ANTONIO PASINI

Some fixed point theorems of the mappings of partially ordered sets


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1. Introduction.

In this paper we give new simple proofs of some fixed point theorems, and strengthen others. The methods we shall use base themselves on two « strong » induction principles, we stated and utilized in [5]. We shall show, moreover, that one of them is equivalent to Axiom of Choice.

Let's now recall some results on the fixed points of a function defined from a partially ordered set \( P; < \) into itself.

**Proposition A.** Let \( P; < \) be a nonempty partially ordered set every well ordered subset of which has an upper bound. And let \( f \) be a function from \( P \) into \( P \) such that \( x < f(x) \) for every \( x \); then \( f \) has a fixed point.

The preceding result is proved in [2] by using Axiom of Choice. As a corollary we get:

**Proposition B.** Let \( P; < \) be a nonempty partially ordered set every well ordered subset of which has a least upper bound, and let \( f \) be a function just like proposition A's one; then \( f \) has a fixed point.

The preceding proposition, however, may be proved independently, and without using Axiom of Choice. (see [1] and [2]).


Lavoro eseguito nell'ambito dei gruppi di ricerca matematica del C.N.R.
PROPPOSITION C (see [3]). Let \( \langle P; \leq \rangle \) be a nonempty partially ordered set every well ordered subset \( B \) of which has an upper bound which is minimal in the set of upper bounds of \( B \). Let \( f \) be a mapping from \( P \) into \( P \) such that \( a \leq f(a) \) for some \( a \) in \( P \), and such that \( x \leq f(x) \leq y \) implies \( f(x) \leq f(y) \), for every \( x \) and \( y \) in \( P \); moreover \( \{ x, f(x) \} \) has a greatest lower bound in \( \langle P; \leq \rangle \); then \( f \) has a fixed point.

PROPPOSITION D (see [1]; see also [2]). Let \( \langle P; \leq \rangle \) be a non-empty partially ordered set every well ordered subset of which has a least upper bound. And let \( f \) be a mapping from \( P \) into \( P \) such that \( a \leq f(a) \) for some \( a \) in \( P \), and such that \( x \leq y \) implies \( f(x) \leq f(y) \), for every \( x \) and \( y \) in \( P \). Then \( f \) has a fixed point.

Proposition \( D \) is proved in [1] and in [2] without using Axiom of Choice.

PROPPOSITION E (see [4]). Let \( \langle P; \leq \rangle \) be a nonempty well ordered set every subset of which has a least upper bound. Let \( f \) be just like in proposition \( D \); then the set \( \{ p | f(p) = p \} \) is nonempty and has a maximal element.

Proposition \( E \) is proved in [4] without any use of Axiom of Choice. But this is possible because \( \langle P; \leq \rangle \) is well ordered.

We shall prove (but postpone the proofs):

THEOREM A. Let \( \langle P; \leq \rangle \) be a nonempty partially ordered set every well ordered subset \( B \) of which has an upper bound which is minimal in the set of upper bounds of \( B \). And let \( f \) be a function from \( P \) into \( P \) such that:

1. \( x \leq f(x) \leq y \) implies \( f(x) \leq f(y) \) for every \( x \) and \( y \) in \( P \).
2. \( \{ x, f(x) \} \) has a greatest lower bound in \( P \), for every \( x \) in \( P \).
3. There is an element \( a \) of \( P \) such that \( a \leq f(a) \).

Then the set \( \{ p | f(p) = p \} \) is nonempty and has a maximal element.

Theorem \( A \) strengthens proposition \( C \). Theorem \( A \)'s proof makes use of Axiom of Choice.

THEOREM B. Let \( \langle P; \leq \rangle \) be a nonempty partially ordered set every well ordered subset of which has a least upper bound. And let \( f \) be a mapping from \( P \) into \( P \) such that \( x \leq f(x) \leq y \) implies \( f(x) \leq f(y) \), for every \( x \) and \( y \) in \( P \). And let be \( a \leq f(a) \) for some \( a \) in \( P \). Then the set \( \{ p | f(p) = p \} \) (is nonempty and) has a maximal element.
REMARK. Theorem B generalizes propositions E and D: in the fact (we require on P more general conditions than in E and) what we require on f is less strong than isotonicity: clearly if f is isomone \( x < f(x) < y \) implies \( f(x) < f(y) \), but there are non isomone functions f such that \( x < f(x) < y \) implies \( f(x) < f(y) \), and is a \( < f(a) \) for a convenient a in P. Let consider, for instance, the ordered set \( \langle 1, 2, 3 \rangle ; \langle \rangle \), where \( 1 < 2 < 3 \), and let’s pose \( f(1) = 3, f(3) = 3, f(2) = 1 \).

In theorem B we doesn’t need Axiom of Choice in proving \( \{ p | f(p) = p \} \neq 0 \). So we get a proof of proposition D which doesn’t use Axiom of Choice. Moreover, if we assume \( \langle P; \rangle \) is a well ordered set (as in proposition E), our methods give a proof of the existence of a maximal element in \( \{ p | f(p) = p \} \), which doesn’t use Axiom of Choice.

**THEOREM C.** Let \( \langle P; \rangle \) and f be just like in proposition A. Then \( \{ p | f(p) = p \} \) (is nonempty and) has a maximal element.

2. «Strong» transfinite induction principles.

Let be given a class A and a limit ordinal number \( \alpha_0 \). Let \( \alpha \) be a nonzero ordinal number less than \( \alpha_0 \), and \( \Phi_\alpha \) be a function from the set \( [0, \alpha) \) (i.e.: the set of all ordinal numbers less than \( \alpha \) into A. Let \( P(x, y) \) be a first order quantification scheme free on \( x, y \), belonging to a language which formalizes set theory. And let \( P(\alpha, \Phi_\alpha) \) be the sentence we give from \( P(x, y) \) by substituting \( x \) with \( \alpha \) and \( y \) with \( \Phi_\alpha \). Moreover let \( P(\alpha, \Phi_\alpha) \) be of the form:

\[
(\forall \gamma_1) \ldots (\forall \gamma_n) \left( (\gamma_1 < \alpha) \land \ldots \land (\gamma_n < \alpha) \right) \rightarrow Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n)
\]

where \( \gamma_1, \ldots, \gamma_n \) vary on \( [0, \alpha) \) and \( Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n) \) satisfies the following condition:

**CONDITION A.** For any choice of \( \gamma_1, \ldots, \gamma_n \) less than \( \alpha \), \( Q(\Phi_\alpha, \gamma_1, \ldots, \gamma_n) \) is true only if for every \( \beta \) greater than zero, less than \( \alpha \), and such that is an upper bound for \( \gamma_1, \ldots, \gamma_n \), \( Q(\Phi_\alpha[0, \beta), \gamma_1, \ldots, \gamma_n) \) is true (where \( \Phi_\alpha[0, \beta) \) is the restriction of \( \Phi_\alpha \) to \( [0, \beta) \)).

We proved in [5] (section 2, corollaries 2.1 and 2.2) the following results:
**LEMMA A.** If for every $\alpha$ less than $\alpha_0$ and greater than 0, for every function $\Phi_\alpha$ such that $P(\alpha, \Phi_\alpha)$ is true, there is only one function $\Phi_{\alpha+1}$ such that $P(\alpha + 1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1}[0, \alpha] = \Phi_\alpha$, and if there is a function $\Phi_1$ such that $P(1, \Phi_1)$ is true, then for every $\beta$ (less than $\alpha_0$ and greater than 0) there exists a function $\Phi_\beta$ such that $P(\beta, \Phi_\beta)$ is true.

Lemma A’s proof doesn’t need Axiom of Choice. Let’s now suppose $\Lambda$ is a set. Then we have:

**LEMMA B.** If for every $\alpha$ less than $\alpha_0$ and greater than 0, for every function $\Phi_\alpha$ such that $P(\alpha, \Phi_\alpha)$ is true, there is a function $\Phi_{\alpha+1}$ such that $P(\alpha + 1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1}[0, \alpha] = \Phi_\alpha$, and if there is a function $\Phi_1$ such that $P(1, \Phi_1)$ is true, then for every $\beta$ (less than $\alpha_0$ and greater than 0) there exists a function $\Phi_\beta$ such that $P(\beta, \Phi_\beta)$ is true.

In lemma B’s proof we made use of Axiom of Choice (formulated as well ordering principle); conversely we shall see now that lemma B implies the well ordering principle, and then the Axiom of Choice. Let $A$ be a nonempty set, and $a$ an element of $A$; let’s consider the set $A^* = A \cup \{a\}$, and let $\alpha_0$ be a limit ordinal of power greater than $2|A^*|$ (where $|A^*|$ is the cardinality of $A^*$) (1). Let $P(\alpha, \Phi_\alpha)$ be the following statement:

For every $\gamma < \alpha$, if $\{\Phi_\alpha(\delta)|\delta < \gamma\}$ is properly contained in $A$, then $\Phi_\alpha(\gamma) \in A$, and $\Phi_\alpha(\delta) \neq \Phi_\alpha(\gamma)$ for every $\delta < \gamma$;

It’s easily seen that this statement satisfies condition A. Let now $\Phi_\alpha$ be a function from $A^*$ into $A^*$ such that $P(\alpha, \Phi_\alpha)$ is true. If $\Phi_\alpha(\delta) = \{A\}$ for a $\delta < \alpha$, then let $\delta_0$ be the least $\delta$ such that $\Phi_\alpha(\delta) = \{A\}$. Then $\phi_\alpha(\gamma) \in A^*$, by proposition $P(\alpha, \Phi_\alpha)$; so we get a contradiction, and have to admit $\{\Phi_\alpha(\gamma)|\gamma < \delta_0\} = A$. So we can define $\Phi_{\alpha+1}$ by:

$(\Phi_{\alpha+1}[0, \alpha] = \Phi_\alpha$, and $\Phi_{\alpha+1}(a) = \{A\}$. And it’s easily seen that $P(\alpha + 1, \Phi_{\alpha+1})$ is true. Now we have to consider the second case, that’s when $\{A\} \notin \{\Phi_\alpha(\delta)|\delta < \alpha\}$. We have two subcases: $\{\Phi_\alpha(\delta)|\delta < \alpha\} = A$ and $\{\Phi_\alpha(\delta)|\delta < \alpha\} \neq A$. In the first subcase we pose $\Phi_{\alpha+1}(\alpha) = \{A\}$; in the second subcase we equate $\Phi_{\alpha+1}(\alpha)$ to an arbitrary element of $A - \{\Phi_\alpha(\delta)|\delta < \alpha\}$. And it’s trivial to see that $P(\alpha + 1, \Phi_{\alpha+1})$ is true (in the first as well as in the second subcase). So, by lemma B, if $\beta$

(1) It’s known that this can be done without any use of Axiom of Choice.
is an ordinal of power greater than $2^{\lambda^*}$ and less than $\sigma_0$ (Obviously we can always choose $\sigma_0$ such that such a $\beta$ exists), there is a $\Phi_\beta$ such that $P(\beta, \Phi_\beta)$ is true. Let's suppose, by absurd, $\{\Phi_\beta(\gamma) | \gamma < \beta\}$ properly contained in $A$. Then $\Phi_\beta$ is injective, and so has as many values as the power of $\beta$; this gets a contradiction. So we must have $\{\Phi_\beta(\gamma) | \gamma < \beta\} \supseteq A$. And it's trivial to check that $\Phi_\beta$ defines a well ordering on $A$; as we wished to prove. Then:

**Lemma B is equivalent to Axiom of Choice.**

### 3. Proofs.

**Proof of Theorem A.** Let $a$ be an element of $P$ such that $f(a) > a$. Let $\alpha_0$ be a limit ordinal whose power is greater than $2^{|P|}$ ($|P|$ is $P$'s cardinality). Let $\Phi_\alpha$ indicate a function from $[0, \alpha)(0 < \alpha < \sigma_0)$ into $P$, and let $P(\alpha, \Phi_\alpha)$ be the following statement:

$\Phi_\alpha(0) = a$. And for every ordinal number $\gamma$, if $\gamma < \alpha$, then is $\gamma + 1 < \alpha$,

$\Phi_\alpha(\gamma + 1) = f(\Phi_\alpha(\gamma))$, and if $\gamma$ is a limit ordinal, then the set $B_\gamma$ of upper bounds of $\{\Phi_\alpha(\delta) | \delta < \gamma\}$ is nonempty and has a minimal element, and $\Phi_\alpha(\gamma)$ is such an element.

It's easily seen that $P(\alpha, \Phi_\alpha)$ satisfies condition A. Let's now suppose $\Phi_\alpha$ is a function verifying $P(\alpha, \Phi_\alpha)$. Now we prove that there is a function $\Phi_{\alpha+1}$ such that $P(\alpha + 1, \Phi_{\alpha+1})$ is true and $\Phi_{\alpha+1}(0, \alpha) = \Phi_\alpha$. We have to distinguish two cases: $\alpha = \delta + 1$ for some ordinal number $\delta$, or $\alpha$ is a limit ordinal. First we verify $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$ is a well ordered subset of $\langle P; < \rangle$. We start by proving (by transfinite induction on $\gamma < \alpha$) the following statement $S(\gamma)$:

$\Phi_\alpha(\delta) < \Phi_\alpha(\gamma)$ for every $\delta$, with $\delta < \gamma$.

Let's now suppose $S(\gamma)$ true for any $\gamma < \mu$ ($\mu$ is a given ordinal number less than $\alpha$). We shall prove $S(\mu)$ is true. We must distinguish two cases: there is an ordinal number $\nu$ such that $\mu = \nu + 1$, or $\mu$ is a limit ordinal. If $\mu = \nu + 1$, then $\Phi_\alpha(\mu) = f(\Phi_\alpha(\nu))$. We have two subcases: $\nu = \lambda + 1$ or $\nu$ is a limit ordinal; if $\nu = \lambda + 1$, $\Phi_\alpha(\nu) = f(\Phi_\alpha(\lambda))$, and then $\Phi_\alpha(\mu) = f(\Phi_\alpha(\nu))$. But $f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\nu) > \Phi_\alpha(\lambda)$ (by inductive hypothesis). So $\Phi_\alpha(\lambda) < f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\nu)$; then we get $f(\Phi_\alpha(\lambda)) < f(\Phi_\alpha(\nu))$, that's $\Phi_\alpha(\mu) > \Phi_\alpha(\nu)$. And this suffices to prove $S(\mu)$.
Let's now consider the second subcase: that's when $v$ is a limit ordinal. We have $\Phi_\alpha(v) > \Phi_\alpha(\lambda)$ for every $\lambda < v$, and $f(\Phi_\alpha(\lambda)) = \Phi_\alpha(\lambda + 1) > \Phi_\alpha(v)$ (by inductive hypothesis). And $\Phi_\alpha(\lambda) < f(\Phi_\alpha(\lambda))$ (by inductive hypothesis). So $\Phi_\alpha(\lambda) < f(\Phi_\alpha(\lambda))$: then we get $f(\Phi_\alpha(\lambda)) < f(\Phi_\alpha(v))$: then $\Phi_\alpha(\lambda) < \Phi_\alpha(\mu)$. So: $\Phi_\alpha(\mu) > \Phi_\alpha(\lambda)$ for every $\lambda < v$; hence $\Phi_\alpha(\mu)$ is an upper bound of $\{\Phi_\alpha(\lambda) | \lambda < v\}$. But $\Phi_\alpha(\mu) = f(\Phi_\alpha(v))$: then does exists the greatest lower bound of $\{\Phi_\alpha(\nu), \Phi_\alpha(\mu)\}$, and it's clearly an upper bound of $\{\Phi_\alpha(\lambda) | \lambda < v\}$; but $\Phi_\alpha(v)$ is a minimal upper bound of $\{\Phi_\alpha(\lambda) | \lambda < v\}$. Then $\Phi_\alpha(v) = \text{g.l.b.} \{\Phi_\alpha(v), \Phi_\alpha(\mu)\}$, that's $\Phi_\alpha(v) < \Phi_\alpha(\mu)$, as we wished to prove; finally we have $S(\mu)$.

Let's now consider the second case, that's when $\mu$ is a limit ordinal; then, by $P(\alpha, \Phi_\alpha)$, $\Phi_\alpha(\mu)$ is an upper bound for $\{\Phi_\alpha(\lambda) | \lambda < \mu\}$, and $S(\mu)$ is trivially verified; then $S(\gamma)$ is true for every $\gamma < \alpha$. This implies $\Phi_\alpha$ is a monotone function from $[0, \alpha)$ into $\mathbb{P}$; and therefore $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$ is a well ordered set; then, by hypothesis, does exists a minimal upper bound $b$ of $\{\Phi_\alpha(\gamma) | \gamma < \alpha\}$, and we can pose $\Phi_{\alpha + 1}(\alpha) = b$.

And so, by lemma $B$, for every $\beta < \alpha$ is definable a $\Phi_\beta$ such that $P(\beta, \Phi_\beta)$ is true. We have already seen that $\Phi_\beta$ is a monotone function from $[0, \beta)$ into $\langle P; < \rangle$. Let's now suppose $x \neq f(x)$ for every $x$; in $P$; then $\Phi_\beta$ has as many values as the power of $\beta$. And we get an absurd because we can choose $\beta$ with power greater than $|P|$. So we must admit $\Phi_\beta = f(\Phi_\beta)$ for a convenient $\Phi_\beta \in P$.

We remark that we can choose $\Phi_\beta$ greater than (or equal to) $\alpha$.

We shall now prove that the set of fixed points of $f$ has a maximal element. Let's consider another proposition $P'(\alpha, \Phi'_\alpha)$ as follows:

for every $\gamma_1$ and $\gamma_2$ less than $\alpha$, if $\gamma_1 < \gamma_2$, then, if $\Phi'_\alpha(\gamma_1)$ is maximal in $\{p | f(p) = p\}$, is $\Phi'_\alpha(\gamma_1) = \Phi'_\alpha(\gamma_2)$. And, if it isn't maximal in $\{p | f(p) = p\}$, is $\Phi'_\alpha(\gamma_1) < \Phi'_\alpha(\gamma_2)$. Moreover, for every $\gamma$ less than $\alpha$, $\Phi'_\alpha(\gamma)$ is a fixed point for $f$. And is $\Phi'_\alpha(0) = \Phi_\alpha(0)$ (where $\Phi_\alpha(0)$ is the before found fixed point of $f$).

It's a trivial question to check that condition $A$ is satisfied by $P'(\alpha, \Phi'_\alpha)$. Now we must see that, if there's a function $\Phi'_\alpha$ such that $P'(\alpha, \Phi'_\alpha)$ is true, then we can construct a function $\Phi'_\alpha(\alpha + 1)$ such that $P'(\alpha + 1, \Phi'_\alpha(\alpha + 1))$ is true and $\Phi'_\alpha(\alpha + 1 | [0, 1]) = \Phi'_\alpha(\alpha)$. We distinguish two cases: when is $\alpha = \delta + 1$ for a convenient $\delta$, and when $\alpha$ is a limit ordinal.

Let's suppose is $\alpha = \delta + 1$; we have two subcases: $\Phi'_\alpha(\delta)$ is maximal in $\{p | f(p) = p\}$, or not. If $\Phi'_\alpha(\delta)$ is such a maximal element, then we set $\Phi'_\alpha(\alpha + 1) = \Phi'_\alpha(\delta)$ (and $\Phi'_\alpha(\alpha + 1 | [0, \alpha]) = \Phi'_\alpha(\alpha)$). It's easily seen that $P'(\alpha + 1, \Phi'_\alpha(\alpha + 1))$ is true. Let's now consider the subcase when $\Phi'_\alpha(\delta)$
isn’t maximal in \( \{ p | f(p) = p \} \); then there is a fixed point \( q \) of \( f \) such that \( q > \Phi_{\alpha}(\delta) \). If we set \( \Phi'_{\alpha+1}(\alpha) = q \) (and \( \Phi'_{\alpha+1}(\delta, \alpha) = \Phi_{\alpha}^{\delta} \)) we get a function satisfying \( P'(\alpha + 1, \Phi'_{\alpha+1}) \). Let’s now consider the case when \( \alpha \) is a limit ordinal. We have two subcases: there is an ordinal \( \gamma \) less than \( \alpha \) such that \( \Phi_{\alpha}(\gamma) \) is maximal in \( \{ p | f(p) = p \} \), or there’s no such ordinal. In the first subcase is \( \Phi'_{\alpha}(\gamma) = \Phi'_{\alpha}(\gamma) \) for every \( \lambda \) less than \( \alpha \) and greater than \( k \). So we can set \( (\Phi'_{\alpha+1}(\delta, \alpha) = \Phi'_{\alpha}(\gamma) \). \( \Phi'_{\alpha+1}) \), so defined, verifies \( P'(\alpha + 1, \Phi'_{\alpha+1}) \). In the second subcase \( \{ \Phi'_{\alpha}(\gamma) | \gamma < \alpha \} \) is a well ordered set. Then by hypothesis, it has a minimal upper bound \( b \). We shall see now that there is a fixed point \( q \) greater than \( b \). We have \( \Phi'_{\alpha}(\gamma) = f(\Phi_{\alpha}(\gamma)) \), and \( b > \Phi'_{\alpha}(\gamma) \), for every \( \gamma \) less than \( \alpha \). That’s: \( \Phi'_{\alpha}(\gamma) < f(\Phi_{\alpha}(\gamma)) < b \). We get: \( f(\Phi_{\alpha}(\gamma)) < f(b) \); that’s \( \Phi'_{\alpha}(\gamma) < f(b) \), for every \( \gamma \) less than \( \alpha \). So \( f(b) \) in an upper bound for the set \( \{ \Phi'_{\alpha}(\gamma) | \gamma < \alpha \} \); but the set \( \{ b, f(b) \} \) has a greatest lower bound. This, and the minimality of \( b \) in the set of upper bounds of \( \{ \Phi_{\alpha}(\gamma) | \gamma < \alpha \} \), give \( b < f(b) \). By the first part of the proof of this theorem, we can find a fixed point \( q \), such that \( q > b \). If we set \( (\Phi'_{\alpha+1}(\delta, \alpha) = \Phi'_{\alpha}(\gamma) \) and \( \Phi'_{\alpha+1}(\delta, \alpha) = q \), we have a function \( \Phi_{\alpha+1} \) satisfying \( P'(\alpha + 1, \Phi'_{\alpha+1}) \). Then, by lemma \( B \), for every nonzero ordinal \( \beta \) less than \( \alpha \), we have a function \( \Phi'_{\beta} \) such that \( P'(\beta, \Phi'_{\beta}) \) is true. Let’s choose \( \beta \) of power greater than \( \vert P \vert \). If no \( \Phi'_{\beta}(\gamma) \) is maximal in the set of fixed points of \( f \), \( \Phi'_{\beta} \) is an injection, and therefore has as many values as the power of \( \beta \); therefore we get an absurd. And we have to admit that there is an ordinal \( \gamma \) such that \( \Phi'_{\beta}(\gamma) \) is maximal in \( \{ p | f(p) = p \} \). As we wished to prove.

**Proof of Theorem B.** The proof is quite similar to theorem A’s one; then various details will be omitted. \( \alpha_{0} \) is an ordinal number of power greater than \( 2^{\vert \beta \vert} \), \( \Phi_{\alpha}^{\beta} \) a function from \( [0, \alpha) \) into \( P \), \( a \) an element of \( P \) such that \( a < f(a) \), and \( P(\alpha, \Phi_{\alpha}^{\beta}) \) is the following statement:

\[
\Phi_{\alpha}(0) = a. \quad \text{And for every ordinal number } \gamma, \text{ if } \gamma < \alpha, \text{ then if } \gamma + 1 < \alpha, \Phi_{\alpha}^{\beta}(\gamma + 1) = f(\Phi_{\alpha}^{\beta}(\gamma)), \text{ and if } \gamma \text{ is a limit ordinal, then does exists in } P \text{ the least upper bound of } \{ \Phi_{\alpha}^{\beta}(\lambda) | \lambda < \gamma \}, \text{ say it l.u.b. } (\Phi_{\alpha}^{\beta}(\lambda) | \lambda < \gamma) = \Phi_{\alpha}^{\beta}(\gamma).
\]

It’s easily seen that \( P(\alpha, \Phi_{\alpha}^{\beta}) \) satisfies condition A. Let now \( \Phi_{\alpha}^{\beta} \) be a function verifying \( P(\alpha, \Phi_{\alpha}^{\beta}) \). We shall show that there is (only) one function \( \Phi_{\alpha+1} \) such that \( P(\alpha + 1, \Phi_{\alpha+1}) \) is true and is \( \Phi_{\alpha+1}(\delta, \alpha) = \Phi_{\alpha}^{\delta} \). We have to distinguish two cases: \( \alpha = \delta + 1 \) for some ordinal number \( \delta \), or \( \alpha \) is a limit ordinal. If \( \alpha = \delta + 1 \) we define \( \Phi_{\alpha+1} \) by pos-
And it's easily seen that \( \phi^{++1} \) is true. Let's now suppose \( \alpha \) is a limit ordinal. We have to see \( \Phi_\alpha^\beta(\lambda) \leq \alpha \) is a well ordered subset of \( \{ \Phi; \prec \} \). As we did do in theorem A, let's consider the following statement \( S(\gamma) \):

\[
\Phi_\alpha^\beta(\delta) \prec \Phi_\alpha^\beta(\gamma) \quad \text{for every } \delta(\delta < \gamma < \alpha).
\]

Now, given \( \mu < \alpha \), let's suppose that \( S(\gamma) \) is true for every \( \gamma < \mu \). We shall prove that \( S(\mu) \) is true. We must distinguish two cases:

- \( \mu = \lambda + 1 \) or \( \mu \) is a limit ordinal. Let’s now suppose \( \mu = \lambda + 1 \). If \( \lambda = \alpha \), \( \Phi(\mu) \) is proved in quite a similar manner as in theorem A’s proof. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal. As in Theorem A’s proof, we get for every \( \lambda < \nu \), then \( \Phi_\alpha^\beta(\lambda) > . \) Hence \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is trivial, by \( \Phi(\mu) \). So \( S(\mu) \) is true in any case. Hence \( S(\gamma) \) is true for every \( \gamma < \alpha \); this implies that \( \Phi_\alpha^\beta \) is a monotone function from the well ordered set \( \{0, \alpha\} \) into \( \{ \Phi; \prec \} \); then \( \Phi_\alpha^\beta(\gamma) \prec \Phi_\alpha(\nu) \). Hence \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is trivial, by \( \Phi(\mu) \). So \( S(\mu) \) is true in any case. Hence \( S(\gamma) \) is true for every \( \gamma < \alpha \); this implies that \( \Phi_\alpha^\beta \) is a monotone function from the well ordered set \( \{0, \alpha\} \) into \( \{ \Phi; \prec \} \); then \( \Phi_\alpha^\beta(\gamma) \prec \Phi_\alpha^\beta(\nu) \). Hence \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is true. Let’s now consider the second case, that’s when \( \mu \) is a limit ordinal: now \( S(\mu) \) is true.
is not maximal in \( \{ x | f(x) = x \} \), is \( \psi_a(\gamma + 1) = \text{g.l.b.} \{ x | f(x) = x \} \) and \( x > \psi_a(\gamma) \). And if \( \gamma \) is a limit ordinal, then \( \psi_a(\gamma) = P(\text{l.u.b.} \cdot \psi_a(\delta)) \).

It may be seen that \( P(\alpha, \psi_a) \) so defined verifies condition A, and every \( \psi_a \) such that \( P(\alpha, \psi_a) \) is true may be extended in a unique manner to a \( \psi_{a+1} \) such that \( P(\alpha + 1, \psi_{a+1}) \) is true. So we can find, by lemma A (and, therefore, without using Axiom of Choice) a \( \psi_\beta \) for every \( \beta < \alpha_0 \). If \( |\beta| = |\beta| \), is \( \psi_\beta(\gamma) \) maximal in \( \{ x | f(x) = x \} \) for an ordinal \( \gamma \) (\( |\beta| \) is the power of \( \beta \)).

**Proof of Theorem C.** Let \( \alpha_0 \) be a limit ordinal whose power is greater than \( 2^{|P|} \). Let \( \Phi_\alpha \) indicate a function from \([0, \alpha) (0 < \alpha < \alpha_0) \) into \( P \). Let \( P(\alpha, \Phi_\alpha) \) be the following statement:

for every ordinal number \( \gamma \) in \([0, \alpha) \), if \( \gamma + 1 < \alpha \), \( \Phi_\alpha(\gamma + 1) = f(\Phi_\alpha(\gamma)) \), and, if \( \gamma \) is a limit ordinal, then does exist an upper bound of \( \{ \Phi_\alpha(\lambda) | \lambda < \gamma \} \) in \( \langle P ; < \rangle \), and \( \Phi_\alpha(\gamma) \) is such an upper bound.

It’s easily seen that \( P(\alpha, \Phi_\alpha) \) verifies condition A. Moreover, given \( \Phi_\alpha \) verifying \( P(\alpha, \Phi_\alpha) \), there is a \( \Phi_{\alpha+1} \) verifying \( P(\alpha + 1, \Phi_{\alpha+1}) \) and such that \( \Phi_{\alpha+1}(0, \alpha_0) = \Phi_\alpha \). We must distinguish two cases: \( \alpha = \delta + 1 \) for a convenient ordinal \( \delta \), or \( \alpha \) is a limit ordinal. If \( \alpha = \delta + 1 \), we pose \( \Phi_{\alpha+1}(\alpha) = f(\Phi_\alpha(\delta)) \); if \( \alpha \) is a limit ordinal, then we set \( \Phi_{\alpha+1}(\alpha) \) equal to an upper bound of \( \{ \Phi_\alpha(\gamma) | \gamma < \alpha \} \); and such an upper bound does exist because is \( \Phi_\alpha(\gamma) < \Phi_\alpha(\lambda) \) whenever is \( \gamma < \lambda \), and so \( \Phi_\alpha \) realizes an order-homomorphism from the well-ordered set \([0, \alpha) \) into \( \langle P ; < \rangle \), and then \( \{ \Phi_\alpha(\gamma) | \gamma < \alpha \} \) is a well-ordered subset of \( \langle P ; < \rangle \). Then it follows from lemma B that for every \( \beta < \alpha_0 \) there is a function \( \Phi_\beta \) such that \( P(\beta, \Phi_\beta) \) is true. As \( \alpha_0 \) has power greater than \( 2^{|P|} \), we can choose \( \beta \) in \([0, \alpha_0) \) of power greater than to \( 2^{|P|} \). Let’s now suppose (by absurd) \( x < f(x) \) for every \( x \) in \( P \); then we get \( \Phi_\beta(\gamma) < \Phi_\beta(\mu) \) for every choose of \( \gamma, \mu \) such that \( \gamma < \mu \). So \( \Phi_\beta \) is injective, and there fore has more than \( 2^{|P|} \) values. But, as \( \Phi_\beta \)'s values are in \( P \), \( \Phi_\beta \) has at most \( |P| \) values; we get an absurd. This constrains us to admit there is a fixed point of \( f \), say it \( p \).

Now we shall prove that the nonempty set of fixed points of \( f \) has a maximal element. Let’s consider the statement \( P'(\alpha, \Phi'_\alpha) \) as follows:

For every \( \gamma \) less than \( \alpha \), \( \Phi'_\alpha(\gamma) \) is a fixed point for \( f \). And is \( \Phi'_\alpha(0) = p \). And, for every \( \gamma_1, \gamma_2 \) less than \( \alpha \), if \( \gamma_1 < \gamma_2 \). Then, if
\( \Phi'_\alpha(\gamma_1) \) is maximal in \( \{ x | f(x) = x \} \), \( \Phi'_\alpha(\gamma_1) = \Phi'_\alpha(\gamma_2) \); and, if it isn’t such a maximal element, \( \Phi'_\alpha(\gamma_1) < \Phi'_\alpha(\gamma_2) \).

\( P'(\alpha, \Phi'_{\alpha}) \) verifies condition A. Now we shall see that, given a function \( \Phi'_{\alpha} \) such that \( P'(\alpha, \Phi'_{\alpha}) \) is true, we can construct a function \( \Phi'_{\alpha+1} \) which extends \( \Phi'_{\alpha} \) and such that \( P'(\alpha + 1, \Phi'_{\alpha+1}) \) is also true. If \( \alpha \) isn’t a limit ordinal, proof is just like in theorem A. If \( \alpha \) is a limit ordinal, but there’s a \( \gamma \) less than \( \alpha \) and such that \( \Phi'_\alpha(\gamma) \) is maximal in \( \{ x | f(x) = x \} \), proof is just like in theorem A. The only case we must check is when \( \alpha \) is a limit ordinal and \( \alpha \) isn’t maximal in the set of fixed points of \( f \), for every \( \gamma \) less than \( \alpha \). In this case the set \( \{ \Phi'_\alpha(\gamma) | \gamma < \alpha \} \) is a well ordered set. Then there exists an upper bound \( b \) of it. In the ordered set \( Q \) of all elements of \( P \) greater or equal to \( b \), \( f \) has a fixed point \( q \). If we set \( \Phi'_{\alpha+1}(\alpha) = q \), we get a function as we required. Now the existence of a maximal element in \( \{ x | f(x) = x \} \) follows as in theorem A.

As we wished to prove.

REMARK. Let’s assume on \( P \) and \( f \) the hypothesis of proposition B. Then, if we substitute in the preceding proof the statement \( P(\alpha, \Phi'_{\alpha}) \) with the statement:

for every ordinal \( \gamma \) in \([0, \alpha)\), if \( \gamma + 1 < \alpha \), \( \Phi'_\alpha(\gamma + 1) = f(\Phi'_\alpha(\gamma)) \) and, if \( \gamma \) is a limit ordinal, \( \Phi'_\alpha(\gamma) = \text{l.u.b.}(\Phi'_\alpha(\lambda) | \lambda < \nu) \)

we have a proof of proposition A which utilizes only lemma A, and therefore doesn’t need Axiom of Choice.

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