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A common fixed point theorem

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A Common Fixed Point Theorem.

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In this note, we shall prove a fixed point theorem which is a generalization of M. G. Maia's theorem [1].

THEOREM. *Let X be a metric space with two metrics d and δ. If X satisfies the following conditions:*

1) $d(x, y) \leqslant \delta(x, y)$ *for every x, y in X,*

2) *X is complete with respect to d,*

3) *two mappings $f, g\colon X \to X$ are continuous with respect to the metric d, and*

$$\delta(f(x), g(y)) \leqslant \alpha\delta(x, y) + \beta[\delta(x, f(x)) + \delta(y, g(y))] + \gamma[\delta(x, g(y)) + \delta(y, f(x))]$$

for every x, y in X, where α, β, γ are non-negative and $\alpha + 2\beta + 2\gamma < 1$, then f, g have a unique common fixed point.

PROOF. Le x_0 be a point in X, put

$$x_1 = f(x_0),\ x_2 = g(x_1),\ \ldots,\ x_{2n} = g(x_{2n-1}),\ x_{2n+1} = f(x_{2n}),\ \ldots$$

Then

$$\delta(x_n, x_{n+1}) \leqslant \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)^n \delta(x_0, x_1)\,.$$

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(see I. Rus [2], p. 20). Therefore, by $d \leqslant \delta$,

$$d(x_n, x_{n+1}) \leqslant \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n \delta(x_0, x_1) .$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence with respect to d. Since X is complete with respect to d, $\{x_n\}$ has a limit point y_0 in X, i.e. $x_n \xrightarrow{d} y_0$: Hence, by the continuity of f with respect to the metric d,

$$y_0 = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}) = f\left(\lim_{n\to\infty} x_{2n}\right) = f(y_0) .$$

Similary we have $y_0 = g(y_0)$. Therefore y_0 is a common fixed point of f and g.

Let z_0 be a common fixed point of f, g, then

$$\begin{aligned} \delta(y_0, z_0) &= \delta\big(f(y_0), g(z_0)\big) \\ &\leqslant \alpha\delta(y_0, z_0) + 2\gamma\delta(y_0, z_0) , \end{aligned}$$

which implies $y_0 = z_0$. Hence f, g have a unique common fixed point in X, which completes the proof.

REMARK. In Theorem, let $f(x) = g(x)$, $\beta = \gamma = 0$, then we have a theorem by M. G. Maia [1].

REFERENCES

[1] M. G. MAIA, *Un'osservazione sulle contrazioni metriche*, Rend. Sem. Mat. Padova, **40** (1968), pp. 139-143.

[2] I. A. RUS, *Teoria punctului fix*, II. Cluj (1973).

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