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On the Algebraic Compactness of Some Quotients of Product Groups.

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It is well known that, given a countable family $(G_n)_{n\in\mathbb{N}}$ of abelian groups, the group $\prod_n G_n/\bigoplus_n G_n$ is algebraically compact (see [B] and [H]). This result has been generalized by Fuchs ([F], Theorem 42.1).

Fuchs's generalization suggests the following enquiry about the situation which arises when the « countability » assumption is removed from the hypotheses. In this direction O. Gerstner has proved that the direct product of infinitely many copies of the integers modulo the corresponding direct sum is algebraically compact only when the index set is countable ([G], Satz 2).

This result shall be extended here to arbitrary non algebraically compact groups. Related theorems shall also be obtained.

Let us fix some notations. Let X be a set, φ a (perhaps improper) filter on X, A an abelian group; denote by A^X the group of all functions from X to A, with pointwise addition, by $\Sigma_A(\varphi)$ the subgroup of all functions in A^X which vanish on some element of φ (hence $\Sigma_A(\varphi) = A^X$ iff φ is improper), by A_φ the group $A^X/\Sigma_A(\varphi)$. Also, let φ^* be the filter generated by the intersections of countable subsets of φ. Thus φ^* is proper iff φ has the countable intersection property (c.i.p.). that is, every countable subset of φ has non-empty intersection. Moreover, [F]. Theorem 42.1, shows that $\Sigma_A(\varphi^*)/\Sigma_A(\varphi)$ is algebraically compact.

THEOREM 1. *Let ψ be a filter that properly contains φ^*. If $\Sigma_A(\psi)/\Sigma_A(\varphi)$ is algebraically compact, then A is algebraically compact.*

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PROOF. Let

$$\sum_{j\in\mathbf{N}} n_{ij}x_j = a_i \qquad (i\in\mathbf{N}) \tag{1}$$

be a system of linear equations in A, and assume that every finite subsystem of (1) is solvable in A. Pick $V\in\psi\setminus\varphi^*$, and let α_i denote the function which is a_i on $X\setminus V$ and 0 on V, and put $\bar{a}_i = \alpha_i + \Sigma_A(\varphi)$ In this way we obtain

$$\sum_{j\in\mathbf{N}} n_{ij}\bar{x}_j = \bar{a}_i \tag{2}$$

linear system in $\Sigma_A(\psi)/\Sigma_A(\varphi)$. Clearly (2) is finitely solvable, and hence solvable, by the hypothesis. Then there exist $f_j\in\Sigma_A(\psi)$ $(j\in\mathbf{N})$ such that the sets $Z_i = \{x\in X\colon \sum_{j\in\mathbf{N}} n_{ij}f_j(x) = \alpha_i(x)\}$ belong to φ for every $i\in\mathbf{N}$. Since $V\notin\varphi^*$, $Z_0 = \bigcap_i Z_i$ is not contained in V. Take $\xi\in Z_0\setminus V$. Then $(f_j(\xi))_{j\in\mathbf{N}}$ is a solution of (1) in A.

Plainly, all subgroups $\Sigma_A(\varphi)$ are pure in A^X. Then:

COROLLARY 1. *Let φ be a filter on X, with the countable intersection property. Then $A_\varphi = A^X/\Sigma_A(\varphi)$ is algebraically compact iff A is algebraically compact.*

Observe that the direct sum $A^{(X)}$ is the group $\Sigma_A(\varphi_0)$, where φ_0 is the filter of cofinite subsets of X. Then:

COROLLARY 2. *Let X be uncountable. Then $A^X/A^{(X)}$ is algebraically compact iff A is algebraically compact.*

Stronger results may be obtained for certain torsion-free groups. First we need a simple lemma.

LEMMA. *Let φ be a filter on X, closed under countable intersection, p a prime. Then $p^\omega A = 0$ (resp. $A^1 = 0$) implies $p^\omega A_\varphi = 0$ (resp: $A^1_\varphi = 0$).*

PROOF. Assume that $f\notin\Sigma_A(\varphi)$ is of infinite p-height modulo $\Sigma_A(\varphi)$. Then for every $k\in\mathbf{N}$ there exists $f_k\in A^X$ such that $Z_k = \{x\in X\colon f(x) = p^k f_k(x)\}\in\varphi$. Thus $Z_0 = \bigcap_k Z_k\in\varphi$. Since $f\notin\Sigma_A(\varphi)$, there is $\xi\in Z_0$ such that $f(\xi)\neq 0$. Clearly $f(\xi)\in p^\omega A$. The proof for A^1 is almost identical.

REMARK. If φ is not closed under countable intersection, and A is unbounded, then $A^1_\varphi \neq 0$. Thus the hypothesis on φ cannot be weakened.

THEOREM 2. *Let A be a (necessarily torsion-free) group such that $p^\omega A = q^\omega A = 0$ for at least two different primes p, q. Let φ be a filter on X. Then every algebraically compact subgroup of $A^\varphi = A^X/\Sigma_A(\varphi)$ is contained in $\Sigma_A(\varphi^*)/\Sigma_A(\varphi)$ which is then the largest algebraically compact subgroup of A_φ.*

PROOF. What we have to prove amounts to show that $A_{\varphi^*} = A^X/\Sigma_A(\varphi^*)$ contains no non-zero algebraically compact subgroup. And in fact, by the lemma, $p^\omega A_{\varphi^*} = q^\omega A_{\varphi^*} = 0$, which implies that no non-zero r-adic module can be a subgroup of A_{φ^*}, for no prime r.

THEOREM 3. *Let A be a countable reduced torsion free group. Let φ be a filter on X. Then the largest algebraically compact subgroup of A_φ is $\Sigma_A(\varphi^*)/\Sigma_A(\varphi)$.*

PROOF. By the lemma A_{φ^*} is reduced. Since A_{φ^*} is clearly torsion-free (unless it is 0) we have to prove that A_{φ^*} contains no copy of the p-adic integers, for no prime p. Assume the contrary, and let C be a copy of J_p (p-adic integers) contained in A_{φ^*}; C is the completion in the p-adic topology of some infinite cyclic subgroup $\langle \alpha + \Sigma_A(\varphi^*)\rangle$, $\alpha \in A^X$.

Let $a_1, a_2, \ldots$ be an enumeration of the non-zero elements in the range of α. Put $V_i = \alpha^{-1}(a_i)$. Then $Z_0 = \bigcap_i (X \setminus V_i) = \alpha^{-1}(0) \notin \varphi^*$, since $\alpha \notin \Sigma_A(\varphi^*)$. Since φ^* is closed under countable intersection, $X \setminus V_j \notin \varphi^*$ for some j. Put $V = V_j$, $a = a_j$. We prove that the homomorphism $\omega: \mathbf{Z} \to A$ given by $\omega(n) = na$ may be extended to an homomorphism $\bar{\omega}: J \to A$ ($J = z$-adic integers), which is impossible, since A is countable reduced torsion free. Take a sequence (x_n) in $\mathbf{Z}$ such that $x_{n+1} - x_n \in n!\mathbf{Z}$ for every $n \in \mathbf{N}$. Then $(x_n\alpha + \Sigma_A(\varphi^*))$ is a Cauchy sequence in the z-adic topology of C, which coincides with the p-adic topology of C. Hence there exist $f \in A^X$ and $g_1, g_2, \ldots, \in A^X$ such that $f + \Sigma_A(\varphi^*) \in C$, $g_n + \Sigma_A(\varphi^*) \in C$ for all $n \in \mathbf{N}$, and such that the sets

$$W_n = \{t \in X : f(t) - x_n\alpha(t) = n!g_n(t)\}$$

belong to φ^* for every $n \in \mathbf{N}$. Thus $W = \bigcap_n W_n \in \varphi^*$, and hence $W \cap V \neq \emptyset$, since $X \setminus V \notin \varphi^*$. Pick $\xi \in W \cap V$. We have $f(\xi) - x_n a =$

$= n!g_n(\xi)$ for every $n \in \mathrm{N}$. Then $f(\xi)$ is the limit of $(x_n a)$ in the z-adic topology of A. Letting $\bar{\omega}(\lim x_n) = \lim x_n a$, we obtain the required extension.

Finally, let A denote an arbitrary abelian group, φ a filter on X. Since $\Sigma_A(\varphi^*)/\Sigma_A(\varphi)$ is a pure algebraically compact subgroup of A_φ, it is a direct summand of A_φ, and its complements are isomorphic to $A_{\varphi^*} = A^X/\Sigma_A(\varphi^*)$.

In seeking informations on the algebraically compact subgroups of A_{φ^*}, we first observe that φ^* may be assumed to be free, i.e. $\bigcap \varphi^* = \emptyset$. For, otherwise, A_{φ^*} is the direct sum of A^S, with $S = \bigcap \varphi^*$, and of the group $A^{X \setminus S}/\Sigma_A(\psi)$, where $\psi = \{Z \cap (X \setminus S) \colon Z \in \varphi^*\}$; and ψ is closed under countable intersection and free (perhaps improper).

If φ^* is a free filter, closed under countable intersection, then, under the very mild restriction that the cardinality of X be non-measurable, the group A_{φ^*} contains products of infinitely many copies of A, as we shall now prove. Thus, if A has torsion, A_{φ^*} contains direct products of infinitely many finite cyclic groups, and even unbounded such direct products, if the torsion subgroup of A is unbounded.

PROPOSITION. *Let X be a set of non measurable cardinality, φ a free filter on X, closed under countable intersection.*

Then A_φ contains a subgroup isomorphic to the direct product of countably many copies of A.

PROOF. By the hypotheses φ is not an ultrafilter. Then there exists $X_1 \subset X$ such that neither X_1 nor $X \setminus X_1$ belong to φ. Thus φ « traces » on $X \setminus X_1$ a free filter φ_1; and φ_1 is still closed under countable intersection, and free. Again, there exists $X_2 \subset X \setminus X_1$ such that neither X_2, nor $(X \setminus X_1) \setminus X_2$ is in φ_1. By induction, we obtain a countable family X_1, $X_2, \ldots$ of pairwise disjoint subsets of X, such that φ traces on all of them. The subgroup G of A^X consisting of all functions which are constant on every X_i and 0 on $X \setminus \left(\bigcup_i X_i\right)$ is clearly isomorphic to A^{N}. And since $Z \cap X_i \neq \emptyset$ for every $Z \in \varphi$ and $i \in \mathrm{N}$, $G \cap \Sigma_A(\varphi) = 0$.

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