial conditions at the point \( x_0 \) in \( D \). (The proof of this fact for the real domain given in [7; p. 34] is easily seen to be valid for the complex simply-connected region \( D \), where of course, the contour of integration is any rectifiable path in \( D \) from \( x_0 \) to \( x \)).

If \( \{\psi_1, ..., \psi_n\} \) is a fundamental set of solutions for \( \Omega(y) = 0 \), then the Green's function \( H(x, \zeta) \) is a function of the form \( \sum_{j=1}^{n} \psi_j(x)w_j(\zeta) \). In this paper, we determine the asymptotic behavior of \( H(x, \zeta) \) by determining the asymptotic behavior near \( \infty \) of the functions \( w_j(\zeta) \), when \( \{\psi_1, ..., \psi_n\} \) is a particular fundamental set whose existence was proved in [1, 2] and whose asymptotic behavior in subsectorial regions is known. The asymptotic behavior of \( \{\psi_1, ..., \psi_n\} \) is as follows: Associated with \( \Omega \) is a polynomial \( P(\alpha) \) of degree \( p \leq n \) ([2; § 3(e)]). If \( \alpha_1, ..., \alpha_r \) are the distinct roots of \( P(\alpha) \) with \( \alpha_j \) of multiplicity \( m_j \), then \( \psi_1, ..., \psi_{p} \) are solutions of \( \Omega(y) = 0 \) where each \( \psi_j \) is \( \sim \) to a constant multiple of a distinct function of the form \( x^m(\log x)^{m-1} \), where \( 1 \leq m \leq m_j \). For the remaining solutions \( \psi_{p+1}, ..., \psi_{n} \), each \( \psi_k \) is \( \sim \) to a function of the form \( \exp \int V_k \) where each \( V_k \) is \( \sim \) to a function of the form \( c_kx^{-1+d_k} \) for \( d_k > 0 \) and complex non-zero \( c_k \). (The functions \( c_kx^{-1+d_k} \) involved can be determined in advance by an algorithm. For a complete discussion, see § 4 below).

If the above fundamental set \( \{\psi_1, ..., \psi_n\} \) is used to calculate the Green's function, \( H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x)w_j(\zeta) \), directly from the definition of \( H(x, \zeta) \) (see § 3 below), the asymptotic behavior of the functions \( w_j(\zeta) \) is difficult to determine since each \( w_j \) depends on the quotient of the Wronskian of \( \{\psi_1, ..., \psi_n\} - \{\psi_j\} \) by the Wronskian of \( \{\psi_1, ..., \psi_n\} \). However, in this paper we do succeed in determining the asymptotic behavior of the functions \( w_j(\zeta) \) by taking advantage of a factorization result proved in [1]. It was shown in [1; § 7] that under a simple change of dependent variable and multiplication by a suitable function, the operator \( \Omega \) is transformed into an operator \( \Phi \) which possesses an exact factorization into first order operators \( f_j \) of the form \( f_j(y) = y - (y'/f_j) \), where the asymptotic behavior of the functions \( f_1, ..., f_n \) involved is known.
precisely. Since the Green's functions $K(x, \zeta)$ for a factored operator $\Phi=\Phi_1\Phi_2$ is related to the Green's functions $K_1$ and $K_2$ for $\Phi_1$ and $\Phi_2$, respectively, by $K(x, \zeta) = \int_{\zeta}^{x} K_2(x, s)K_1(s, \zeta)ds$ (see [7; p. 41] for the proof in the real domain and § 8 B below for the proof in the complex domain), we are in a position to use an inductive proof to determine the behavior of the Green's function for $\Phi$ (see § 6 below), and this easily leads to a result for $\Omega$. In this connection, we make use of results in [3, 4] in determining the asymptotic behavior of the integrals which arise.

Our main result (§ 5) states that if $\Omega$ has been suitably normalized by dividing through by a known function of the form $cx^n$, and if the distinct roots $\alpha_0, ... , \alpha_r$ of $P(\alpha)$ also have distinct real parts, then there exists a fundamental set of solutions $\{\psi_1, ... , \psi_n\}$ for $\Omega(y)=0$ having the asymptotic behavior which was previously described such that the asymptotic behavior of each function $\psi_j(x)$, in the Green's function $H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x)\psi_j(\zeta)$ for $\Omega$, is related to the asymptotic behavior of the corresponding function $\psi_i(x)$ as follows: If $1 \leq j \leq p$, we know $\psi_j(x)$ is $\sim$ to a function of the form $a_jx^{\alpha_j}(\log x)^{m-1}$ where $1 \leq m \leq m_i$ and $a_j$ is a constant. We prove that $\psi_j(\zeta)$ is $\sim$ to a constant multiple of $\zeta^{r-1-a_j}(\log \zeta)^{m_i-m}$. For $p+1 \leq k \leq n$, we know $\psi_k(x)$ is $\sim$ to a function of the form $\exp \int_{\zeta}^{x} V_k$. We prove that $\psi_k(\zeta)$ is $\sim$ to a function of the form $\exp \int_{\zeta}^{x} U_k$, where $U_k \sim -V_k$, and in fact, we obtain more detailed information on $U_k$. (The condition concerning distinctness of the real parts of the $\alpha_i$ is needed in the proof since it guarantees that any two of the functions $\psi_1, ... , \psi_p$ are comparable with respect to the order relation $\prec$ (see § 2 (b))). Since the functions $\psi_1, ... , \psi_n$ comprise a fundamental set of solutions of the equation $\Omega'(y)=0$ where $\Omega'$ is the adjoint of $\Omega$ (see [7; p. 38]), we have therefore succeeded in also determining the asymptotic behavior of a fundamental set of solutions of the adjoint equation $\Omega^*(y)=0$.

In § 8, we prove certain results which are needed in the proof of the main theorem.
2. Concepts from [5] and [8].

(a) [8; § 94]. Let $-\pi \leq a < b \leq \pi$. For each non-negative real valued function $g$ on $(0, (b-a)/2)$, let $E(g)$ be the union (over $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \arg (x - h(\delta)) < b - \delta$ where $h(\delta) = g(\delta) \exp (i(a+b)/2)$. The set of all $E(g)$ (for all choices of $g$) is denoted $F(a, b)$ and is a filter base which converges to $\infty$. Each $E(g)$ is simply-connected by [8; § 93]. If $V(x)$ is analytic in $E(g)$ then the symbol $\int V$ will stand for a primitive of $V(x)$ in $E(g)$. A statement is said to hold except in finitely many directions (briefly e.f.d.) in $F(a, b)$ if there are finitely many points $r_1 < r_2 < ... < r_q$ in $(a, b)$ such that the statement holds in each of $F(a, r_1), F(r_1, r_2), ..., F(r_q, b)$ separately.

(b) [8; § 13]. If $f$ is analytic in some $E(g)$, then $f \to 0$ in $F(a, b)$ means that for any $\varepsilon > 0$, there is a $g_1$ such that $|f(x)| < \varepsilon$ for all $x \in E(g_1)$. $f < 1$ in $F(a, b)$ means that in addition to $f \to 0$, all functions $\theta_i f \to 0$ where $\theta_i$ is the operator $\theta_i f = (x \log x ... \log_{i-1} x)'^i$. Then $f_i < f_2$, $f_i \sim f_2$, $f_i \leq f_2$, $f_i \geq f_2$ mean respectively, $f_1/f_2 < 1$, $f_1 - f_2 < f_2$, $f_1 \sim f_2$ for some constant $c \neq 0$, and finally either $f_i < f_2$ or $f_i \approx f_2$. If $f \sim c$, we write $f(\infty) = c$, while if $f < 1$, we write $f(\infty) = 0$. The relation $\ll$ has the property ([8; § 28]) that if $f < 1$ then $0 f < 1$ for all $j$. If $f \sim K x^{\alpha_0}(\log x)^{\delta_0}$ for complex $\alpha_0$ and $K$ and real $\alpha_1$, then $\delta_0(f)$ will denote $\alpha_0$. It is easily verified that for every $\varepsilon > 0$, $x^{\alpha_0 - \varepsilon} < x^{\alpha_0} < x^{\alpha_0 + \varepsilon}$, from which it easily follows that if $\Re(\delta_0(f)) < \Re(\delta_0(h))$ then $f < h$. If $f \sim c x^{-1+d}$ where $c$ is a non-zero constant and $d \geq 0$ then, the indicial function of $f$ is the function on $(-\pi, \pi)$ defined by $IF(f)(\varphi) = \cos (d \varphi + \arg c)$. Finally, a function $h$ is called trivial if $h \ll x^\alpha$ for all real $\alpha$.

(c) [8; § 49] (and [10; § 53]). A logarithmic domain of rank zero (briefly, an $LD_0$) over $F(a, b)$ is a complex vector space $L$ of functions (each analytic in some $E(g)$), which contains the constants, and such that any finite linear combination of elements of $L$, with coefficients which are functions of the form $c x^\alpha$ (for real $\alpha$), is either $\sim$ to a function of this latter form or is trivial.
(d) [5; § 3]. If \( G(z) = \sum_{j=0}^{n} b_j(x)z^j \), where the \( b_j \) belong to an \( LD_0 \), then a function \( N(x) \) of the form \( cx^a \) (for real \( a \)) is called a **critical monomial** of \( G \), if there is a function \( h \sim N \) such that \( G(h) \) is not \( \sim G(N) \). (An algorithm for finding all critical monomials can be found in [5; § 26]). The critical monomial \( N \) of \( G \) is called **simple** if \( N \) is not a critical monomial of \( \partial G/\partial z \).

3. The Green’s function.

If \( \Omega(y) = \sum_{j=0}^{n} a_j(x)y^{(j)} \) where the coefficients \( a_j(x) \) are analytic in a simply-connected region \( D \), and \( a_n(x) \) has no zero in \( D \), then the one-sided Green’s function for \( \Omega \) is the function \( H(x, \zeta) \) on \( D \times D \) defined as follows: If \( B = \{ \psi_1, ..., \psi_n \} \) is a fundamental set of solutions in \( D \) for \( \Omega(y) = 0 \), and if \( W \) is the Wronskian of \( B \) while \( W_j \) is the Wronskian of \( B - \{ \psi_j \} \), then \( H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x)v_j(\zeta) \) where

\[
v_j(\zeta) = (-1)^{n+j}W_j(\zeta)/(a_n(\zeta)W(\zeta)).
\]

(Remark: It follows from the uniqueness theorem for solutions of linear differential equations that the Green’s function is independent of which fundamental set is used, since it is easily verified (as in [7; p. 33]) that no matter which fundamental set is used, the corresponding \( H(x, \zeta) \) is a solution of \( \Omega(y) = 0 \) for each \( \zeta \in D \), satisfying the following initial conditions at \( x = \zeta : \partial^kH(x, \zeta)/\partial^k = 0 \) for \( 0 \leq k \leq n-2 \); \( \partial^{n-1}H(x, \zeta)/\partial x^{n-1} = \) \((1/a_n(\zeta))\).)

4. Results from [1] and [2].

Let \( \Omega(y) \) be an \( n^{th} \) order linear differential polynomial, coefficients in an \( LD_0 \) over \( F(a, b) \). If \( \theta \) is the operator \( \theta y = xy' \), \( \Omega(y) \) may be written \( \Omega(y) = \sum_{j=0}^{n} B_j(x)\theta^jy \) where the functions \( B_j \) belong to an \( LD_0 \). We assume \( B_n \) is non-trivial. By dividing through by the highest power of \( x \) which
is \sim\) to a coefficient \(B_j\), we may assume that for each \(j\), \(B_j \leq 1\) and there is an integer \(p \geq 0\) such that \(B_p \approx 1\) while for \(j > p\), \(B_j < 1\). Let \(q = \min \{j : B_j \approx 1\}\). By dividing through by \(B_q(\infty)\), we may assume \(B_q \sim 1\). Let \(P(\alpha) = \sum_{j=0}^{n} B_j(\infty)\alpha^j\) and let \(\alpha_1, ..., \alpha_r\) be the distinct non-zero roots of \(P(\alpha)\) with \(\alpha_j\) of multiplicity \(m_j\). (Thus \(q + \sum m_i = p\)). Define 

\[M_1, ..., M_p\] 
as follows: \(M_j = (\log x)^{j-1}\) if \(1 \leq j \leq q\); \(M_{q+j} = x^{\alpha_j}(\log x)^{j-1}\) if \(1 \leq j \leq m_1\), and in general, \(M_{q+m_1 + ... + m_k + j} = x^{\alpha_{k+1}}(\log x)^{j-1}\) for \(1 \leq k < r\) and \(1 \leq j \leq m_{k+1}\). Define a sequence of integers \(p = t(0) < t(1) < ... < t(\sigma) = n\) as follows: \(t(0) = p\) and if \(t(j)\) has been defined and is less than \(n\), let \(t(j+1)\) be the largest \(k\) such that \(t(j) < k \leq n\) and such that \(B_i \leq B_k\) for all \(i\), \(t(j) < i \leq n\). Let \(G(z) = \sum_{j=0}^{\sigma} z^{t(j)}B_{t(j)}z^{t(j)-p}\), and assume that the critical monomials \(N_1, ..., N_{n-p}\) of \(G\) are each simple (§ 2 (d)), and are arranged so that \(N_j \leq N_{j+1}\) for each \(j\). Then e.f.d. in \(F(a, b)\), the following conclusions hold:

(a) Each \(N_j\) is of the form \(c_j x^{j-1+d_j}\) where \(c_j\) is a non-zero constant and \(d_j > 0\).

(b) The equation \(\Omega(y) = 0\) possesses a linearly independent set of solutions \(\{g_1, ..., g_p\}\) where \(g_j \sim M_j\) for \(1 \leq j \leq p\).

(c) If we set \(h_j = (\log x)^{-q}g_j\) for \(1 \leq j \leq p\) and define functions \(f_1, ..., f_p, \Psi_0, ..., \Psi_{p-1}\) recursively by the formulas, \(\Psi_0 = h_1\) and \(f_{j+1} = \Psi_j/\Psi_i\) where \(\Psi_j = (f_j ... f_1)h_{j+1}\) (recalling that \(f_j(y) = y - (y'/f_j)\)), then there exist functions \(f_{p+1}, ..., f_n\) with \(f_k \sim N_{k-p}\) such that,

(i) The equation \(\Omega(y) = 0\) possesses solutions \(g_{p+1}, ..., g_n\) such that \(g_k\) is of the form \(g_k = R_k \exp \int f_k\) where \(R_k \sim (\log x)^{q} \prod_{j=1}^{k-1} (f_j/(f_j-f_k))\) for \(p+1 \leq k \leq n\).

(ii) The solutions \(g_1, ..., g_n\) form a fundamental set of solutions for \(\Omega(y) = 0\).

(iii) If \(\Phi_0(z) = (1/q!)\Omega((\log x)^q z)\), then for some function \(E \sim 1\), the operator \(\Phi_0\) possesses the exact factorization \(\Phi_0 = Ef_n ... f_1\) where \(f_j(y) = y - (y'/f_j)\).
(iv) If $h_k = (\log x)^{-q} g_k$ for $1 \leq k \leq n$, then $f_k \ldots f_1(h_k) = 0$ for each $k \in \{1, \ldots, n\}$.

(v) The functions $f_1, \ldots, f_p$ have the following asymptotic behavior:

\[ f_i \sim -(q-i+1)x^{-(i+1)}(\log x)^{-1} \text{ if } 1 \leq j \leq q; \quad f_{q+j} \sim \alpha_{q+1} x^{-1} \text{ if } 1 \leq j \leq m_{q+1}, \]

and in general, $f_{q+m_j+\ldots+m_k+j} \sim \alpha_{k+1} x^{-1}$ for $1 \leq k < r$ and $1 \leq j \leq m_{k+1}$.

(REMARK. (a) is proved in [1; § 5]; (b) is proved in [2; §§ 5, 7, 10]; For (c), (i) is proved in [1; § 9] in light of [1; § 8]; (ii) is proved in [1; § 9]; (iii) and (v) are proved in [1; § 7]; (iv) for $1 \leq k \leq p$ follows from the definition of $f_j$, while for $p+1 \leq k \leq n$, it is proved in [1; § 9]).

In view of the above results, and with the above notation, we can make the following definition:

**Definition.** A fundamental system of solutions $(\psi_1, \ldots, \psi_n)$ of $\Omega(y) = 0$ is called asymptotically canonical if $\psi_j \approx M_j$ for $1 \leq j \leq p$ while for $p+1 \leq k \leq n$, $\psi_j$ is $\approx$ to a function of the form $R_k \exp \int f_k$.

5. **The Main Theorem.**

Let $\Omega(y)$ be an $n^{th}$ order linear differential polynomial with coefficient in an $LD_0$ over $F(a, b)$. By dividing through by a convenient function of form $cx^\theta$ (as in § 4), we may assume $\Omega(y) = \sum_{j=0}^{n} B_j(x) \theta^j y$, where $\theta$ is the operator $\theta y = xy'$, and where the coefficients $B_j$ belong to an $LD_0$ over $F(a, b)$ and have the following asymptotic properties: $B_j \neq 1$ for each $j$; For some integers $0 \leq q \leq p$, $B_p \approx 1$, $B_q \sim 1$ and $B_j < 1$ if $j > p$ or $j < q$. Let $B_n$ be non-trivial in $F(a, b)$. Let $P(\alpha) = \sum_{j=0}^{n} B_j(\infty) \alpha^j$ and let $P$ have the property that if $\alpha$ and $\beta$ are roots of $P$ with $\alpha \neq \beta$, then $\alpha$ and $\beta$ have distinct real parts. Let $\alpha_1, \ldots, \alpha_r$ be the distinct non-zero roots of $P$, with $\alpha_i$ of multiplicity $m_i$, and let $M_1, \ldots, M_p$ be as in § 4. Let $G(z)$ be the polynomial constructed as in § 4, and assume, as in § 4, that the critical monomials $N_1, \ldots, N_{n-p}$ of $G(z)$ are each simple.
and are arranged so that \( N_j \leq N_{j+1} \) for each \( j \). Define functions \( u(x_1), \ldots, u_n(x) \) e.f.d. in \( F(a, b) \) as follows: 

\[
u_j(x) = x^{-1}(\log x)^{q-1} \text{ if } 1 \leq j \leq q; \]

\[
u_{q+j}(x) = x^{-1-s_k}(\log x)^{m_j-1} \text{ for } 1 \leq j \leq m_1, \text{ and in general } \nu_{q+m_j+\ldots+m_{k+j}}(x) = x^{-1-s_k}(\log x)^{m_{k+1}-1} \text{ for } 1 \leq k < r \text{ and } 1 \leq j \leq m_{k+1}; \]

For \( p+1 \leq k \leq n \), let \( u_k(x) \) be a function of the form \( u_k(x) = E_k(x) \exp \left(-\int f_k\right) \) where \( E_k = f_k \prod_{j=k+1}^{n} \frac{f_j}{f_j-f_k} \), the \( f_j \) being in \( \S \ 4 \). Then e.f.d. \( F(a, b) \), the following conclusions hold:

1. The equation \( \Omega(y) = 0 \) possesses an asymptotically canonical fundamental system of solutions \( (\psi_1, \ldots, \psi_n) \) in the sense of \( \S \ 4 \) (i.e. \( \psi_j \approx M_j \) for \( 1 \leq j \leq p \), while \( \psi_k \approx R_k \exp \left(\int f_k \right) \) for \( p+1 \leq k \leq n \)) such that such that the one-sided Green’s function for \( \Omega \) is of the form \( H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x)w_j(\zeta) \) where \( w_j = u_j \) for each \( j = 1, \ldots, n \).

2. The equation \( \Omega^*(y) = 0 \), where \( \Omega^* \) is the adjoint of \( \Omega \), possesses a fundamental set of solutions \( (\psi_1^*, \ldots, \psi_n^*) \) where \( \psi_j^* \sim u_j \) for each \( j = 1, \ldots, n \).

**Remark.** It suffices to prove Part (1), since (2) will follow from (1) (see [7; p. 38]). In view of \( \S \ 4 \) (c) (iii) we first prove a lemma concerning the Green’s function for a factored operator, \( \Phi = f_n \ldots f_1 \). The proof will make use of results proved in \( \S \ 8 \), and the proof of the main theorem will be concluded in \( \S \ 7 \).

6. **Lemma.** Let \( 0 \leq q \leq p \leq n \), and let \( m_1, \ldots, m_r \) be positive integers such that \( q + \sum_{j=1}^{n} m_j = p \). Let \( \alpha_1, \ldots, \alpha_r \) be distinct non-zero complex numbers such that \( \text{Re}(\alpha_j) < \text{Re}(\alpha_{j+1}) \) for each \( j \). If \( q > 0 \), assume also that \( \text{Re}(\alpha_j) \neq 0 \) for each \( j \). Let \( M_1, \ldots, M_p \) be as defined in \( \S \ 4 \). Let \( I \) be an open subinterval of \( (-\pi, \pi) \) and let \( h_1, \ldots, h_p \) be functions such that \( h_j \sim (\log x)^{-\alpha} M_j \) in \( F(I) \) for \( 1 \leq j \leq p \). Let \( f_1, \ldots, f_p, \Psi_0, \ldots, \Psi_{p-1} \) be
defined as in § 4 (c) and let $f_1, f_p$ have the asymptotic behavior described in § 4 (c) (v). Let $N_1, \ldots, N_{n-p}$ be distinct functions, each of the form $c_j x^{-1+d_j}$ for complex $c_j \neq 0$ and $d_j > 0$, arranged so that $N_j \preceq N_{j+1}$ for each $j$. For $p+1 \leq k \leq n$, let $f_k$ be a function $\sim N_{k-p}$ in $F(I)$ and let $h_k$ be a function of the form $h_k = A_k \exp \int f_k$ where $A_k \sim \prod_{j=1}^{k-1} (f_j/f_1 - f_k)$ in $F(I)$. Assume that $h_1, \ldots, h_n$ are linearly independent and that for each $j \in \{1, \ldots, n\}$, $f_j \cdots f_1(h_1) = 0$ (where $f_j(y) = y - (y'/f_j)$). Let $\Phi = f_n \cdots f_1$ and let $u_1, \ldots, u_n$ be as in § 5. Then, e.f.d. in $F(I)$, there exists a fundamental set of solutions $\{\varphi_1, \ldots, \varphi_n\}$ of $\varphi(y) = 0$, such that $\varphi_j \approx h_j$ for $j = 1, \ldots, n$ and such that the one-sided Green's function for $\Phi$ is of the form $H_0(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x) v_j(\zeta)$ where $v_j \approx u_j$ for $j = 1, \ldots, n$.

**Proof.** The proof will be by induction on $n$. We consider first the case $n = 1$. Here $\Phi = f_1$, and since $f_1(h_1) = 0$, we have by § 8 A that the Green's function for $\Phi$ is,

$$H_0(x, \zeta) = h_1(x) v_1(\zeta) \quad \text{where} \quad v_1(\zeta) = f_1(\zeta)/h_1(\zeta).$$

We distinguish the two cases $p < n$ and $p = n$. If $p < n$ then $p = 0$ (since $n = 1$). Thus by § 5, $u_1(\zeta) = E_1(\zeta) \exp (-\int f_1 \xi)$ where $E_1 = f_1$. But since $f_1(h_1) = 0$, clearly $h_1(\zeta) = \exp \int f_1$ and hence by (1), $v_1 = u_1$ so the result holds if $p < n$. If $p = n = 1$, we distinguish the two subcases $q < p$ and $q = p$. If $q < p$, then $q = 0$. Hence $h_1 \sim x^{-a}$ and $f_1 \sim \alpha x^{-1}$. Thus by (1), $v_1(\zeta) \approx \zeta^{-1-a_1}$, so $v_1 \approx u_1$. If $q = p$, then $h_1 \sim (\log x)^{-1}$ and $f_1 \sim -x^{-1} (\log x)^{-1}$. Hence by (1), $v_1(\zeta) \approx \zeta^{-1}$, so again $v_1 \approx u_1$. Thus the lemma holds for $n = 1$.

Now let $n > 1$, and assume that the lemma holds for $n-1$. Let $h_1, \ldots, h_n$ and $\Phi = f_n \cdots f_1$ be given as in the statement of the lemma. (We show that the conclusion of the lemma holds for $\Phi$). It follows from the hypothesis, that $h_1, \ldots, h_{n-1}$ are solutions of $\Phi(y) = 0$ where

$$\Phi_1 = f_{n-1} \cdots f_1.$$
We distinguish the two cases, $p = n$ and $p < n$.

**Case I.** $p = n$. In this case, we will distinguish three subcases.

**Subcase A.** $q < p$ and $m_r = 1$. Then $h_n \sim x^{\sigma_r} (\log x)^{-q}$. It is easily verified that using the given solutions $h_1, \ldots, h_{n-1}$ of $\Phi_1(y) = 0$, the operator $\Phi_l$ satisfies the induction hypothesis, where the corresponding functions $u_l$ are precisely $u_1, \ldots, u_{n-1}$ as defined in the statement of the lemma (see § 5). Hence by the inductive assumption, there exists e.f.d. in $F(l)$, a fundamental set of solutions $\{\varphi_1, \ldots, \varphi_{n-1}\}$ of $\Phi_1(y) = 0$ such that $\varphi_i \approx h_i$ for each $i$ and such that the Green's function for $\Phi_l$ is of the form $H_l(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x) w_j(\zeta)$ where $w_i \approx u_i$ for each $i$. Now by definition of $f_n$, we have $f_n(\Psi_{n-1}) = 0$, where $\Psi_{n-1} = f_{n-1} \ldots f_1(h_n)$. In view of the asymptotic relations for the $f_j$ given in § 4 (c) (v), it is easily verified using [1; § 6 (B), (D)] that

\[ \Psi_{n-1} = x^{\sigma_r}. \]

Since $f_n(\Psi_{n-1}) = 0$ and $f_n \sim \alpha_n x^{-1}$, it follows from § 8 A that the Green's function for the operator $f_n$ is $H_2(x, \zeta) = \Psi_{n-1}(x) w(\zeta)$ where (using (3)), $w(\zeta) \approx \zeta^{-1-\sigma_r}$. Since $\Phi = f_n \Phi_1$ (by (2)), we have by § 8 B that the Green's function for $\Phi$ is

\[ H_0(x, \zeta) = \int_{\zeta}^{x} H_1(x, s) H_2(s, \zeta) ds. \]

Hence,

\[ H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x) w(\zeta) \int_{\zeta}^{x} w_j(s) \Psi_{n-1}(s) ds. \]

Now $w_i \approx u_i$, so in view of (3), $w_j(s) \Psi_{n-1}(s) \approx s^{\sigma_r} u_j(s)$. Hence by the asymptotic relations for the $u_i$ (see § 5), clearly for $1 \leq j \leq n-1, (\delta_0(w_j \Psi_{n-1}))$ is either $\alpha_j - 1$ or $\alpha_j - 1 - \alpha_k$ for some $k < r$. Since $\alpha_r \neq 0$ and $\alpha_k \neq \alpha_r$ for $k < r$, we have that $\delta_0(w_j \Psi_{n-1}) \neq -1$ for each $j$. Thus by § 8 D (a), for each $j = 1, \ldots, n-1$, there exists e.f.d. in $F(l)$, a function $Q_j(s) \approx s^{\sigma_r + 1} u_j(s)$ such that $Q_j = w_j \Psi_{n-1}$ Hence the right side of (4) is

\[ \int_{\zeta}^{x} \sum_{j=1}^{n-1} \varphi_j(x) w(\zeta) \int_{\zeta}^{x} w_j(s) \Psi_{n-1}(s) ds. \]
\[ \sum_{j=1}^{n-1} \varphi_j(x) H(x) \delta_j(Q_j(x) - Q_{j-1}(\zeta)), \] so (4) may be written,

\[ H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x) v_j(\zeta) + V(x) w(\zeta) \]

where \( v_j(\zeta) = -w(\zeta) Q_j(\zeta) \) and \( V(x) = \sum_{j=1}^{n-1} \varphi_j(x) Q_j(x) \). Since \( w(\zeta) \approx \zeta^{-a-1} \), and \( Q_j(\zeta) \approx \zeta^{s_{q+1}} u_j(\zeta) \), clearly \( v_j = u_j \) for \( 1 < j < n - 1 \). Furthermore, since \( u_n(\zeta) \approx \zeta^{-a-r} \), we have \( w \approx u_n \). Hence in view of (5), the conclusion of the lemma will hold for \( \Phi \), if it can be shown that \( \{ \varphi_1, \ldots, \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \) and that

\[ V(x) = h_n(x). \]

To prove (6), we note first that \( \varphi_1, \ldots, \varphi_{n-1} \) are independent solutions of \( \Phi(y) = 0 \), since they form a fundamental set for \( \Phi(y) = 0 \). Hence in view of (5), we have by § 8 (C) that \( \varphi_1, \ldots, \varphi_{n-1}, V \) form a fundamental \( \Phi(y) = 0 \). Since \( h_1, \ldots, h_n \) also form a fundamental set, there exist constant \( \beta_k \) and \( \gamma_k \) such that,

\[ V = \sum_{k=1}^{n} \beta_k h_k, \]

and

\[ h_n = \sum_{k=1}^{n-1} \gamma_k \varphi_k + \gamma_n V. \]

Now by hypothesis, for \( q + 1 \leq j \leq n - 1 \), we have \( \text{Re} (\delta_0(h_j)) < \text{Re} (\alpha_i) \). Thus \( h_j < h_n \) (see § 2 (b)), and since \( \varphi_j \approx h_j \), we have \( \varphi_j < h_n \) also. Hence \( U = \sum_{j=q+1}^{n-1} \beta_j h_j \) is \( \varphi_j < h_n \), and \( W = \sum_{j=q+1}^{n-1} \gamma_j \varphi_j < h_n \), and so (7) and (8) may be written,

\[ V = \beta_n h_n + \sum_{i=1}^{q} \beta_i h_i + U, \] where \( U < h_n \), and

\[ h_n = \gamma_n V + \sum_{i=1}^{q} \gamma_i \varphi_i + W \] where \( W < h_n \).
Now if \( q = 0 \), then (6) will follow from (9) if \( \beta_n \neq 0 \). But this is clear, for if \( \beta_n = 0 \), then by (9), \( V < h_n \), and hence from (10) we would obtain \( h_n < h_n \) (since \( q = 0 \)) which is a contradiction. Now consider the case \( q > 0 \). Then by assumption, either \( \text{Re} (\alpha_r) > 0 \) or \( \text{Re} (\alpha_r) < 0 \). If \( \text{Re} (\alpha_r) > 0 \), then for \( 1 \leq i \leq q \), \( h_i < h_n \) (and hence \( \varphi_i < h_n \)) since \( \delta_0 (h_i) = 0 < \text{Re} (\alpha_r) \). Thus again, (6) will follow from (9) if \( \beta_n \neq 0 \). But if \( \beta_n = 0 \), then from (9), \( V < h_n \) and so from (10) we would obtain \( h_n < h_n \) which is impossible. If \( \text{Re} (\alpha_r) < 0 \), we consider each term \( \varphi_i Q_i \) in \( V \). Since \( \varphi_i \approx h_i \), we have for \( 1 \leq j \leq n - 1 \), \( \varphi_i Q_j \approx x^{q_j} (1 + \log x)^{-q} M_j u_j \). By the asymptotic relations for \( M_j \) and \( u_j \), clearly \( \delta_0 (M_j u_j) = -1 \), and hence,

\[
\delta_0 (\varphi_i Q_i) = \alpha_r \quad \text{for} \quad 1 \leq j \leq n - 1.
\]

Since \( \text{Re} (\alpha_r) < 0 \) and \( \delta_0 (h_k) = 0 \) for \( 1 \leq k \leq q \), we thus obtain \( V < h_k \) and \( h_n < h_k \) for \( 1 \leq k \leq q \). Thus from (9), \( \sum_{i=1}^{q} \beta_i h_i < h_k \) for each \( k \leq q \).

Since \( h_1 < h_2 < \ldots < h_q \), this implies \( \beta_i = 0 \) for \( 1 \leq i \leq q \), for in the contrary case, setting \( j_0 = \max \{ i : 1 \leq i \leq q, \beta_i \neq 0 \} \), we would obtain the contradiction, \( h_{j_0} \approx \sum_{i=1}^{q} \beta_i h_i < h_{j_0} \). Thus from (9), \( V = \beta_n h_n + U \), so (6) will hold if \( \beta_n \neq 0 \). But if \( \beta_n = 0 \), then \( V < h_n \), so since \( \text{Re} (\alpha_r) < 0 \) and \( \varphi_k \approx h_k \), it would follow by (10), that \( \sum_{i=1}^{q} \gamma_i \varphi_i < \varphi_k \) for \( 1 \leq k \leq q \). This would imply, as above that \( \gamma_i = 0 \) for \( 1 \leq i \leq q \), so from (10) (and \( V < h_n \)) we would again obtain the contradiction \( h_n < h_n \). Thus \( \beta_n \neq 0 \) so (6) holds. Thus in this subcase, the conclusion of the lemma holds for \( \Phi \).

**Subcase B.** \( q < p \) and \( m_r > 1 \). Since \( p = n \), we have \( h_n \approx x^{\sigma_r} (\log x)^{-q + m_r} \). For convenience, let \( \sigma(j) = q + m_1 + \ldots + m_{r-1} + j \) for \( 0 \leq j \leq m_r \). As in Subcase A, \( h_1, \ldots, h_{n-1} \) form a fundamental set for \( \Phi(y) = 0 \), and we want to calculate the corresponding functions \( u_k \) for \( h_1, \ldots, h_{n-1} \). Now the \( \alpha_i \) and \( m_i \) involved in \( h_1, \ldots, h_{n-1} \) are the same as in the statement for the lemma, and so the corresponding functions \( u_k \) for \( k \leq \sigma(0) \), are precisely \( u_1, \ldots, u_{\sigma(0)} \) as defined in the statement of the lemma. The remaining solutions in \( \{ h_1, \ldots, h_{n-1} \} \) are \( h_{\sigma(j)} \) for \( 1 \leq j \leq m_r - 1 \). Thus the corresponding functions \( u_k \) for these solutions
are obtained by using \( m' = m_r - 1 \) in place of \( m_r \) in the definition of \( u_{\sigma(j)} \) given in § 5. Since \( u_{\sigma(j)} = x^{-1-\alpha_r} (\log x)^{m_r-j} \), using \( m' \) in place of \( m_r \) clearly results in \( (\log x)^{-1} u_{\sigma(j)} \) as the corresponding \( u \) for \( h_{\sigma(j)} \). Hence, by applying the inductive assumption to \( \Phi_1 \), there exists e.f.d. in \( F(I) \), a fundamental set \( \{ \varphi_1, ..., \varphi_{n-1} \} \) for \( \Phi_1(y) = 0 \) such that \( \varphi_i \approx h_i \) for each \( i \), and such that the Green’s function for \( \Phi_1 \) is of the form

\[
H_1(x, \zeta) = \sum_{k=1}^{n-1} \varphi_k(x) \psi_{k-1}(\zeta) \int_{\zeta}^{x} w_k(s) \psi_{n-1}(s) \, ds.
\]

Now for \( 1 \leq k \leq \sigma(0) \), \( w_k \approx u_k \) and hence \( w_k(s) \psi_{n-1}(s) \approx s^{\alpha} u_k(s) \). Hence as in Subcase A, \( \delta_0(w_k \psi_{n-1}) \neq -1 \), and thus by § 8 D (a), for \( 1 \leq k \leq \sigma(0) \), there exists e.f.d. in \( F(I) \), a function \( Q_k(s) \approx s^{\alpha} u_k(s) \) such that \( Q' = w_k \psi_{n-1} \). Now for \( \sigma(1) \leq k \leq n-1 \), say \( k = \sigma(j) \) where \( 1 \leq j \leq m_r - 1 \), we have \( w_k \approx (\log x)^{-1} u_k \). Since \( u_k \approx x^{-1-\alpha} (\log x)^{m_r-j} \), and also that \( m_r - j - 1 > -1 \) (since \( j < m_r \)), and so by § 8 D (b), for \( k = \sigma(j) \) there exists e.f.d. in \( F(I) \), a function \( Q_k(s) \approx (\log x)^{m_r-j} \) such that \( Q' = w_k \psi_{n-1} \). Hence the right side of (12) is

\[
H_{\delta}(x, \zeta) = \sum_{k=1}^{n-1} \varphi_k(x) \nu_k(\zeta) + V(x) \psi(\zeta),
\]

where \( \nu_k(\zeta) = -w(\zeta) Q_k(\zeta) \) and \( V(x) = \sum_{k=1}^{n-1} \varphi_k(x) Q_k(x) \). Now for \( 1 \leq k \leq \sigma(0) \), \( Q_k(\zeta) \approx \zeta^{\alpha} u_k(\zeta) \) and so \( \nu_k \approx u_k \) since \( w(\zeta) \approx \zeta^{-1-\alpha} \). For \( \sigma(1) \leq k \leq n-1 \), say \( k = \sigma(j) \), we have \( Q_k(\zeta) \approx (\log \zeta)^{m_r-j} \). Thus \( \nu_k(\zeta) \approx \zeta^{-1-\alpha} (\log \zeta)^{m_r-j} \) and so again \( \nu_k \approx u_k \). Furthermore \( w \approx u_n \), and so in view of (13), the conclusion of the lemma will hold for \( \Phi \), if it can be shown that \( \{ \varphi_1, ..., \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \) and that

\[
V \approx h_n.
\]
The proof of (14) is very similar to the proof of (6) in Subcase A. As in Subcase A, there exist constants $\beta_k$ and $\gamma_k$ such that (7) and (8) hold. By hypothesis for $q+1 \leq k \leq \sigma(0)$, $\Re(\delta_0(h_k)) < \Re(\alpha_r)$ so $h_k < h_n$. For $\sigma(1) \leq k \leq n-1$, say $k = \sigma(j)$ (where $1 \leq j \leq m_r - 1$), we have $h_k \sim \sim x^\alpha (\log x)^{-q+j-1}$ so $h_k < h_n$ since $j < m_r$. Thus setting $U = \sum_{j=q+1}^{n-1} \beta_j h_j$ and $W = \sum_{j=q+1}^{n-1} \gamma_j \varphi_j$, we have $U < h_n$ and $W < h_n$, and so we obtain (9) and (10). The proof now proceeds exactly as in Subcase A to establish (14). (We remark that the relation (11) which is needed in the proof is easy to verify, as in Subcase A, by using the definition of $Q_j$.)

**Subcase C.** $q = p$. Thus $q = n$ by this case. As before, $h_1, ..., h_{n-1}$ form a fundamental set for $\Phi_1(y) = 0$ given by (2). Now $h_j \sim (\log x)^{-q+j-1}$ for $1 \leq j \leq n-1$, and this does not fit the induction hypothesis for $\Phi_1$. (i.e. Since $\Phi_1 = f_{q-1} ... f_1$, the corresponding $q$ for $\Phi_1$ is $q - 1$, and hence in order to apply the inductive assumption to $\Phi_1$, the $j^{th}$ solution must be $\sim (\log x)^{-(q-1)M_j}$ which is clearly not the case for $h_j$.) To remedy this, we set $\Lambda(z) = \Phi_1((\log x)^{-1}z)$. Then for $1 \leq j \leq n-1$, the functions $h_j^\# = (\log x)h_j$ solve $\Lambda(z) = 0$. Clearly, $h_j^\# \sim (\log x)^{-q+j}$, so,

\begin{equation}
(15) \quad h_j^\# \sim (\log x)^{-(q-1)M_j} \text{ for } 1 \leq j \leq n-1.
\end{equation}

Define functions $U_1, ..., U_{n-1}, \psi_0, ..., \psi_{n-2}$ recursively by $\psi_0 = h_1^\#$ and $U_{j+1} = \psi_j / \psi_j$ where $\psi_j = U_j ... U_1(h_j^\#)$. Then clearly,

\begin{equation}
(16) \quad (\dot{U}_j ... \dot{U}_1)(h_j^\#) = 0 \text{ for } 1 \leq j \leq n-1.
\end{equation}

In view of (15), it follows easily using $[1; \S 6 (A), (D)]$ that for $1 \leq j \leq n-1$,

\begin{equation}
(17) \quad U_j \sim -(q-j)x^{-1}(\log x)^{-1} \text{ and } \psi_{j-1} \approx h_j^\#.
\end{equation}

Let $\Lambda_1 = \dot{U}_{n-1} ... \dot{U}_1$. In view of (15), (16), (17), it is clear that $\Lambda_1$, with the solutions $h_1^\#, ..., h_{n-1}^\#$, satisfies the inductive assumption using $q - 1$ for $q$. The corresponding functions $u_j$ are clearly obtained by using $q - 1$ for $q$ in the definition of $u_j$ given in $\S 5$. Since
using \( q^{-1} \) for \( q \) clearly results in \( (\log x)^{-1}u_j \) as the corresponding \( u \) for \( h_j^* \). Hence by the inductive assumption, there exists e.f.d. in \( F(I) \), a fundamental set \( \{ \varphi_1^*, \ldots, \varphi_{n-1}^* \} \) for \( \Lambda_1(y)=0 \) such that \( \varphi_j^* \approx h_j^* \) for each \( j \), and such that the Green's function for \( \Lambda_1 \) is of the form

\[
K(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j^*(x)w_j(\zeta),
\]

where \( w_j = (\log x)^{-1}u_j \) for \( 1 \leq j \leq n-1 \). We now prove,

\[
\Lambda = a(x)\Lambda_1, \quad \text{where} \quad a(x) = \Phi_1((\log x)^{-1}).
\]

To prove (19), we apply the division algorithm for linear differential operators ([9; § 2]), and divide \( \Lambda \) by \( U_1 \). Since \( U_1 \) is of order one, there exist an operator \( \Gamma_1 \) and a function \( b_1(x) \) such that \( \Lambda = \Gamma_1 U_1 + b_1 \). Since \( \Phi_1 \) is of order \( n-1 \), clearly \( \Lambda \) is of order \( n-1 \) and hence \( \Gamma_1 \) is of order \( n-2 \) by [9; § 5 (a)]. Since \( \Lambda(h_1^*)=0 \) and \( U_1(h_1^*)=0 \) (by (16)), we have \( b_1h_1^*=0 \). Since \( h_1^* \neq 0 \) by (15), \( b_1=0 \) so \( \Lambda = \Gamma_1 U_1 \). Dividing \( \Gamma_1 \) by \( U_2 \), there exists an operator \( \Gamma_2 \) of order \( n-3 \) and a function \( b_2 \) such that \( \Gamma_1 = \Gamma_2 U_2 + b_2 \). Since \( \Lambda(h_2^*)=0 \) and \( U_2(h_2^*)=0 \) (by (16)), we have \( b_2 U_1(h_2^*)=0 \). Since \( U_1(h_2^*)=\psi_1 \) and \( \psi_1 \neq 0 \) by (17) we obtain \( b_2=0 \), so \( \Lambda = \Gamma_2 U_2 U_1 \). Continuing this way, we clearly obtain \( \Lambda = \Gamma_{n-1} U_1 \) where \( \Gamma_{n-1} \) is an operator of order zero. Thus for some function \( a(x) \), \( \Lambda(z)=a(x)\Lambda_1(z) \). Evaluating at \( z=1 \) (and noting that \( U_j(1)=1 \)), we obtain (19).

From (19) and the definition of \( \Lambda \), we have, \( \Lambda_1(z)=(1/a(x)) \Phi_1((\log x)^{-1}z) \). Thus by § 8 (A), the Green's function \( H_1(x, \zeta) \) for \( \Phi_1 \) is related to the Green's function \( K(x, \zeta) \) for \( \Lambda_1 \) by \( K(x, \zeta) = (a(\zeta)(\log x)H_1(x, \zeta) \). Thus from (18), we obtain,

\[
H_1(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)(w_j(\zeta)/a(\zeta)),
\]

where \( \varphi_j(x) = (\log x)^{-1}\varphi_j^*(x) \). Since \( \varphi_j^* \approx h_j^* \), clearly,

\[
\varphi_j \approx h_j \quad \text{for} \quad 1 \leq j \leq n-1.
\]
Now by (19), \(a(x) = \hat{f}_{n-1} \ldots \hat{f}_1((\log x)^{-1})\), and by definition, \(\Psi_{n-1} = \hat{f}_{n-1} \ldots \hat{f}_1(h_n)\), where by assumption, \(h_n \sim (\log x)^{-1}\). Since \(q = n\), we have \(f_i \sim -(q - j + 1)x^{-1}(\log x)^{-1}\), and so it easily follows using \([1; \text{§ 6 (D)}]\), that

\[
(22) \quad a(x) \approx (\log x)^{-1} \quad \text{and} \quad \Psi_{n-1} \approx (\log x)^{-1}.
\]

Since \(f_n(\Psi_{n-1}) = 0\) and \(f_n \sim x^{-1}(\log x)^{-1}\), it follows from § 8 A that the Green’s function for \(f_n\) is \(H_2(x, \zeta) = \Psi_{n-1}(x)w(\zeta)\) where (using (22)), \(w(\zeta) \approx \zeta^{-1}\). Since \(\Phi = f_n \Phi_1\), we have by § 8 B and (20) that the Green’s function for \(\Phi\) is,

\[
(23) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)w(\zeta) \int_\zeta^x (w_j(s)\Psi_{n-1}(s)/a(s))ds.
\]

Now \(w_j(s) \approx (\log s)^{-1}u_j(s)\) and \(\Psi_{n-1}(s)/a(s) \approx 1\) by (22). Hence since \(u_j(s) \approx s^{-1}(\log s)^{q-j}\), we have \(w_j(s)\Psi_{n-1}(s)/a(s) \approx s^{-1}(\log s)^{q-j-1}\) for \(1 \leq j \leq n-1\). Since \(q = n\) and \(j < n\), \(q - j - 1 \leq -1\). Thus by § 8 D (b), for each \(j = 1, \ldots, n-1\), there exists e.f.d. in \(F(I)\), a function \(Q(s) \approx (\log s)^{q-j}\) such that \(Q' = w_j\Psi_{n-1}/a.\) Hence the right side of (23) is \(\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta)\) \((Q_j(x) - Q_j(\zeta))\) and so (23) my be written,

\[
(24) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)v_j(\zeta) + V(x)w(\zeta),
\]

where \(v_j(\zeta) = -w(\zeta)Q_j(\zeta)\) and \(V(x) = \sum_{j=1}^{n-1} \varphi_j(x)Q_j(x)\). Since \(w(\zeta) \approx \zeta^{-1}\), \(v_j(\zeta) \approx \zeta^{-1}(\log \zeta)^{q-j}\) so \(v_j \approx u_j\) for \(1 \leq j \leq n-1\). Furthermore \(w \approx u_n\), so in view of (21) and (24), the conclusion of the lemma will hold for \(\Phi\), if it can be shown that \(\{\varphi_1, \ldots, \varphi_{n-1}, V\}\) is a fundamental set of solutions for \(\Phi(y) = 0\) and that

\[
(25) \quad V \approx h_n.
\]

To prove (25), we note first that since \(\{\varphi_1, \ldots, \varphi_{n-1}\}\) is a fundamental set for \(\Lambda_1(y) = 0\), clearly \(\{\varphi_1, \ldots, \varphi_{n-1}\}\) is a fundamental set for
\( \Phi_1(y) = 0. \) Since \( \Phi = \int \Phi_1, \{ \phi_1, \ldots, \phi_{n-1} \} \) is therefore an independent set of solutions of \( \Phi(y) = 0, \) and hence in view of (24), it follows from § 8c that \( \phi_1, \ldots, \phi_{n-1}, V \) form a fundamental set for \( \Phi(y) = 0. \) Since \( h_n \) is a solution \( \Phi(y) = 0 \) by hypothesis, there exist constants \( \gamma_i \) such that,

\[
(26) \quad h_n = \sum_{j=1}^{n-1} \gamma_j \phi_j + \gamma_n V.
\]

Since \( n = q, \ h_j \sim (\log x)^{-q + j - 1} \) and so \( h_j < h_n \) for \( j < n. \) Since \( \phi_j \approx h_j \) by (21), \( \phi_j < h_n \) for \( j < n. \) Thus \( \gamma_n \neq 0, \) for otherwise by (26), we would obtain the contradiction \( h_n < h_n. \) Hence \( \gamma_n \neq 0, \) and so \( h_n \approx V \) by (26). This proves (25), and so the conclusion of the lemma holds for \( \Phi \) in Subcase C, which completes Case I.

**Case II.** \( p < n. \) Then \( h_n = A_n \exp \int f_n. \) Now \( h_1, \ldots, h_{n-1} \) form a fundamental set for \( \Phi_1(y) = 0 \) (see (2)), and we want to calculate the corresponding functions \( u_j \) for \( h_1, \ldots, h_{n-1}. \) Since \( p \leq n - 1, \) the \( \alpha_j \) and \( m_j \) involved in \( h_1, \ldots, h_p \) are the same as in the statement of the lemma, and so the corresponding functions \( u_j \) are precisely \( u_1, \ldots, u_p \) as defined in the statement of the lemma. For the remaining solutions \( h_{p+1}, \ldots, h_{n-1}, \) the corresponding functions \( u_k \) are clearly obtained by using \( n - 1 \) in place of \( n \) in the definitions \( u_{p+1}, \ldots, u_{n-1} \) given in the statement of the lemma (i.e. § 5). Since for \( p + 1 \leq k \leq n - 1, \) \( u_k \) is defined as

\[
E_k(x) \exp (-\int f_k) \text{ where } E_k = f_k \prod_{i=k+1}^n \left( f_i/(f_i-f_k) \right), \text{ using } n - 1 \text{ for } n \text{ clearly results in } E_k \# \exp (-\int f_k), \text{ where } E_k = f_k \prod_{i=k+1}^{n-1} \left( f_i/(f_i-f_k) \right), \text{ as the corresponding } u \text{ for } h_k. \end{equation}

Hence by applying the inductive assumption to \( \Phi_1, \) there exists e.f.d. in \( F(I), \) a fundamental set \( \{ \phi_1, \ldots, \phi_{n-1} \} \) for \( \Phi_1(y) = 0 \) such that \( \phi_j \approx h_j \) for each \( j, \) and such the Green's function for \( \Phi_1 \) is of the form

\[
H_1(x, \zeta) = \sum_{j=1}^{n-1} \phi_j(x) w_j(\zeta) \text{ where } w_j \approx u_j \text{ for } 1 \leq j \leq p, \text{ while } w_k(\zeta) \approx E_k \#(\zeta) \exp (-\int f_k) \text{ for } p + 1 \leq k \leq n - 1. \end{equation}

Let \( z_0(x) \) be a function of the form \( \exp \int f_n. \) Since \( f_n(z_0) = 0, \) it follows from § 8 A that the
Green’s function for $f_n$ is $H_2(x, \zeta) = z_0(x)w(\zeta)$, where

\begin{equation}
(27) \quad w(\zeta) = f_n(\zeta)/z_0(\zeta).
\end{equation}

Since $\Phi = \int \Phi_1$, we have by § 8 (B) that the Green’s function for $\Phi$ is,

\begin{equation}
(28) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)w(\zeta) \int_\zeta^x w_j(s)z_0(s)ds.
\end{equation}

Now for $1 \leq j \leq p$, $w_j \approx u_j$ so $w_j$ is $\approx$ to a function of the form $x^j(\log x)^r$. Since $p < n$, $f_n$ is $\sim$ to a function of the form $cx^{-1+d}$ where $d > 0$. Thus clearly (see § 2 (b)), $IF(f_n)$ has only finitely many zeros on $(-\pi, \pi)$. Since $z_0(s) = \exp \int f_n$, it follows from [3; § 10 (b)] that for $1 \leq j \leq p$, there exists e.f.d. in $F(I)$, a function of the form $Q_j(s) = a_j(s)z_0(s)$ where $a_j \sim w_j/f_n$, such that $Q'_j = w_jz_0$. For $p + 1 \leq k \leq n - 1$, $w_k(s)z_0(s)$ is $\approx$ to a function of the form $E_k(s)\exp \int (f_n - f_k)$. Now for $p + 1 \leq k < j \leq n$, $f_j - f_k \sim f_j$ (since $N_{k-p} \leq N_{j-p}$ and $N_{k-p} \neq N_{j-p}$), and so it easily follows that $E_k^*$ is $\approx$ to a function of the form $x^j(\log x)^r$. Since $f_n - f_k \approx f_n$, $IF(f_n - f_k)$ has only finitely many zeros. Thus it follows from [3; § 10 (b)] that for $p + 1 \leq k \leq n$, there exists e.f.d. in $F(I)$, a function of the form $Q_k(s) = T_k(s)\exp \int (f_n - f_k)$, where $T_k \approx E_k^*/(f_n - f_k)$ such that $Q'_k = w_kz_0$. Hence the right side of (28) is $\sum_{j=1}^{n-1} \varphi_j(x)w(\zeta)(Q_j(x) - Q_j(\zeta))$, so (28) can be written,

\begin{equation}
(29) \quad H_0(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)\nu_j(\zeta) + V(x)w(\zeta),
\end{equation}

where $\nu_j(\zeta) = -w(\zeta)Q_j(\zeta)$ and $V(x) = \sum_{j=1}^{n-1} \varphi_j(x)Q_j(x)$. Now in view of (27), for $1 \leq j \leq p$, $\nu_j \approx f_n a_j$. Since $a_j \sim w_j/f_n$, and $w_j \approx u_j$, we have $\nu_j \approx u_j$. 

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By (27), \( w(\zeta) \approx f_n(\zeta) \exp (- \int \zeta) \). Thus for \( p+1 \leq k \leq n \), clearly \( v_k(\zeta) \approx (f_k E_k^*/(f_n - f_k)) \exp (- \int \zeta) \) and hence \( v_k(\zeta) \approx E_k(\zeta) \exp (- \int \zeta) \). Thus \( v_k \approx u_k \). Furthermore by (27), \( w(\zeta) \approx u_n(\zeta) \), so in view of (29), the conclusion of the lemma will hold for \( \Phi \) if it can be shown that \( \{ \varphi_1, ..., \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \)

and that

\[ V \approx h_n. \]

To prove (30), we note that since \( \{ \varphi_1, ..., \varphi_{n-1} \} \) is a fundamental set for \( \Phi(y) = 0 \), and since \( \Phi = f_n \Phi_1 \), it follows from (29) and § 8 D that \( \{ \varphi_1, ..., \varphi_{n-1}, V \} \) is a fundamental set for \( \Phi(y) = 0 \). Since \( \Phi(h_n) = 0 \), there exist constants \( \beta_i \) such that

\[ h_n = \sum_{j=1}^{n-1} \beta_j \varphi_j + \beta_n V, \]

whence

\[ (h_n - \beta_n V)/h_n = \sum_{j=1}^{n-1} \beta_j (\varphi_j/h_n). \]

We now calculate each term \( \varphi_j Q_i \) in \( V \). For \( 1 \leq j \leq p \), \( \varphi_j Q_j = \varphi_j a_j z_0 \).

For \( p+1 \leq k \leq n-1 \), we have \( \varphi_k \approx h_k \), \( h_k = A_k \exp \int f_k \) and \( Q_k = T_k \exp \int (f_n - f_k) \). Since \( z_0 = \exp \int f_n \), it follows easily that \( \varphi_k Q_k = \Delta_k A_k T_k z_0 \) where \( \Delta_k \approx 1 \). Thus clearly,

\[ V = U z_0, \]

where \( U = \sum_{j=1}^{p} \varphi_j a_j + \sum_{k=p+1}^{n-1} \Delta_k A_k T_k \). Now \( A_k \approx \prod_{i=1}^{k-1} (f_i/(f_j - f_k)) \). Since \( f_j - f_k \) is \( \approx f_k \) if \( j < k \) and \( k \geq p + 1 \), it follows easily that

\[ A_k \approx \text{a function of the form } x^\lambda (\log x)^\gamma. \]
In particular, $A_k$ is a some power of $x$. Since $\Delta_k T_k = E_k^* / (f_n - f_k)$, it follows similarly $\Delta_k T_k$ is a some power of $x$ for $p + 1 \leq k \leq n - 1$. Since $\varphi_j \approx h_j$ and $a_i \approx u_i / f_n$ for $j \leq p$, it follows easily that $\varphi_j$ and $a_i$ are each a some power of $x$. Thus each term in $U$ is a some power of $x$, so clearly,

\begin{equation}
U(x) < x^\sigma \text{ for some real number } \sigma.
\end{equation}

Since $h_n = A_n \exp \int f_n$, clearly $h_n = cA_n z_0$ for some $c \neq 0$. Hence in view of (33), the left side of (32) is $(cA_n - \beta_n U) / (cA_n)$, which by (34) and (35) is clearly a some power of $x$. Thus by (32),

\begin{equation}
\sum_{j=1}^{n-1} \beta_j (\varphi_j / h_n) < x^\lambda
\end{equation}

for some real number $\lambda$.

Consider $\sum_{j=1}^{p} \beta_j \varphi_j$. Now by hypothesis, $\Re(\alpha_i) < \Re(\alpha_i)$ if $i < j$, and if $q > 0$, $\Re \alpha_j \neq 0$. It easily follows (since $\varphi_j \approx h_j$) that for $1 \leq i < j \leq p$, either $\varphi_i < \varphi_j$ or $\varphi_i < \varphi_j$ (see § 2 (b)). Hence clearly, if not all of $\beta_1$, ..., $\beta_p$ are zero, then there exists an index $j_0 \in \{1, ..., p\}$ such that $\beta_{j_0} \neq 0$ and $\varphi_i < \varphi_i$ if $i \leq p$ and $i \neq j_0$. Thus

\begin{equation}
\sum_{j=1}^{p} \beta_j \varphi_j = \varphi_{j_0} (\beta_{j_0} + b(x)) \text{ where } b < 1.
\end{equation}

(If all of $\beta_1$, ..., $\beta_p$ are zero, set $\beta_{j_0}$ and $b$ equal zero so (37) still holds.) For $p + 1 \leq k \leq n - 1$, set $D_k = \varphi_k / h_n$. Then we may write,

\begin{equation}
\sum_{j=1}^{n-1} \beta_j (\varphi_j / h_n) = (1 / h_n) \sum_{j=1}^{p} \varphi_j \beta_j + \sum_{k=p+1}^{n-1} \beta_k D_k.
\end{equation}

Now for $p + 1 \leq k \leq n$, clearly $IF(f_k)$ has only finitely many zeros (see § 2 (b)). For $p + 1 \leq j < k \leq n$, $f_j - f_k = f_k$ so $IF(f_j - f_k)$ also has only finitely many zeros. Thus if we let $\Gamma$ be the union of all zeros in $I$ of all the above functions $IF(f_k)$ and $IF(f_i - f_k)$, then $\Gamma$ is a finite set, say $\varepsilon_1 < ... < \varepsilon_m$. If $I = (\varepsilon_0, \varepsilon_{m+1})$, then letting $J$ be any subinterval of any
of the intervals \((\varepsilon_j, \varepsilon_{j+1})\) such that \(\{\varphi_1, ..., \varphi_{n-1}, V\}\) exist on \(F(J)\), we have that (36) is valid on \(F(J)\) and all \(IF(f_k)\) and \(IF(f_j-f_k)\) as above, are nowhere zero on \(J\). Now clearly, since \(\varphi_k = h_k\), we have \(D_k \approx (A_k/A_n) \exp \int (f_k-f_n)\). In view of (34) and the fact that \(IF(f_k-f_n)\) is nowhere zero on \(J\), it follows from [3; § 10 (a)], that for each \(k \in \{p+1, ..., n-1\}\),

(39) Either \(D_k\) is trivial in \(F(J)\) (i.e. \(D_k < x^a\) for all \(a\)) or \(1/D_k\) is trivial in \(F(J)\).

Since \(h_k = A_k \exp \int f_k\), it follows similarly using (34) and [3; § 10 (a)] that for each \(k \in \{p+1, ..., n\}\),

(40) Either \(h_k\) is trivial or \(1/h_k\) is trivial in \(F(J)\).

Finally, if \(j\) and \(k\) are distinct elements of \(\{p+1, ..., n-1\}\), then since \(D_j/D_k \approx (A_j/A_k) \exp \int (f_j-f_k)\), it follows as above that

(41) Either \(D_j/D_k\) is trivial or \(D_k/D_j\) is trivial in \(F(J)\). We now return to (36) and prove,

(42) For each \(j \in \{p+1, ..., n-1\}\) such that \(1/D_j\) is trivial in \(F(J)\), we have \(\beta_j = 0\).

We prove (42) by contradiction. We assume the contrary and let \(i_0\) be an index such that \(1/D_{i_0}\) is trivial but \(\beta_{i_0} \neq 0\). Let \(L\) be the set of all \(j \in \{p+1, ..., n-1\}\) for which \(\beta_j \neq 0\). For \(i\) and \(j\) in \(L\) with \(i \neq j\), we have by (41) that either \(D_i < D_j\) or \(D_j < D_i\). Since \(L\) is a finite set, clearly there exists \(k_0 \in L\) such that \(D_i < D_{k_0}\) if \(i \in L - \{k_0\}\). If \(k_0 = i_0\) then \(1/D_{k_0}\) is trivial. If \(k_0 \neq i_0\) then \(D_{i_0} < D_{k_0}\) so again,
By the property of $k_0$, we can write

$$\sum_{j=p+1}^{n-1} \beta_j D_i = \beta_k D_k (1 + t)$$

where $t < 1$.

Hence by (36), (37) and (38), we obtain in $F(J)$,

$$\left(\frac{\varphi_{j_0}}{h_n}(\beta_{j_0} + b) + \beta_{k_0} D_k (1 + t)\right) < x^\lambda.$$

Now $D_k h_n = \varphi_{k_0}$. Thus dividing (44) by $D_{k_0}$ and using (43),

$$\left(\frac{\varphi_{j_0}}{\varphi_{k_0}}(\beta_{j_0} + b) + \beta_{k_0} (1 + t)\right)$$

is trivial in $F(J)$.

Since $\beta_{k_0}$ is a non-zero constant, $\beta_{k_0} \approx 1$. If $\beta_{j_0} = 0$ (and $b = 0$), then (45) is clearly impossible. If $\beta_{j_0} \neq 0$, then since $\beta_{k_0} \approx 1$, we have from (45) that $\beta_{k_0} \varphi_{j_0} / \varphi_{k_0} \approx - \beta_{k_0}$. Thus $\varphi_{j_0} / \varphi_{k_0} \approx 1$ and so $h_{j_0} \approx h_{k_0}$. This is clearly impossible since $h_{j_0}$ is $\sim$ to a function of the form $x^a (\log x)^m$ (since $j_0 \leq p$), while by (40), either $h_{k_0}$ or $1 / h_{k_0}$ is trivial. This contradiction proves (42), which in view of (39) clearly implies,

$$\sum_{k=p+1}^{n-1} \beta_k D_k$$

is trivial in $F(J)$.

If $\beta_{j_0} = 0$ (and $b = 0$) in (37), then by (46), the left side of (38) is trivial. Thus by (32), $(h_n - \beta_n V) / h_n$ is trivial and hence is $< 1$ in $F(J)$.

Thus $\beta_n \neq 0$ and $h_n \approx V$ proving (30). If $\beta_{j_0} \neq 0$, then $\sum_{j=1}^{p} \beta_j \varphi_j \approx \varphi_{j_0}$. But in view of (46), we have by (38) and (36) that $1 / h_n \sum_{j=1}^{p} \beta_j \varphi_j < x^\lambda$. Hence $\varphi_{j_0} / h_n < x^\lambda$, so $1 / h_n < x^\lambda / \varphi_{j_0}$. But $\varphi_{j_0} \approx h_{j_0}$ and so (since $j_0 \leq p$), $\varphi_{j_0}$ is $\approx$ to a function of the form $x^a (\log x)^m$. Thus $1 / h_n$ is $< \text{some power of } x$. Hence by (40), $1 / h_n$ must be trivial in $F(J)$. Thus $(1 / h_n) \sum_{j=1}^{p} \beta_j \varphi_j$

is trivial, so by (46), the left side of (38) is trivial. Hence by (32), $(h_n - \beta_n V) / h_n$ is trivial, whence $< 1$, and so again $\beta_n \neq 0$ and $h_n \approx V$ in $F(J)$ proving (30). Thus in Case II, the conclusion of the lemma holds for $\Phi$, and so the lemma is established by induction.
7. Conclusion of proof of § 5.

Let $\Omega$, $q$, $p$, $M_i$ and $u_k$ be as in § 5, where the roots $\alpha_i$ are arranged so that $\text{Re} (\alpha_i) < \text{Re} (\alpha_{i+1})$. By § 4, e.f.d. in $F(a, b)$, the operator $\Phi_0(z) = (1/q!) \Omega((\log x)^q z)$ possesses a factorization $\Phi_0 = E f_1 \cdots f_i$ (with $f_i$ as in § 4 (c)), and there exists a fundamental set $\{g_1, \ldots, g_n\}$ for $\Omega(y) = 0$.

with $g_j \sim M_j$ for $1 \leq j \leq p$ and $g_k \sim R_k \exp \int \frac{x}{j_k}$ for $k > p$, such that if $h_i = (\log x)^{-q} g_i$ for each $j$, then $E f_1 \cdots f_i$ satisfies the hypothesis of § 6 relative to the solution $h_1, \ldots, h_n$. Hence by § 6, e.f.d. in $F(a, b)$, there exists a fundamental set $\{\psi_1, \ldots, \psi_n\}$ for $\Phi(y) = 0$ such that $\psi_j \approx h_j$ and such that the Green's function for $\Phi$ is $H_0(x, \zeta) = \sum_{j=1}^{n} \psi_j(x) v_j(\zeta)$ where $v_j = u_j$ for each $j$. By § 8 A, the Green's function for $\Omega$ is $H(x, \zeta) = (\log x)^q H_0(x, \zeta)/(q! E(\zeta))$. Thus $H(x, \zeta) = \sum_{j=1}^{n} \psi_j(x) w_j(\zeta)$, where $\psi_j(x) = (\log x)^q \psi_j(x)$ and $w_j(\zeta) = v_j(\zeta)/(q! E(\zeta))$. Then clearly, $\{\psi_1, \ldots, \psi_n\}$ is a fundamental set for $\Omega(y) = 0$ and $\psi_j \approx g_j$ (since $\psi_j \approx h_j$). Hence, $(\psi_1, \ldots, \psi_n)$ is an asymptotically canonical fundamental system for $\Omega$ in the sense of § 4. Finally, since $E \sim 1$, clearly $w_j \approx u_j$. This concludes the proof of the main theorem.

8. Results needed in the proof of §§ 6.

A. Lemma. Let $f$ and $E$ be analytic functions having no zeros in a simply-connected region $D$. Then:

(a) If $h(z)$ is analytic function in $D$ such that $\hat{h}(h) = 0$ and $h \not\equiv 0$, then the Green's function for $\hat{f}$ is $K(x, \zeta) = h(x) w(\zeta)$, where $w(\zeta) = -f(\zeta)/h(\zeta)$.

(b) If $\Omega(y) = \sum_{j=0}^{n} a_j(x) y^{(j)}$, where the $a_j(x)$ are analytic in $D$ and $a_n(x)$ has no zeros in $D$, and if $\Lambda(z) = E(x) \Omega(f(x) z)$, then the Green's function $H(x, \zeta)$ for $\Omega$ is related to the Green's function $H_1(x, \zeta)$ for $\Lambda$ by $H_1(x, \zeta) = H(x, \zeta)/(f(x) E(\zeta))$.

Proof. Since $\{h\}$ is a fundamental set for $\hat{f}(y) = y - (y'/f) = 0$, Part (a) follows from the definition of $K(x, \zeta)$.
For (b), set $H_2(x, \zeta) = \Phi(\zeta)f(x)H_1(x, \zeta)$. As in § 3, for each $\zeta \in D$, $H_1(x, \zeta)$ is a solution of $A(z) = 0$ satisfying the following initial conditions at $x = \zeta$: $\frac{\partial^k H_1(x, \zeta)}{\partial x^k} = 0$ for $k \leq n - 2$, while $\frac{\partial^{n-1} H_1(x, \zeta)}{\partial x^{n-1}} = 1/(E(\zeta)f(\zeta)a_n(\zeta))$ since $E|a_n$ is the leading coefficient of $\Lambda$. It is then easily verified that for each $\zeta$, $H_2(x, \zeta)$ is a solution of $\Omega(y) = 0$ satisfying the same initial conditions at $x = \zeta$ as the solution $H(x, \zeta)$ (see § 3). Hence $H_2 = H$ by the uniqueness theorem for linear differential equations.

**B. LEMMA.** Let $\Phi_1(y) = \sum_{j=0}^{n} a_j(x)y^{(j)}$ and $\Phi_2(y) = \sum_{j=0}^{m} b_j(x)y^{(j)}$, where the $a_j$ and $b_j$ are analytic in a simply-connected region $D$, and $a_n$ and $b_m$ have no zeros in $D$. Let $\Phi_3 = \Phi_2(\Phi_1)$ and for $k = 1, 2, 3$ let $H_k(x, \zeta)$ be the Green’s function for $\Phi_k$. Then $H_3(x, \zeta) = \int_{\zeta}^{x} H_1(x, s)H_2(s, \zeta)ds$, the contour of integration being any rectifiable path in $D$ from $\zeta$ to $x$.

**PROOF.** Set $K(x, \zeta) = \int_{\zeta}^{x} H_1(x, s)H_2(s, \zeta)ds$. By the property of the Green’s function given in § 1, $K(x, \zeta)$ is for each $\zeta$, a solution of $\Phi_1(y) = H_2(x, \zeta)$, and hence (see § 3), $K(x, \zeta)$ is a solution of $\Phi_3(y) = 0$. Furthermore, using the initial conditions at $x = \zeta$ satisfied by $H_1$ and $H_2$ (see § 3), a straightforward calculation shows for each $\zeta \in D$, the solution $K(x, \zeta)$ of $\Phi_3(y) = 0$ satisfies the same initial conditions at $x = \zeta$ as the solution $H_3(x, \zeta)$ (see § 3). Thus by the uniqueness theorem for linear differential equations $K = H_3$ proving Lemma B.

**C. LEMMA.** Let $\Phi(y) = \sum_{j=0}^{n} a_j(x)y^{(j)}$, where the $a_j$ are analytic in $D$ and $a_n$ is nowhere zero in $D$. Then if the Green’s function for $\Phi$ can be written in the form $H(x, \zeta) = \sum_{j=1}^{n} \varphi_j(x)\psi_j(\zeta)$, where $\varphi_1, ..., \varphi_{n-1}$ are linearly independent solutions of $\Phi(y) = 0$, then $\{\varphi_1, ..., \varphi_n\}$ form a fundamental set of solutions for $\Phi(y) = 0$.

**PROOF.** We complete $\{\varphi_1, ..., \varphi_{n-1}\}$ to a fundamental set $\{\varphi_1, ..., \varphi_{n-1}, g\}$ for $\Phi(y) = 0$. Then by definition (§ 3), $H(x, \zeta) = \sum_{j=1}^{n-1} \varphi_j(x)\psi_j(\zeta) +$
\[ g(x) \nu_n(\zeta), \text{ and it is proved in [7; p. 38], that } \{ \nu_1, ..., \nu_n \} \text{ form a fundamental set for the adjoint equation } \Phi'(y) = 0. \text{ Now for each } \zeta \in D, H(x, \zeta) \text{ solves } \Phi(y) = 0, \text{ so clearly,} \]

\[ (47) \quad \Phi(\phi_n(x) \nu_n(\zeta)) = 0 \text{ for each } \zeta \]

If \( \nu_n(\zeta) = 0 \), then from the two representations for \( H(x, \zeta) \), and the independence of \( \{ \phi_1, ..., \phi_{n-1}, g \} \), we would obtain \( \nu_n(\zeta) = 0 \) which would contradict the independence of \( \{ \nu_1, ..., \nu_n \} \). Thus for some \( \zeta \in D \), \( \nu_n(\zeta) \neq 0 \) and so from (47), \( \nu_n \) is a solution of \( \Phi(y) = 0 \). To show \( \{ \phi_1, ..., \phi_n \} \) is independent, we assume the contrary. Then since \( \{ \phi_1, ..., \phi_{n-1} \} \) is independent, we would have a relation of the form

\[ \phi_n = \sum_{i=1}^{n-1} c_i \phi_i. \]

Thus, \( H(x, \zeta) = \sum_{i=1}^{n-1} \phi_i(x)(\nu_i(\zeta) + c_i \nu_n(\zeta)) \), which together with the other representation for \( H \) and the independence \( \{ \phi_1, ..., \phi_{n-1}, g \} \), again the contradiction \( \nu_n(\zeta) = 0 \), thus proving Lemma C.

**D. Lemma.** Let \( R(x) \) be a function such that in some \( F(I) \), \( R(x) \approx x^\alpha (\log x)^\beta \) for some complex number \( \alpha \) and real number \( \beta \). Then:

(a) If \( \alpha \neq -1 \), then e.f.d. in \( F(I) \), there exists a function \( Q(x) \approx x R(x) \) such that \( Q' = R \).

(b) If \( \alpha = -1 \) but \( \beta \neq -1 \), then e.f.d. in \( F(I) \), there exists a function \( Q(x) \approx (\log x)^{\beta+1} \) such that \( Q' = R \).

**Proof.** Under the change of variable \( y = x^\alpha z \) and division by \( x^{\alpha-1} \), the equation \( y' = R(x) \) is trasformed into,

\[ (48) \quad xz' + \alpha z = T(x), \text{ where } T(x) = x^{-\alpha} R(x). \]

Thus \( T(x) \approx (\log x)^\beta \). If \( \alpha \neq -1 \), then by [4; § 3], equation (48) possesses, e.f.d. in \( F(I) \), a solution \( z_0(x) \approx T(x) \). Part (a) then follows by taking \( Q(x) = x^\alpha z_0(x) \). If \( \alpha = -1 \) but \( \beta \neq -1 \), then by [4; § 3], equation (48) possesses, e.f.d. in \( F(I) \), a solution \( z_1(x) \approx (\log x)T(x) \). Part (b) then follows by taking \( Q(x) = x^{-1} z_1(x) \).

**Remark.** In the case where \( \alpha \) is real, Lemma D also follows from [6; Lemma \( \zeta \), p. 272].
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