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Distributional boundary values in $\mathcal{D}'_{L^p}$, III

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1. Introduction.

In Carmichael [5, 6] we have obtained results in which distributions in $\mathcal{D}_{L^p}$ are related to and represented as boundary values of analytic functions. In the present paper we shall continue our investigation of this topic.

All terminology concerning cones $C \subseteq \mathbb{R}^n$ and compact subcones in this paper will be the same as that in Carmichael [4, p. 845] or [6, p. 252]. In particular we call the readers attention to the function $u_C(t)$, the indicatrix of the cone $C$, the number $\rho_C$, which characterizes the nonconvexity of $C$, and the tubular cone $T^C = \mathbb{R}^n + iC \subseteq \mathbb{C}^n$, the definition of which can be found in the above references.

Let $C$ be an open cone; and let $f(z)$, $z = x + iy \in \mathbb{C}^n$, satisfy

\begin{equation}
|f(z)| \leq K(C')(1 + |z|)^N \exp \left[ 2\pi(A + \sigma) |y| \right], \quad z \in T^C = \mathbb{R}^n + iC',
\end{equation}

for all real numbers $\sigma > 0$, where $C'$ is an arbitrary compact subcone of $C$, $A$ is a nonnegative real number, $N$ is any real number, and $K(C')$ is a constant depending on $C'$. The functions which we have studied in [5, 6] in relation to the $\mathcal{D}_{L^p}$ distributions have been analytic functions in the octants

$$0_\delta = \{ z : \delta_j \text{ Im } (z_j) > 0, \delta = (\delta_1, ..., \delta_n), \delta_j = \pm 1, j = 1, ..., n \}$$

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or in the general tubular cone $T^C = \mathbb{R}^n + iC$ which satisfy boundedness conditions similar to (1). In [4, 7, 8, 9] we have related analytic functions having a growth condition as in (1) to other spaces of distributions. Letting $\mathcal{S}'$ denote the Schwartz space of tempered distributions, we have obtained the following result which has importance in quantum field theory and which will be useful in this paper.

**Theorem 1.** Let $C$ be an open connected cone. Let $f(z)$ be analytic in $T^C = \mathbb{R}^n + iC$ and satisfy (1). Let $f(z) \to U$ in the $\mathcal{S}'$ topology as $y \to 0$, $y \in C' \subset C$. Then $U \in \mathcal{S}'$; there exists an element $V \in \mathcal{S}'$ such that $\text{supp}(V) \subseteq \{ t : u(t) \leq A \}$ and $U = \hat{V}$; and $f(z) = \langle V, e^{2\pi i (z, t)} \rangle$, $z \in T^C$, $C' \subset C$.

**Proof.** See [4, Theorem 2]. Here $\text{supp}(V)$ is the support of $V$, $\hat{V}$ denotes the Fourier transform of $V$, and $C'$ is an arbitrary compact subcone of $C$.

Korányi [14] and Stein, Weiss, and Weiss [16] have defined the classical Hardy $H^p(T^C)$ spaces, $1 \leq p \leq \infty$, for functions analytic in tube $T^C$. We note that $H^p(T^C) \subset \mathcal{D}'_L \subset \mathcal{S}'$, $1 \leq p \leq \infty$.

In this paper we shall obtain distributional boundary value results concerning the space of functions $H^p(T^C)$, and the boundary values will be seen to be elements of $\mathcal{D}'_L$. As in Carmichael [5, 6], the topology which we shall use will be that of $\mathcal{S}'$. In section 2 we shall obtain results similar to Theorem 1 for functions $f(z) \in H^\infty(T^C)$ and for functions $f(z) \in H^p(T^C)$, $1 \leq p < \infty$, which satisfy (1). Under these assumptions more can be said about the function $f(z)$ than in Theorem 1; we shall see that the convergence of $f(z)$ to an element in $\mathcal{D}'_L \subset \mathcal{S}'$ can be proved and that $f(z)$ can be represented by the Poisson integral of its boundary value as well as the Fourier-Laplace transform $\langle V, e^{2\pi i (z, t)} \rangle$ of $V \in \mathcal{S}'$. Further, for suitable choices of $p$, $f(z) \in H^p(T^C)$ can also be represented by the Cauchy integral of its boundary value. If $f(z) \in H^2(T^C)$ it is known that $f(z)$ has each of the above representations. In the results of this paper we extend the values of $p$ for which $H^p(T^C)$ functions have each of these representations. Further, we prove a version of Fatou's theorem in which more is concluded about the $H^\infty(T^C)$ function and its boundary value than in the classical setting for tube domains. In section 3 we shall obtain converse results to those of section 2 in which an $H^p(T^C)$ function is manufactured from a distrib-
In particular we are interested in obtaining a converse to the classical Fatou theorem. Section 4 will be devoted to obtaining generalizations to disconnected tubular cones.

In the remainder of this introductory section we shall introduce the $n$ dimensional notation and definitions to be used throughout this paper. The $n$ dimensional notation and the definition of the derivative $D^\alpha$, $\alpha$ being an $n$-tuple of nonnegative integers, will be the same as in Carmichael [4]. $T^c$ will always represent the subset of $C^n$ defined by $T^c = R^n + iC$, where $C$ is a cone. If $C$ is connected, $T^c$ will be called a tubular radial domain; while if $C$ is not connected, we shall refer to $T^c$ as a tubular cone. The function spaces $S$ and $\mathcal{S}$ and the distribution spaces $S'$ and $\mathcal{D}'$ are defined in Schwartz [15]; and all definitions of terms concerning distributions, such as support and convolution, are those of Schwartz. The Fourier and inverse Fourier transforms of $L^1$ functions and $S'$ distributions are defined in Carmichael [4]. The Fourier transform of a function $\varphi(t)$ will be denoted by $\mathcal{F}[\varphi(t); x]$ or $\tilde{\varphi}(x)$; similarly we denote the inverse Fourier transform as $\mathcal{F}^{-1}[\varphi(t); x]$. The Fourier and inverse Fourier transforms of $V \in S'$ are denoted $\hat{V}$ and $\mathcal{F}^{-1}(V)$, respectively.

A sequence $\{\varphi_\lambda\} \in S$ converges to $\varphi \in S$ in $S$ as $\lambda \to \lambda_0$ if

$$
\lim_{\lambda \to \lambda_0} \sup_{x} |x^\beta D^\alpha (\varphi_\lambda(x) - \varphi(x))| = 0,
$$

where $\alpha$ and $\beta$ are arbitrary $n$-tuples of nonnegative integers. Let $z \in T^c$, $C$ being an open connected cone. By $f(z) \to V$ in the topology of $S'$ as $y = \text{Im}(z) \to 0$, $y \in C$, we mean that $\langle f(z), \varphi(x) \rangle \to \langle V, \varphi(x) \rangle$ as $y \to 0$, $y \in C$, where $\varphi$ is any element of $S$. We note that the boundary value $V$ is obtained on the distinguished boundary of $T^c$, $\{z = x + iy : x \in R^n, y \neq 0\}$, which is not necessarily the topological boundary unless $n = 1$.

On several occasions in this paper we shall make use of Theorem 4 in [4]. We note that this result holds for $A = 0$ as well as for $A > 0$; the proof for $A = 0$ is exactly the same. With this in mind, we shall assume henceforth that Theorem 4 in [4] holds for all real numbers $A \geq 0$. Unless otherwise specified, $g(x) \in L^p(f(z) \in H^p(T^c))$, $1 \leq p \leq \infty$, means throughout this paper $g(x) \in L^p(f(z) \in H^p(T^c))$ for some $p$. 


1 \leq p \leq \infty$. The definition of the $H^p(T^c)$ spaces, $1 \leq p \leq \infty$, which we shall use in this paper is given in [16].

2. Distributional boundary values of $H^p$ functions.

Let $C$ be an open connected cone, and let $0(C)$ denote the convex envelope (hull) of $C$. If $f(z)$ is analytic in $T^c$, then by Bochner's theorem on analytic extension [3, Chapter V], $f(z)$ has an analytic extension to $T^{0(c)}$. Further, if $f(z) \in H^p(T^c)$, then its extension is in $H^p(T^{0(c)})$ and

$$\sup_{y \in C} \int_{R^n} |f(x+iy)|^p \, dx = \sup_{y \in 0(C)} \int_{R^n} |f(x+iy)|^p \, dx.$$ 

(See [16, p. 1036]). Thus it suffices to assume that $C$ is convex.

For $z \in T^c$, we define the Cauchy kernel $K(z-t)$ by

$$K(z-t) = \int_{C^*} e^{2\pi i (z-t, \eta)} \, d\eta,$$

where $C^* = \{ \eta : u_C(\eta) \leq 0 \}$ is the dual cone of $C$. If $C^*$ contains an entire straight line, then by a result of Vladimirov [18, Lemma 1, p. 222] the cone $C^*$ lies in some $(n-1)$ dimensional plane; and $K(z-t) = 0$. To avoid this triviality we assume throughout this section that the cone $C$ is open, convex, and has the property that $C^*$ contains no entire straight line.

From the Cauchy kernel we define the Poisson kernel corresponding to $T^c$ by

$$Q(z; t) = \frac{K(z-t)K(z-t)}{K(2iy)}.$$

If $T^c$ is the upper half plane in $C^1$, then $K(z-t)$ and $Q(z; t)$ are $\frac{1}{2 \pi i}$ $\frac{1}{z-t}$ and $\frac{1}{\pi} \frac{y}{(t-x)^2 + y^2}$, respectively, which are the classical Cauchy and Poisson kernels.
Let \( g \in L^p, 1 \leq p \leq \infty \). Then

\[
\int_{\mathbb{R}^n} g(t)K(z-t)dt \quad \text{and} \quad \int_{\mathbb{R}^n} g(t)Q(z; t)dt, \quad z \in T^c,
\]

are the Cauchy and Poisson integrals, respectively, of \( g \). We can now prove

**Theorem 2.** Let \( f(z) \in H^p(T^c), 1 \leq p < \infty \); and let \( f(z) \) satisfy (1). There exists a function \( g(x) \in L^p, 1 \leq p \leq \infty \), such that \( f(z) \to g(x) \) in the topology of \( S' \) (as well as in the \( L^p \) norm topology) as \( y = \text{Im} (z) \to 0, y \in C \); and there exists an element \( V \in S' \) with \( \text{supp} (V) \subseteq S_A = \{ t : u(t) \leq A \} \) such that \( g(x) = \hat{V} \) and

\[
f(z) = \langle V, e^{2\pi i (z \cdot t)} \rangle = \int_{\mathbb{R}^n} g(t)Q(z; t)dt, \quad z \in T^c.
\]

**Proof.** Combining Propositions 4 and 3 (c) in Korányi [14], we obtain the existence of a function \( g(x) \in L^p \), such that

\[
in L^p \text{ as } y \to 0, \quad y \in C. \quad \text{Let } \varphi \in \mathcal{S}. \quad \text{By Hölder's inequality,}
\]

\[
\frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p < \infty. \quad \text{If } p = 1 \text{ we have}
\]

\[
| \langle f(z), \varphi(x) \rangle - \langle g(x), \varphi(x) \rangle | \leq \| f(z) - g(x) \|_{L^p} \| \varphi \|_{L^q}.
\]

where \( | \varphi(x) | \leq K \). Since \( \varphi \in \mathcal{S} \subseteq L^q \) for all \( q, 1 \leq q \leq \infty \), then by (3), (4), and the fact that \( f(z) \to g(x) \) in \( L^p, 1 \leq p < \infty \), as \( y \to 0, y \in C \), we have that \( f(z) \to g(x) \) in \( S' \) as \( y \to 0, y \in C \). Having obtained this \( S' \) boundary value, we now apply Theorem 1 and obtain an element \( V \in S' \) with \( \text{supp} (V) \subseteq S_A \) such that \( g(x) = \hat{V} \) and \( f(z) = \langle V, e^{2\pi i (z \cdot t)} \rangle, \quad z \in T^c, \quad C' \subseteq C \). But under these conditions on \( V \), we have by [4, Theorem 4] that

\[
\langle V, e^{2\pi i (z \cdot t)} \rangle \text{ is analytic in } T^c.
\]

Thus by the identity theorem for analytic
functions, \( f(z) = \langle V, e^{-ni(z,t)} \rangle, \ z \in \mathbb{C} \). Again applying Propositions 4 and 3 (c) of Korányi [14], we have

\[
f(z) = \int_{\mathbb{R}^n} g(t)Q(z; t)dt, \ z \in \mathbb{C}.
\]

and (2) is obtained.

We now restrict \( p \) to \( 1 \leq p \leq 2 \) in Theorem 2 and obtain an interesting corollary. First, however, we prove the following lemma.

**Lemma 1.** Let \( f \in L^p, \ 1 \leq p \leq 2 \). Let \( g \in L^q, \ \frac{1}{p} + \frac{1}{q} = 1 \); and assume that \( \mathcal{F}^{-1}[g(t); x] \) exists classically and belongs to \( L^p, \ 1 \leq p \leq 2 \), \( \frac{1}{p} + \frac{1}{q} = 1 \). Then

\[
\mathcal{F}^{-1}(f \ast g) = \mathcal{F}^{-1}(f)\mathcal{F}^{-1}(g)
\]

in \( S' \).

**Proof.** Since \( f \in L^p, \ 1 \leq p \leq 2 \), then \( \mathcal{F}^{-1}[f(t); x] \) exists classically and is an element of \( L^q, \ \frac{1}{p} + \frac{1}{q} = 1 \). By hypothesis \( \mathcal{F}^{-1}[g(t); x] \in L^p, \ 1 \leq p \leq 2 \). Thus \( \mathcal{F}^{-1}[f(t); x] \mathcal{F}^{-1}[g(t); x] \in L^1 \subset S' \). Further, it is known that \( f \ast g \) exists as a classical convolution, is continuous, and is an element of \( L^r, \ \frac{1}{r} = \frac{1}{p} + \frac{1}{q} = 1 \), (i.e. \( L^\infty \)). Thus \( f \ast g \in S' \), and \( \mathcal{F}^{-1}(f \ast g) \in S' \). Since both sides of (5) are well defined as elements of \( S' \), (5) follows by a result of Schwartz [15, Chapter VII] which states that the inverse Fourier transform converts convolution into multiplication in \( S' \) when the algebraic operations are well defined in \( S' \).

**Corollary 1.** Let \( f(z) \in H^p(T^C), \ 1 \leq p \leq 2 \); and let \( f(z) \) satisfy (1) for \( A = 0 \). There exists a function \( g(x) \in L^p, \ 1 \leq p \leq 2 \), such that \( f(z) \to g(x) \) in the \( S' \) topology (as well as the \( L^p \) norm topology) as \( y \to 0, \ y \in C \); and there exists a function \( h(t) \in L^q, \ \frac{1}{p} + \frac{1}{q} = 1 \), with \( \text{supp}(h) \subseteq C^* = \{ t : u_C(t) \leq 0 \} \) almost everywhere such that \( g = \hat{h} \) in \( S' \).
and

(6) \[ f(z) = \langle h(t), e^{2\pi i (z, t)} \rangle = \int_{\mathbb{R}^n} g(t)K(z-t)dt = \int_{\mathbb{R}^n} g(t)Q(z, t)dt, \]

where the equality (6) is in $S'$. 

\textbf{PROOF.} From Theorem 2 we obtain the function $g(x) \in L^p$, $1 \leq p \leq 2$, and an element $V \in S'$ with $\text{supp} (V) \subseteq C^*$ and $g = \hat{V}$. Thus $V = \mathcal{F}^{-1}(g)$ in $S'$, Since $g(x) \in L^p$, $1 \leq p \leq 2$, $h(t) = \mathcal{F}^{-1}[g(x); t]$ exists classically and is an element of $L^q$, $\frac{1}{p} + \frac{1}{q} = 1$. Thus $V = h(t)$ in $S'$, and $\text{supp} (h) \subseteq C^*$ almost everywhere. Let $\varphi \in S$. Performing a change of order of integration we obtain

(7) \[ \langle \langle h(t), e^{2\pi i (x, t)} \rangle, \varphi(x) \rangle = \langle h(t), e^{-2\pi i (y, t)} \varphi(t) \rangle = \langle \mathcal{F}[I_{C^*}(t)e^{-2\pi i (y, t)}]h(t), \varphi(x) \rangle, \]

where $I_{C^*}(t)$ is the characteristic function of $C^*$. Now $I_{C^*}(t)e^{-2\pi i (y, t)} \in L^p$ for all $p$, $1 \leq p \leq \infty$. In particular if $1 \leq p \leq 2$, then

\[ \mathcal{F}[I_{C^*}(t)e^{-2\pi i (y, t)}; x] \in L^q \text{ for all } q, \frac{1}{p} + \frac{1}{q} + 1. \]

We now apply Lemma 1 to obtain

\[ \mathcal{F}^{-1}(g \ast \mathcal{F}[I_{C^*}(t)e^{-2\pi i (y, t)}; x]) = h(t)I_{C^*}(t)e^{-2\pi i (y, t)} \]

in $S'$. Thus

\[ \mathcal{F}(I_{C^*}(t)e^{-2\pi i (y, t)}h(t)) = g \ast \mathcal{F}[I_{C^*}(t)e^{-2\pi i (y, t)}; x] = g \ast \int_{C^2} e^{2\pi i (z, t)}dt \]

in $S'$. Returning to (7) we have

(8) \[ \langle \langle h(t), e^{2\pi i (z, t)} \rangle, \varphi(x) \rangle = \langle g \ast \int_{C^2} e^{2\pi i (z, t)}d(t), \varphi(x) \rangle \]

\[ = \langle \langle g(t), K(z-t) \rangle, \varphi(x) \rangle. \]
Combining (8) with (2) we thus obtain (6), and the proof is complete.

The results obtained in Theorem 2 and Corollary 1 are reminiscent of classical results for $H^p$ spaces of functions analytic in a half plane in $\mathbb{C}$. Hille and Tamarkin [11, Theorem 2] have shown that if $f(z)$ is analytic in the half plane $\text{Im}(z) > 0$ and has a limit function $F(x) \in L^p$, and if $f(z)$ is represented by the Cauchy integral of $F(x)$, then it is also represented by the Poisson integral of $F(x)$ and vice versa. Hille and Tamarkin ([11, Theorem 3] and [12, Theorem]) have also obtained results relating analytic functions which have boundary values and which are represented by the Cauchy (Poisson) integral of their boundary values with a Fourier transform which vanishes on a half line. (For related results we also refer to [15]). Of course the Hille and Tamarkin theorems hold for the $H^p$ spaces of functions analytic in a half plane. Stein and Weiss have shown that if $f(z) \in H^\gamma(\mathbb{C}^p)$, then equality (6) holds [17; Theorem 3.1, p. 101; Theorem 3.6, p. 103; Theorem 3.9, p. 106]. In Theorem 2 and Corollary 1 we have obtained conditions under which these classical results of Hille and Tamarkin are extended to the $H^p(\mathbb{C}^p)$ spaces for other values of $p$.

We shall now obtain a result similar to Theorem 2 for $H^\infty(\mathbb{C})$. In this version of Fatou's theorem we are able to say more about the element of $H^\infty$ and its boundary value than in the classical setting for tubular radial domains.

**Theorem 3.** Let $f(z) \in H^\infty(\mathbb{C})$. There exists a function $g(x) \in L^\infty$ such that $f(z) \rightarrow g(x)$ in the $S'$ topology (as well as in the weak-star topology of $L^\infty$) as $y \rightarrow 0, y \in \mathbb{C}$; and there exists an element $V \in \mathcal{D}'_{L^2}$ with $g(x) = V$ and $\text{supp}(V) \subseteq \mathbb{C}^+ = \{t : u_C(t) \leq 0\}$ such that

$$f(z) = \langle V, e^{2\pi i \langle z, t \rangle} \rangle = \int_{\mathbb{R}^n} g(t)Q(z; t)dt, z \in \mathbb{C}.$$

**Proof.** Combining Propositions 4 and 3 (d) in Koranyi [14], we obtain the existence of a function $g(x) \in L^\infty$ such that $f(z) \rightarrow g(x)$ in the weak-star topology of $L^\infty$ as $y \rightarrow 0, y \in \mathbb{C}$. This convergence and the Lebesgue dominated convergence theorem imply immediately that $f(z) \rightarrow g(x)$ in $S'$ as $y \rightarrow 0, y \in \mathbb{C}$. These same results of Korányi also
We now put $F(z) = \frac{f(z)}{1 + z_1^2 \ldots z_n^2}$.

Since $f(z) \in H^m(T^C)$, then $F(z) \in H^2(T^C)$; and by a result of Bochner [2, section 3]. (See also Vladimirov [18, pp. 224-227]), there exists a function $\psi(t) \in L^2$ with supp $(\psi) \subseteq C^*$ such that

$$F(z) = \int_{\mathbb{R}^n} \psi(t)e^{2\pi i (z,t)} dt, \quad z \in T^C.$$

We now put $V = (1 + D^{(2, \ldots, 2)})\psi(t)$. Then supp $(V) = $ supp $(\psi) \subseteq C^*$; and by the Schwartz characterization theorem [15, Théorème XXV, p. 201], $V \in \mathcal{S}'$. A straightforward calculation now gives

$$\langle V, e^{2\pi i (z,t)} \rangle = (1 + z_1^2 \ldots z_n^2) \int_{\mathbb{R}^n} \psi(t)e^{2\pi i (z,t)} dt = f(z).$$

Let $\xi(\eta) \in \mathcal{S}$, the space of infinitely differentiable functions, $\eta \in \mathbb{R}^1$, such that $\xi(\eta) = 1$ for $\eta \geq 0$, $\xi(\eta) = 0$ for $\eta \leq -\varepsilon$, $\varepsilon > 0$; and $0 \leq \xi(\eta) \leq 1$. Put $\gamma(t) = \xi((t, y))$, $y \in C$. Let $\varphi \in \mathcal{S}$. Using (9) we obtain

$$\langle f(z), \varphi(x) \rangle = \langle V, \gamma(t)e^{-2\pi i (y,t)}\tilde{\varphi}(t) \rangle, \quad z \in T^C.$$

It is straightforward to show that $\gamma(t)e^{-2\pi i (y,t)}\tilde{\varphi}(t) \rightarrow \gamma(t)\tilde{\varphi}(t)$ in $\mathcal{S}$ as $y \rightarrow 0$, $y \in C$. Since $V \in \mathcal{D}'_p \subset \mathcal{S}'$ (i.e. is continuous), then

$$\langle V, \gamma(t)e^{-2\pi i (y,t)}\tilde{\varphi}(t) \rangle \rightarrow \langle V, \gamma(t)\tilde{\varphi}(t) \rangle = \langle V, \varphi \rangle$$

as $y \rightarrow 0$, $y \in C$. (11) combined with (10) shows that $f(z) \rightarrow \tilde{V}$ in $\mathcal{S}'$ as $y \rightarrow 0$, $y \in C$. Since the limit in $\mathcal{S}'$ of $f(z)$ is unique, we thus have $g(x) = \tilde{V}$; and the proof is complete.
If \( f(z) \in H^\infty(T^c) \), then by definition \( f(z) \) is bounded for \( z \in T^c \) and hence satisfies (1) for \( A = 0 \). Thus once we obtained the boundary value \( g(x) \) in Theorem 3, we could have immediately applied Theorem 1 to obtain an element \( V \in \mathcal{S}' \) such that \( g(x) = \widehat{V} \) and \( \text{supp} (V) \subseteq C^* \). We see, however, from the proof of Theorem 3 that we can actually make the stronger statement that \( V \in \mathcal{D}'_x (\mathcal{D}'_x \subseteq \mathcal{S}') \).

3. Converse results.

Throughout this section \( C \) will denote an open convex cone which has the property that \( C \) contains no entire straight line.

The following theorem and corollary can be viewed as converses to the combination of Propositions 4 and 3 (c) of Korányi [14] and to Theorem 2 of the present paper for the corresponding values of \( p \).

THEOREM 4. Let \( g(t) \in L^p, \ 1 \leq p \leq 2; \) and let \( \text{supp} (g) \subseteq C^* = \{ t : u_C(t) \leq 0 \} \). There exists a function \( f(z) \in H^q(T^c) \) and a function \( h(x) = \mathcal{F}[g(t); x] \in L^q, \frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq 2, \) such that \( f(z) \to h(x) \) in the \( \mathcal{S}' \) topology (as well as in the \( L^q \) norm topology, \( \frac{1}{p} + \frac{1}{q} = 1, 1 < p \leq 2, \) or in the weak-star topology of \( L^\infty \) if \( p = 1 \)) as \( y \to 0, y \in C \).

PROOF. Let \( I_C(t) \) denote the characteristic function of \( C^* \), and let \( \gamma(t) \) be defined as in the proof of Theorem 3. Put

\[
f(z) = \int_{\mathbb{R}^n} g(t)e^{2\pi i (z \cdot t)} dt = \int_{\mathbb{R}^n} I_C(t)g(t)\gamma(t)e^{2\pi i (z \cdot t)} dt, \ z \in T^c.
\]

Since \( g(t) \in L^p \subseteq \mathcal{S}', \ 1 \leq p \leq 2, \) and \( \text{supp} (g) \subseteq C^* \), then by [4, Theorem 4] \( f(z) \) is analytic in \( T^c \). For the present we let \( y = \text{Im} (z) \in C \) be fixed. We have for \( t \in \mathbb{R}^n \) that

\[
| I_C(t)\gamma(t)e^{-2\pi i (y \cdot t)} | \leq 1;
\]

and \( I_C(t)\gamma(t)e^{-2\pi i (y \cdot t)} \in L^q \) for all \( q, \ 1 \leq q \leq \infty \). By Hölder's inequality and (12), we have for \( g(t) \in L^p, \ 1 \leq p \leq 2, \) that \( I_C(t)\gamma(t)e^{-2\pi i (y \cdot t)}g(t) \in \).
and the Fourier transform can be interpreted in the appropriate limit in the mean sense for $1 < p \leq 2$. Thus by the Fourier transform theory, if $p = 1$, $q = \infty$; and using (12) we have

$$|f(z)| \leq \int |g(t)| dt \leq \infty.$$  

For $1 < p \leq 2$, we have again by the Fourier transform theory and (12) that

$$\int |f(x + iy)|^q dx \leq \|I_c^*(y)\|_{L_p} \|g(t)\|_{L_q} \leq \|g\|_{L_p} \leq \infty,$$

$$\frac{1}{p} + \frac{1}{q} = 1.$$  

But the right hand sides of (13) and (14) are independent of $y \in C$. Thus the estimates in (13) and (14) hold for all $y \in C$; and it follows that $f(z) \in H^q(T^c)$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p \leq 2$. Further, since $g(t) \in L^p$, $\frac{1}{p} + \frac{1}{q} = 1$; and using a proof similar to that in equations (10) and (11), we have $f(z) \rightarrow \gamma(x) = h(x)$ in $S'$ as $y \rightarrow 0$, $y \in C$.

Let $1 < p \leq 2$. As in the proof of Theorem 2, we obtain the existence of a function $\psi(x) \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1$, such that $f(z) \in H^q(T^c)$ converges in the $L^q$ norm topology, and hence in the $S'$ topology, to $\psi(x)$ as $y \rightarrow 0$, $y \in C$. Since the $S'$ limit of $f(z)$ is unique, then $h(x) = \psi(x)$ almost
everywhere. Thus \( f(z) \to h(x) \) in the \( L^q \) norm topology as \( y \to 0, y \in C \).
If \( p = 1 \) and \( q = \infty \), it similarly follows using the proof of Theorem 3
that \( f(z) \in H^\infty (T^C) \) converges in the weak-star topology of \( L^\infty \) to \( h(x) \)
as \( y \to 0, y \in C \); and the proof is complete.

**Corollary 2.** Let \( g(x) \in L^2 \), and let \( V \in S' \) such that \( \text{supp} (V) \subseteq C^* = \{ t : u_C(t) \leq 0 \} \) and \( g(x) = \bar{V} \) in \( S' \). There exists an element
\( f(z) \in H^2 (T^C) \) such that \( f(z) \to g(x) \) in the \( S' \) topology (as well as in
the \( L^2 \) norm topology) as \( y \to 0, y \in C \).

**Proof.** Since \( g(x) \in L^2 \), there exists an element \( h(t) \in L^2 \) such that
\( g(x) = \mathcal{F} [h(t); x] \) and \( h(t) = \mathcal{F}^{-1}[g(x); t] \). For \( \phi \in \mathcal{S} \) we have

\[
\langle V, \phi \rangle = \langle \mathcal{F}^{-1}(g), \phi \rangle = \langle h, \phi \rangle;
\]

so that \( V = h(t) \) in \( \mathcal{S} \) and \( \text{supp} (h) \subseteq C^* \) almost everywhere. We now put

\[
f(z) = \int h(t) e^{2\pi i z \cdot t} dt = \int I_{C^*}(t) h(t) \gamma(t) e^{2\pi i z \cdot t} dt, \quad z \in T^C,
\]

where \( I_{C^*}(t) \) and \( \gamma(t) \) are as in the proof of Theorem 4; and the conclusions
follow from Theorem 4 for this \( f(z) \).

We note that the functions \( f(z) \in H^q (T^C) \) constructed in Theorem 4
and Corollary 2 satisfy the following boundedness condition:

\[
| f(z) | \leq K(C')(1 + |z|)^N(1 + |y|^{-M}), \quad z \in T^C,
\]

where \( C' \) is an arbitrary compact subcone of \( C \), \( K(C') \) is a constant
depending on \( C' \), and \( M \) and \( N \) are nonnegative integers which do not
depend on \( C' \). This result follows from Theorem 4 of Carmichael [4].

In Theorem 4 and Corollary 2 the manufactured function \( f(z) \) has
belonged to certain specified \( H^q (T^C) \) spaces, namely those values of \( q 
\)
satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \), \( 1 \leq p \leq 2 \). In the following theorem we obtain condi-
tions under which the function \( f(z) \) is in \( H^q (T^C) \) for all \( q, 1 \leq q \leq \infty \).
This result generalizes a theorem of Carmichael [10, Theorem II.4].
THEOREM 5. Let $\varphi(t) \in \mathcal{S}$; and let $\text{supp}(\varphi) \subseteq S_A = \{ t : u(t) \leq A \}$, $A \geq 0$. There exists a function $f(z) \in \mathcal{H}^p(T^c)$ for all $p$, $1 \leq p \leq \infty$, such that $f(z) \to \hat{\varphi}(x) e^{\sim z} \in \mathcal{S}$ in the topology of $\mathcal{S}'$ (as well as pointwise) as $y \to 0$, $y \in C$.

PROOF. Let $I_{SA}(t)$ denote the characteristic function of $S_A$. Let $\xi(\eta) \in \mathcal{S}$, $\eta \in \mathbb{R}^n$, such that $\xi(\eta) = 1$ for $\eta \geq -A$, $\xi(\eta) = 0$ for $\eta \leq -A - \varepsilon$, $\varepsilon > 0$; and $0 \leq \xi(\eta) \leq 1$. Put $\gamma(t) = \xi(\langle t, y \rangle)$, $y \in C$. Since $\varphi(t) \in \mathcal{S} \subseteq \mathcal{S}'$ and $\text{supp}(\varphi) \subseteq S_A$, then by [4, Theorem 4],

$$f(z) = \int_{\mathbb{R}^n} \varphi(t) e^{2\pi i \langle z, t \rangle} dt = \int_{\mathbb{R}^n} I_{SA}(t) \varphi(t) \gamma(t) e^{2\pi i \langle z, t \rangle} dt$$

is analytic in $T^c$. For $z \in T^c$ and $t \in \mathbb{R}^n$ we have

$$| I_{SA}(t) \varphi(t) \gamma(t) e^{2\pi i \langle z, t \rangle} | = | I_{SA}(t) \gamma(t) e^{-2\pi i \langle t, y \rangle} \varphi(t) | \leq e^{2\pi A} | \varphi(t) |.$$

Since $\varphi(t) \in \mathcal{S}$ we may apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{y \to 0} \int_{y \in C} \varphi(t) e^{2\pi i \langle z, t \rangle} dt = \hat{\varphi}(x);$$

and $\hat{\varphi}(x) \in \mathcal{S}$. Further, using (15) we have for all $z \in T^c$ that

$$| f(z) | \leq e^{2\pi A} \int_{\mathbb{R}^n} | \varphi(t) | dt \leq K < \infty,$$

$K$ being a constant. Thus $f(z) \in \mathcal{H}^\sim(T^c)$; and from another application of the Lebesgue dominated convergence theorem, we obtain that $f(z) \to \hat{\varphi}(x)$ in the $\mathcal{S}'$ topology as $y \to 0$, $y \in C$.

Now let $\alpha = (\alpha_1, ..., \alpha_n)$ be an arbitrary $n$-tuple of nonnegative integers. Using the facts that $D^\alpha \varphi(t) \in \mathcal{S}$ and $\text{supp}(D^\alpha \varphi(t)) \subseteq S_A$ for any $\alpha$, we integrate by parts in the integral defining $f(z)$ (i.e. $\int_{\mathbb{R}^n} \varphi(t) e^{2\pi i \langle z, t \rangle} dt$) and obtain

$$f(z) = (-2\pi i z_1)^{-\alpha_1} ... (-2\pi i z_n)^{-\alpha_n} \int_{\mathbb{R}^n} D^\alpha \varphi(t) e^{2\pi i \langle z, t \rangle} dt;$$
so that

(17) \[ |z_1|^{s_1} \cdots |z_n|^{s_n} |f(z)| \leq \frac{e^{2\pi A}}{(2\pi)^{a_1 + \cdots + a_n}} \int_{\mathbb{R}^n} |D^a \varphi(t)| \, dt \leq K_a < \infty. \]

From (16) and (17) we obtain

\[
|f(z)| \leq (K + K_a)(1 + |z_1|^{s_1} \cdots |z_n|^{s_n})^{-1} \leq (K + K_a)(1 + |x_1|^{a_1} \cdots |x_n|^{a_n})^{-1},
\]

and this inequality holds for all \(n\)-tuples \( \alpha \) of nonnegative integers. We now choose \( \alpha = (2, \ldots, 2) \). For any \( p, 1 \leq p < \infty \) we thus have

(18) \[ \int_{\mathbb{R}^n} |f(x+iy)|^p \, dx \leq (K + K_2 \cdots K_2)^p \int_{\mathbb{R}^n} (1 + |x_1|^2 \cdots |x_n|^2)^{-p} \, dx. \]

The right hand side of (18) is finite for any \( p, 1 \leq p < \infty \); and for each fixed \( p \), the value of the right hand side of (18) is independent of \( y \in C \). Thus \( f(z) \in H^p(T^C) \) for all \( p, 1 \leq p < \infty \); and we have already seen that \( f(z) \in H^\infty(T^C) \). The proof is complete.

Korányi \[14, Propositions 4 and 3 (d)\] has proved the classical Fatou theorem for functions \( f(z) \in H^\infty(T^C) \). The following theorem is a converse to this result and to Theorem 3 of the present paper.

**Theorem 6.** Let the cone \( C \) be contained in \( \{ y : y_j > 0, j = 1, \ldots, n \} \). Let \( g(x) \in L^\infty \) such that \( g(x) = \hat{V} \) in \( S' \) where \( V \in S' \) and \( \text{supp} \{V\} \subseteq \mathbb{C}^* = \{ t : u_c(t) \leq 0 \} \). There exists a function \( f(z) \in H^\infty(T^C) \) such that \( f(z) \to g(x) \) in the \( S' \) topology (as well as in the weak-star topology of \( L^\infty \)) as \( y \to 0, y \in C' \subseteq C \), where \( C' \) is an arbitrary compact subcone of \( C \).

**Proof.** Put

\[
h(x) = \frac{g(x)}{1 + x_1^2 \cdots x_n^2}.
\]

Since \( g(x) \in L^\infty \), \( h(x) \in L^2 \). By hypothesis \( g(x) = \hat{V} \); so that

\[
V = \mathcal{F}^{-1}(g) = \mathcal{F}^{-1}[(1 + x_1^2 \cdots x_n^2)h(x)] \in S'.
\]
We thus have for \( \varphi \in \mathcal{S} \) that

\[
\langle V, \varphi \rangle = \langle (1+x_1^2 \ldots x_n^2)h(x), \mathcal{F}^{-1}[\varphi(t) \cdot x] \rangle.
\]

Since \( h(x) \in L^2 \), there exists a function \( k(t) \in L^2 \) such that \( h(x) = \mathcal{F}[k(t) \cdot x] \); and

\[
\langle V, \varphi \rangle = \langle \mathcal{F}[k(t) \cdot x], (1+x_1^2 \ldots x_n^2)\mathcal{F}^{-1}[\varphi(t) \cdot x] \rangle = \langle (1+D^{(2, \ldots, 2)})k(t), \varphi \rangle.
\]

Thus \( V = (1+D^{(2, \ldots, 2)})k(t) \), and it follows that \( \text{supp}(k) \subseteq C^* \) almost everywhere. We now put

\[
f(z) = \langle V, e^{2\pi i(z \cdot t)} \rangle = \langle V, \gamma(t)e^{2\pi i(z \cdot t)} \rangle, \quad z \in T^C,
\]

where \( \gamma(t) \) is defined as in the proof of Theorem 3. By \([4, \text{Theorem 4}]\), \( f(z) \) is analytic in \( T^C \); and \( f(z) \rightarrow \hat{V} = g(x) \) in \( \mathcal{S}' \) as \( y \rightarrow 0, \ y \in C' \subseteq C \).

We now prove that \( f(z) \) is bounded for \( z \in T^C \). By a straightforward calculation we have

\[
f(z) = \int_{\mathbb{R}^n} k(t)e^{2\pi i(z \cdot t)}dt + z_1^2 \int_{\mathbb{R}^n} k(t)e^{2\pi i(z_1 \cdot t)}dt. \tag{19}
\]

We put

\[
P(z) = \int_{\mathbb{R}^n} k(t)e^{2\pi i(z \cdot t)}dt.
\]

It is easily seen that \( P(z) \) is bounded for \( z \in T^C \); and again applying \([4, \text{Theorem 4}]\), we have that \( P(z) \) is analytic in \( T^C \). To show that \( z_1^2 \ldots z_n^2 P(z) \) is bounded for \( z \in T^C \) we consider the function

\[
F(\varepsilon, z) = \exp\left[i\varepsilon(z_1^\sigma + \ldots + z_n^\sigma)\right](z_1^2 \ldots z_n^2)P(z), \quad z \in T^C,
\]

where \( 0 < \sigma < 1 \) and \( \varepsilon > 0 \) is fixed for the present. Since \( P(z) \) is analytic in \( T^C \), then \( F(\varepsilon, z) \) is also. By our assumption on the cone \( C \), we have
that $T^C$ is contained in the octant $\{ z : \text{Im}(z_j) > 0, j = 1, \ldots, n \}$. Thus for
\[ z = (z_1, \ldots, z_n) \in T^C, \quad z_j = r_j e^{i\theta_j}, \quad 0 < \theta_j < \pi, \quad j = 1, \ldots, n; \]
and
\[
F(\varepsilon, z) = \exp \left[ \varepsilon (r_1^2 \cdots r_n^2) \exp \left[ -\varepsilon (r_1 \sin \theta_1 + \cdots + r_n \sin \theta_n) \right] \right],
\]
where $M$ is the bound on $P(z)$. Now $0 < \theta_j < \pi, j = 1, \ldots, n$, and $0 < \sigma < 1$ imply $\sin \sigma \theta_j > 0, j = 1, \ldots, n$; and it follows from (20) that $F(\varepsilon, z)$ is bounded for each fixed $\varepsilon > 0$ and for $z \in T^C$. Further, as $y \to 0, y \in C$,
\[
F(\varepsilon, z) \to \exp \left[ \varepsilon (|x_1|^2 + \cdots + |x_n|^2)(\cos \sigma \pi + i \sin \sigma \pi) \right] (x_1^2 \cdots x_n^2) h(x)
\]
in the weak-star topology of $L^\infty$. Since $0 < \sigma < 1$, then $\sin \sigma \pi > 0$; and we have from the definition of $h(x)$ that
\[
|\exp \left[ \varepsilon (|x_1|^2 + \cdots + |x_n|^2)(\cos \sigma \pi + i \sin \sigma \pi) \right] (x_1^2 \cdots x_n^2) h(x) | \leq \frac{(x_1^2 \cdots x_n^2 |g(x)|}{1 + x_1^2 \cdots x_n^2} \leq B,
\]
where $B$ is the bound on $g(x) \in L^\infty$, and this bound in (21) is independent of $\varepsilon$. Thus for each fixed $\varepsilon > 0$, $F(\varepsilon, z) \in H^\infty(T^C)$; and $F(\varepsilon, z)$ converges in the weak-star topology of $L^\infty$ to a bounded measurable function. It follows from Propositions 4 and 3 (d) of Korányi [14] that
\[
F(\varepsilon, z) = \int_{\mathbb{R}^n} \exp \left[ \varepsilon (|t_1|^2 + \cdots + |t_n|^2)(\cos \sigma \pi + i \sin \sigma \pi) \right] (t_1^2 \cdots t_n^2) h(t) Q(z; t) dt.
\]
Thus by (22), (21) and Proposition 2 (b) of Korányi [14], we have
\[
|F(\varepsilon, z)| \leq B \int_{\mathbb{R}^n} Q(z; t) dt = B;
\]
and this bound is independent of \( \varepsilon > 0 \). Returning to the definition of \( F(\varepsilon, z) \) and using (23), we obtain

\[
|z_1^2 \ldots z_n^2 P(z)| \leq B \exp \left[ \varepsilon (r_1^2 \sin \sigma_1 + \ldots + r_n^2 \sin \sigma_n) \right],
\]

\( \varepsilon > 0 \). Since \( z_1^2 \ldots z_n^2 P(z) \) and \( B \) are independent of \( \varepsilon \), we let \( \varepsilon \to 0 \) in (24) and obtain that \( z_1^2 \ldots z_n^2 P(z) \) is bounded by \( B \), the bound on \( g(x) \in L^\infty \), for all \( z \in T^C \). We now conclude from (19) that \( f(z) \in H^\infty(T^C) \).

Using this fact and exactly the same method used in the last paragraph of the proof of Theorem 4, we obtain that \( f(z) \to g(x) \) in the weak-star topology of \( L^\infty \) as \( y \to 0 \), \( y \in C^* \subseteq C \); and the proof is complete.

Results similar to Theorem 6 can be proved using the same methods for the cone \( C \) being contained in any of the \( 2^n \) domains \( \{ y : \delta_j y_j > 0, \delta_j = \pm 1, j = 1, \ldots, n \} \); the choice of \( \{ y : y_j > 0, j = 1, \ldots, n \} \) was purely a matter of convenience. A special case of Theorem 6 has been obtained by Beltrami and Wohlers [1, Theorem 3] for one dimension and functions analytic in a half plane. We now obtain a corollary to Theorem 6.

**Corollary 3.** Let the cone \( C \) be contained in \( \{ y : y_j > 0, j = 1, \ldots, n \} \). Let \( f(z) \) be analytic in \( T^C \) and satisfy (1) for \( A = 0 \). Let \( f(z) \to g(x) \in L^\infty \) in the topology of \( S' \) as \( y \to 0 \), \( y \in C \). Then \( f(z) \in H^\infty(T^C) \).

**Proof.** By Theorem 1, there exists an element \( V \in S' \) with \( \text{supp} \ (V) \subseteq C^* \) and \( \hat{V} = g(x) \) such that \( f(z) = \langle V, e^{z(t, \cdot, \cdot)} \rangle \), \( z \in T^C \). By hypothesis \( f(z) \) is analytic in \( T^C \); and by [4, Theorem 4], \( \langle V, e^{z(t, \cdot, \cdot)} \rangle \) is analytic in \( T^C \). Thus by the identity theorem for analytic functions, \( f(z) = \langle V, e^{z(t, \cdot, \cdot)} \rangle \), \( z \in T^C \); and the conclusion is immediate from Theorem 6.

### 4. Functions analytic in disconnected tubular cones.

Let \( C \) be an open cone which is the finite union of open cones \( C_j, j = 1, \ldots, m \), each of which is convex and has the property that \( \overline{C}_j \) contains no entire straight line. Throughout this section \( T^C \) will denote the tubular cone associated with the open (possibly disconnected) cone \( C \) which satisfies the above property; and we recall that \( 0(C) \) denotes the convex envelope (hull) of \( C \).
Let $f(z)$ be analytic in $T^c = \mathbb{R}^n + iC$, $C = \bigcup_{j=1}^{m} C_j$, and satisfy (1). For each $j = 1, \ldots, m$, suppose that $f(z) \in H^p(T^c_j)$, $1 \leq p < \infty$. By Theorem 2, there exist functions $g_j(x) \in L^p$, $1 \leq p < \infty$, such that $f(z) \to g_j(x)$ in $S'$ as $y \to 0$, $y \in C_j$, $j = 1, \ldots, m$. We now prove the following generalization of Theorem 2.

**Theorem 7.** Let $f(z)$ be analytic in $T^c$ and satisfy (1). For each $i = 1, \ldots, m$, let $f(z) \in H^p(T^c_i)$, $1 \leq p < \infty$. Let the $S'$ boundary values $g_i(x) \in L^p$ of $f(z)$, $z \in T^c_i$, be equal in $S'$. Then $f(z)$ has an analytic extension (denoted $F(z)$) to $T^0(C)$; for any arbitrary compact subcone $C'$ of $0(C)$, $D^\alpha F(z)$ satisfies

$$(25) \quad |D^\alpha F(z)| \leq K(C')(1 + |z|)^\alpha(1 + |y|^{-\alpha}) \exp \left[2\pi A_{pC} |y| \right], \quad z \in T^c,$$

where $\alpha$ is an arbitrary $n$-tuple of nonnegative integers, $K(C')$ is a constant depending on $C'$, and $M$ and $N$ are nonnegative integers which do not depend on $C'$; there exists a function $g(x) \in L^p$, $1 \leq p < \infty$, such that $F(z) \to g(x)$ in the topology of $S'$ as $y \to 0$, $y \in C' \subset 0(C)$; and if $p = 2$, $F(z) \in H^2(T^0(C))$.

**Proof.** By Theorem 2, there exist elements $V_j \in S'$ with $\text{supp}(V_j) \subseteq S_{A,j} = \{t : u_{C_j}(t) \leq A\}$ such that $g_i(x) = \hat{V}_j$ and

$$(26) \quad f(z) = \langle V_j, e^{2\pi i(z,t)} \rangle, \quad z \in T^c_j, \quad j = 1, \ldots, m.$$ 

By assumption, $g_1(x) = \ldots = g_m(x)$ almost everywhere; and we call this common value $g(x)$. Since $V_j = \mathcal{F}^{-1}(g_j)$, $j = 1, \ldots, m$, it follows immediately that $V_1 = \ldots = V_m$; and we call this common value $V$. Thus $g(x) \in L^p$, $1 \leq p < \infty$, $g(x) = \hat{V}$, and $\text{supp}(V) \subseteq \bigcup_{j=1}^{m} S_{A,j}$; so that $V$ vanishes on $\bigcup_{j=1}^{m} \{t : u_{C_j}(t) > A\}$. Now

$$u_C(t) = \max_{j=1, \ldots, m} u_{C_j}(t);$$

and from the definition of $\rho_C$. (See [4, section II]) we have $u_{\rho_C}(t) \leq$
\[ u_{0(c)}(t) \leq \rho_c \max_{j=1, \ldots, m} u_{c_j}(t); \]

and by a lemma of Vladimirov [18, Lemma 3, p. 220], \( \rho_c < +\infty \).

Now consider the set \( J = \{ t : u_{0(c)}(t) > A\rho_c \} \). If \( t \in J \), then by (27),

\[ t \in \{ t : \max_{j=1, \ldots, m} u_{c_j}(t) > A \} \]

Hence \( t \in \bigcup_{j=1}^m \{ t : u_{c_j}(t) > A \} \), and on this set \( V \) vanishes. Thus \( V \) vanishes if \( t \in J \) which implies that

\[ \text{supp}(V) \subseteq \{ t : u_{0(c)}(t) \leq A\rho_c \}. \]

Let \( \xi(\eta) \in E \), \( \eta \in R^1 \), such that \( \xi(\eta) = 1 \) for \( \eta \geq -A\rho_c \), \( \xi(\eta) = 0 \) for \( \eta \leq -A\rho_c - \varepsilon \), \( \varepsilon > 0 \); and \( 0 \leq \xi(\eta) \leq 1 \). Put \( \gamma(t) = \xi(t, y) \), \( y \in 0(C) \). We now put

\[ F(z) = \langle V, e^{2\pi i (z, t)} \rangle = \langle V, \gamma(t) e^{2\pi i (z, t)} \rangle, \ z \in T_0(C). \]

By [4, Theorem 4], \( F(z) \) is analytic in \( T_0(C) \), satisfies (25), and \( F(z) \to \widehat{V} = g(x) \) in \( S' \) as \( y \to 0 \), \( y \in C' \subset 0(C) \). Further since \( V = V_i \), \( j = 1, \ldots, m \), then from (26) we have \( f(z) = F(z) \), \( z \in T^C \); and \( F(z) \) is the analytic extension of \( f(z) \) to \( T_0(C) \).

If \( p = 2 \), then \( g(x) \in L^2 \); and there exists a function \( h(t) \in L^2 \) such that \( g(x) = \mathcal{F}[h(t); x] \). But then \( \widehat{V} = \hat{h} \) in \( S' \). Thus \( V = h \) in \( S' \), and \( \text{supp}(h) \subseteq \{ t : u_{0(c)}(t) \leq A\rho_c \} \) almost everywhere. Letting \( I(t) \) denote the characteristic function of this support set, we have as in (14) that

\[ \int_{\mathbb{R}^n} |F(x + iy)|^2 \, dx = \| I(t) \gamma(t) e^{-2\pi i (y, t)} h(t) \|_{L^2}^2 \leq \exp(4\pi A\rho_c) \| h \|_{L^2}^2 < \infty. \]

The estimate (28) holds for all \( y \in 0(C) \). Thus \( F(z) \in H^2(T_0(C)) \); and the proof is complete.

Since \( f(z) = F(z) \), \( z \in T^C \), then the conclusion of Theorem 7 states that \( f(z) \) satisfies (25) for \( z \in T^C \). Further, for \( p = 2 \), we proved in Theorem 7 that \( F(z) \in H^2(T_0(C)) \); and it follows as before that \( F(z) \to g(x) \) in the \( L^2 \) norm topology as \( y \to 0 \), \( y \in C' \subset 0(C) \), as well as in the \( S' \) topology.
Theorem 7 generalizes Theorem 2. In the same manner a generalization of Theorem 3 can be obtained for disconnected tubular cones, and we leave the formulation of such a result to the interested reader.

We now obtained a generalization of Theorem 6, the converse Fatou theorem

**Theorem 8.** Let the tubular cone \( T^C = \mathbb{R}^n + iC, \quad C = \bigcup_{j=1}^m C_j \), have the property that \( T^{\alpha(C)} \subseteq \{ z : \text{Im } (z_j) > 0, \quad j = 1, \ldots, n \} \). Let \( g_j(x), \quad j = 1, \ldots, m \), be \( L^\infty \) functions such that for each \( g_j(x) \) there exists an element \( V_j \in S' \) with \( \text{supp } (V_j) \subseteq C^*_j = \{ t : u_C(t) \leq 0 \} \) and \( g_j(x) = V_j \). Let \( g(x) = g_1(x) = \ldots = g_m(x) \) in \( S' \). Then there exists a function \( F(z) \in H^\infty(T^{\alpha(C)}) \) such that

\[
\lim_{y \to 0, \quad y \in C'_j \subseteq C_j} F(z) = g_j(x), \quad j = 1, \ldots, m,
\]

in the topology of \( S' \) (as well as in the weak-star topology of \( L^\infty \)) where \( C'_j \) is an arbitrary compact subcone of \( C_j, \quad j = 1, \ldots, m \).

**Proof.** As in the proof of Theorem 7, \( g(x) = g_1(x) = \ldots = g_m(x) \) implies \( V = V_1 = \ldots = V_m \); and \( g(x) = \hat{V} \) where \( \text{supp } (V) \subseteq \{ t : u_C(t) \leq 0 \} \). Here \( g(x) \in L^\infty \). We put

\[
F(z) = \langle V, \ e^{2\pi i(z, t)} \rangle = \langle V, \ \gamma(t)e^{2\pi i(z, t)} \rangle, \quad z \in T^{\alpha(C)},
\]

where \( \gamma(t) \) is defined as in the proof of Theorem 7 for \( A = 0 \). From the assumption on \( T^{\alpha(C)} \) and the proof of Theorem 6, we have \( F(z) \in H^\infty(T^{\alpha(C)}) \). Since \( V = V_j, \quad j = 1, \ldots, m \), then

\[
F(z) = \langle V_j, \ e^{2\pi i(z, t)} \rangle, \quad z \in T^{C_j}, \quad j = 1, \ldots, m.
\]

By [4, Theorem 4], \( F(z) \to \hat{V}_j = g_j(x) \in L^\infty \) in the topology of \( S' \) as \( y \to 0, \quad y \in C'_j \subseteq C_j, \quad j = 1, \ldots, m \). But since \( F(z) \in H^\infty(T^{\alpha(C)}) \), then \( F(z) \in H^\infty(T^{C'_j}), \quad j = 1, \ldots, m \); and arguing as in the last paragraph of the proof of Theorem 4, we have \( F(z) \to g_j(x) \in L^\infty \) in the weak-star topology of \( L^\infty \) as \( y \to 0, \quad y \in C'_j \subseteq C_j, \quad j = 1, \ldots, m \).

We note that the function \( F(z) \in H^\infty(T^{\alpha(C)}) \) constructed in Theorem 8 has the additional property that \( F(z) \to g(x) \) in both the \( S' \) and weak-
star $L^\infty$ topologies as $y \to 0$, $y \in C' \subset 0(C)$. This result follows immediately from Theorem 6. Generalizations of Theorem 4 and Corollary 2 can also be obtained for disconnected tubular cones. Their formulation and proof are similar in form to Theorem 8, and again we leave the details to the interested reader.

REFERENCES


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