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Consistency property and model existence theorem for second order negative languages with conjunctions and quantifications over sets of cardinality smaller than a strong limit cardinal of denumerable cofinality

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Consistency Property and Model Existence Theorem
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Summary - In a second order negative infinitary language $L_{k,k}^{2-}$, where $k$ is
a strong limit cardinal of cofinality $\omega$, we extend Karp's notions of $\omega$-chain
of models and $\omega$-satisfiability; then we introduce an adequate notion of
consistency property and we prove a model existence theorem to the
effect that any set in a consistency property is $\omega$-satisfiable.

Introduction.

Second order positive languages, $L_{\omega ^1}$, i.e. second order languages
where the second order variables are quantified only universally, look
rather interesting from the point of view of interpolation theorems

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and definability, since these problems are mainly concerned with pre-
dicates which can be considered universally quantified second order
variables. Indeed this approach was taken by Chang in his interpo-
lation theorem in [2], and, more explicitly by Maehara and Takeuti in
their interpolation theorems in [9].

Another promising language for interpolation and definability theo-
rems is $L_{k,k}$, i.e. the infinitary language where we allow conjunc-
tions over sets of less than $k$ formulas and quantifications over sets of less
than $k$ variables, where $k$ is a strong limit cardinal of cofinality $\omega$.
Indeed Karp in [7] was able to extend to this language Craig’s inter-
polation theorem overcoming, in a sense, Malitz’s limit to interpola-
tion theorems for infinitary languages, see [11]. Of course, to do this,
Karp could not use only the usual notion of satisfiability, but she
took advantage of $\omega$-chains of models and $\omega$-satisfiability which were
introduced by Karp herself in [6].

The main tool to prove Karp’s interpolation theorem in [7] is an
adequate notion of consistency property. Consistency property for
first order languages are well explained by Smullyan in [13], and are
one of the main tools in Keisler’s book [8] on the model theory
of $L_{\omega_1,\omega}$.

In this paper we will consider the language $L^{2\omega}_{k,k}$, a combination
of the previous mentioned languages, and we will develop adequate
tools in this language in order to be able to prove new interpolation
theorems in a following paper. Namely we will extend the notions
of $\omega$-chain of structures and $\omega$-satisfiability to $L^{2\omega}_{k,k}$, then we will intro-
duce adequate notions of consistency properties and prove the related
model existence theorems.

CHAPTER I

PRELIMINARIES

I.1. – Set theoretic preliminaries.

The development of this paper will be informal, and we will use
informally the basic notion of a set theory. This will be one with
classes, axiom of regularity and axiom of choice, for instance the set
theory in [12].
We will use \( \in, \subseteq, \subset \) to denote the relations of membership, inclusion and proper inclusion, and \( \notin, \not\subseteq, \not\subset \) to denote their negations.

\[ \{ x : Q(x) \} \] will denote the class of the sets \( x \) such that \( Q(x) \). If \( \{ x : Q(x) \} \subseteq y \) where \( y \) is a set, then also \( \{ x : Q(x) \} \) is a set. Most of the time that we will use the notation \( \{ x : Q(x) \} \) it will be easy to show that there is a set containing \( \{ x : Q(x) \} \); this will be so often and so clear that we will not even mention it.

If \( A \) and \( B \) are classes, \( A - B \) will denote the class \( \{ x : x \in A \text{ and } x \notin B \} \), and \( A^c \) will denote the class \( \{ x : x \notin A \} \).

Functions are classes of ordered pairs \( \langle x, y \rangle \) such that if \( \langle x, y \rangle \) and \( \langle x, z \rangle \) belong to the function then \( y = z \). \( \text{dom} f \) and \( \text{rng} f \) will denote the domain and the range of the function \( f \). \( f \cap A \) will denote the restriction of the function \( f \) to the class \( A \), that is \( f \cap A = \{ \langle x, y \rangle : \langle x, y \rangle \in f \text{ and } x \in A \} \). \( \mathcal{A}B \) will denote the set of all functions whose domain is \( A \) and whose range is in \( B \).

\( \cup A \) will denote \( \{ x : \exists y \in A \text{ and } x \in y \} \). If \( A = \{ A_1, A_2, ..., A_n \} \) we will write \( A_1 \cup A_2 \cup ... \cup A_n \) instead of \( \cup A \).

Ordinals and cardinals will be defined as usual. \( |A| \) will denote the cardinality of the set \( A \).

If \( \alpha \) is a cardinal, \( \alpha^+ \) will denote the cardinal successor of \( \alpha \). A limit ordinal is one which is not equal to \( \alpha + 1 \) for any ordinal \( \alpha \). A limit cardinal is one which is not equal to \( \alpha^+ \) for any cardinal \( \alpha \).

The cofinality of a cardinal \( \alpha \) is the least ordinal \( \delta \) such that there are cardinals \( a_i < \alpha \) for \( i < \delta \) and \( \bigcup \{ a_i : i \in \delta \} = \alpha \).

A limit cardinal \( \alpha \) is called a strong limit cardinal if \( 2^\delta < \alpha \) for \( \delta < \alpha \).

We will reserve the letter \( k \) to denote a strong limit cardinal of cofinality \( \omega \). As usual \( \omega \) denotes the first infinite ordinal.

\( \emptyset \) is the symbol that we reserve for the empty set, i.e. the set \( \{ x : x \in A \text{ and } x \notin A \} \) for any \( A \).

\( \times \) is the symbol that we reserve for the operation of composition of two functions \( f_1 \) and \( f_2 \), i.e. \( f_1 \circ f_2 \) is the function \( \{ \langle x, z \rangle : \text{there is } y \text{ such that } \langle x, y \rangle \in f_1 \text{ and } \langle y, z \rangle \in f_2 \} \).

\[ \times \{ A_i : i \in I \} = \{ \{ (i, a_i) : i \in I \} : a_i \in A_i \} \]

I.2. The language.

The languages \( L^2_{\alpha, \omega} \), \( L^2_{\omega, \alpha} \), \( L^2_{\alpha, \alpha} \) will consist of the following symbols:

(a) \( \alpha \) individual variables: \( v_i \) for \( i \in \alpha \) (0-placed variables),
(b) \( \alpha \) \( p \)-placed predicate variables for each \( p \in \omega \) and \( p \neq 0 \):
\[ V_i^p \text{ for } i \in \alpha, \]
(c) connectives: \(-\) and \&,
(d) quantifier: \( \forall \),
(e) truth symbol: \( t \),
(f) identity symbol: \( = \),
(g) auxiliary symbols: \( , \) and (and).

There is no loss in generality assuming that \( L^2_{\alpha, \omega}, L^2_{\alpha, \omega}, L^2_{\alpha, \omega} \) do not have individual and predicate constants, since these can be regarded as specific variables that we decide not to quantify. Indeed it is considering constants in this way that the notion of satisfaction for a language including constants could be extended to that of \( \omega \)-satisfaction for a language including constants (see section 1.3).

The formulas of the language \( L^2_{\alpha, \omega} \) are defined as follows:

(i) \( V_j^p(v_{i_1}, \ldots, v_{i_p}) \) is an atomic formula for all \( p \in \omega, j \in \alpha \) and \( i_1, \ldots, i_p \in \alpha \). \( t \) is an atomic formula. \( v_{i_1} = v_{i_2} \) is an atomic formula for all \( i_1, i_2 \in \alpha \). Atomic formulas are formulas.

(ii) If \( F \) is a formula, then \(-F\) is a formula.

(iii) If \( F \) is a non empty set of less than \( \alpha \) formulas, then \&\( F \) is a formula.

(iv) If \( \bar{v} \) is a set of less than \( \alpha \) individual variables, and \( F \) is a formula, then \( \forall \bar{v}F \) is also a formula.

(v) If \( \bar{V} \) is a set of less than \( \alpha \) predicate variables, and \( F \) is a formula, then \( \forall \bar{V}F \) is also a formula.

(vi) Nothing else is a formula.

The scope of an occurrence of the connective \(-\) in a formula \( F \) is the formula \( G \) which is the second element of the ordered pair \(-G\) where \(-\) is the given occurrence of \(-\).

The scope of an occurrence of the connective \& in a formula \( F \) is the set of formulas \( G \) which is the second element of the ordered pair \&\( G \) where \& is the given occurrence of \&.

The scope of an occurrence of the quantifier \( \forall \) in a formula \( F \) is the formula \( G \) which is the third element of the ordered triple \( \forall \bar{v}G \) or of the ordered triple \( \forall \bar{V}G \) where \( \forall \) is the given occurrence of \( \forall \).
An occurrence of $\forall$ is called \textit{first order} if $\forall$ is followed by a set of individual variables; while it is called \textit{second order} if it is followed by a set of predicate variables.

A \textit{first order formula} (a formula in the language $L_{\omega, \omega}$) is one in which all occurrences of $\forall$ are first order.

An occurrence of a variable in a formula is \textit{bound (free)} if it is (is not) within the scope of a quantifier followed by a set containing the given variable.

If $F$ is a formula and $\bar{v}$ is a set of variables each one of them in a set of variables following a quantifier in $F$ and $f$ is a $1-1$ function that preserves the type of the variables from $\bar{v}$ onto a set $\bar{v}'$ of variables that do not occur in $F$ then the result of substituting $f(v)$ for each occurrence of the variable $v \in \bar{v}$ in some set of variables following a quantifier or in the scope of the same quantifier is still a formula.

We will call this procedure to go from one formula to another a \textit{change of bound variables}.

Clearly substituting any variables $v'$ for a variable $v$ of the same type in a formula $F$ we obtain another formula $F'$, but it may happen that an occurrence of $v$ is free (bound) in $F$ and the corresponding occurrence of $v'$ in $F'$ is bound (free). If this happen we will speak of \textit{capture of variables}.

When performing a substitution of free variables, to avoid a \textit{capture of variables} is to perform a change of bound variables before the substitution such that the range of the function in the change of bound variables is disjoint from the set of the variables introduced with the substitution.

\textit{Immediate subformula} of a formula $F$ is the formula:

$G$ if $F$ is $\neg G$;

$G$ if $F$ is $\& G$ and $G \in \overline{G}$;

$G(\bar{v}/f)$ if $F$ is $\forall \bar{v} G$ for all functions $f$ from $\bar{v}$ into the variables preserving the type of the variables, where $G(\bar{v}/f)$ stands for the formula obtained from $G$ by substituting the variables $f(v)$ for each variable $v \in \bar{v}$ and avoiding the capture of variables.

A \textit{subformula} of a formula is either the formula itself or an immediate subformula of a formula which was already proved to be a subformula of the given formula.

A \textit{weak subformula} is either a subformula or the negation of a subformula.

The \textit{depth} of an occurrence of a subformula in a formula is the
number of connectives and quantifiers in the scope of which the occurrence of the subformula is.

REMARK. The depth of an occurrence of a subformula in a formula is a finite number.

A formula is negative (positive) if all second order quantifiers in it are within the scope of an odd (even) number of negation symbols.

The formulas of $L_{a,s}^{2-}$ ($L_{a,s}^{2+}$) are the negative (positive) formulas of $L_{a,s}^2$.

A quantifier is universal (existential) if it is in the scope of an even (odd) number of negation symbols.

An occurrence of a subformula in a formula is negative (positive) if it is in the scope of an odd (even) number of negation symbols.

The rank of a formula is defined as follows:

- if $F$ is an atomic formula then its rank is 0;
- if $F$ is $\neg G$ then rank $F =$ rank $G + 1$;
- if $F$ is $\& \bar{G}$ then rank $F =$ Max (rank $G$) + 1: $G \in \bar{G}$;
- if $F$ is $\forall \bar{v} G$ then rank $F =$ (rank $G$) + 1 for any set of variables $\bar{v}$.

A sentence, or closed formula, is a formula without free variables.

It is clear that a variable $v$ may occur free and bound in the same formula $F$. In this case we may assume that there are always enough variables and so we can perform a change of bound variables replacing the bound occurrences of $v$ with a variable that does not occur in $F$. And we may assume that either all occurrences of a variables are free in a formula or all occurrences of a variable are bound in a formula. Furthermore, it may be that a bound variable occurs in the sets of variables following two or more different occurrences of the quantifier. Then again under the assumption that there are always enough variables, we can perform a change of bound variables in such a way that no bound variable occurs in more than one set following a quantifier, and we may assume this least clause.

The same can be said of the sets $s$ of formulas, considering how a variable occurs in $s$.

The symbols $\delta_i^j$ with $i \in \alpha$ and $j \in \omega$ are called metavariables.

If used instead of a $j$-placed variable in a formula, they give rise to metaformulas.

It is clear that substituting variables for all metavariables of the
same number of places in a metaformula, we obtain a formula. Every
time we do the above we do it preserving the number of places.
At this point it is clear what we mean by a submetaformula.

1.3. - ω-chains of structures and ω-satisfaction.

From now on, when we will consider the languages $L_{k,k}^2$, $L_{k,k}^{2+}$, $L_{k,k}^{2-}$, $k$ will always be a strong limit cardinal of cofinality $\omega$.
The notions of structure, type of a structure adequate to a given
language, substructure, and satisfaction are defined as usual.
An ω-chain of structures $\mathcal{M}$ is a sequence $\langle M_n : n \in \omega \rangle$ of sets $M_n$ such that for all $n \in \omega$ $M_n \subseteq M_{n+1}$.
A bounded assignment $a$ in $\mathcal{M}$ to a set of variables is a function that maps each $p$-placed variable in the given set into a set of $p$-tuples of elements of $\bigcup \{ M_n : n \in \omega \}$ for $p \neq 0$, and for some fixed $n \in \omega$ each individual variable into $M_n$.
The ω-chain of structures $\mathcal{M}$ w-satisfies a formula $F$ of $L_{k,k}^2$ ($L_{k,k}^{2+}$, $L_{k,k}^{2-}$) (or $F$ is w-satisfied by $\mathcal{M}$) under the bounded assignment $a$ to the variables free in $F$, $\mathcal{M}, a \models^w F$, if one of the following cases holds:

(i) $F$ is $t$,
(ii) $F$ is $V_i(v_{i_1}, \ldots, v_{i_p})$ and $\langle a(v_{i_1}), \ldots, a(v_{i_p}) \rangle \in a(V_i^p)$ where $i, i_1, \ldots, i_p \in k$,
(iii) $F$ is $-G$ and not $\mathcal{M}, a \models^w G$,
(iv) $F$ is $\& G$ and for all $G \in \overline{G}$, $\mathcal{M}, a \models^w G$,
(v) $F$ is $\forall \bar{v} G$ with $\bar{v}$ a set of individual variables and for all bounded assignments $b$ to $\bar{v}$, $\mathcal{M}, (a - a \bar{\bar{v}}) \cup b \models^w G$,
(vi) $F$ is $\forall \bar{V} G$ with $\bar{V}$ a set of predicate variables and for all bounded assignments $b$ to $\bar{V}$, $\mathcal{M}, (a - a \bar{\bar{V}}) \cup b \models^w G$,
(vii) $F$ is $v_{i_1} = v_{i_2}$ and $a(v_{i_1})$ is $a(v_{i_2})$.

A formula $F$ is ω-satisfiable if there are $\mathcal{M}, a$ (bounded assign-
ment) such that $\mathcal{M}, a \models^w F$.
A set $S$ of formulas is ω-satisfiable if there are $\mathcal{M}, a$ (bounded assignment) such that for all $F \in S$, $\mathcal{M}, a \models^w F$.
A formula is ω-valid, $\models^w F$, iff $\mathcal{M}, a \models^w F$ for all $\mathcal{M}$ and for all bounded assignments to the free variables of $F$. 

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\textbf{Remark I.} If $F$ is $- \& \bar{G}$ and $\mathcal{M}$, $a \models^o F$, then there is $G \in \bar{G}$ such that $\mathcal{M}$, $a \models^o -G$.

\textbf{Remark II.} If $a$ and $b$ are bounded assignments in $\mathcal{M}$ and $S$ is a set of variables, then $(a - a \models S) \cup (b - b \models S)$ is also a bounded assignment.

\textbf{Remark III.} If $F$ is $- \forall \bar{v} \bar{G}$ and $\mathcal{M}$, $a \models^o F$, then there is a bounded assignment $b$ to $\bar{v}$ such that $\mathcal{M}$, $(a - a \models \bar{v}) \cup b \models^o -G$.

\textbf{Remark IV.} Similarly if $F$ is $- \forall \bar{V} \bar{G}$ and $\mathcal{M}$, $a \models^o F$, then there is a bounded assignment $b$ to $\bar{V}$ such that $\mathcal{M}$, $(a - a \models \bar{V}) \cup b \models^o -G$.

\textbf{Remark V (Karp [7])}. A formula in $L_{k,k}$ in which all first order quantifiers are followed by finite sets of variables is $\omega$-satisfied in an $\omega$-chain of structures $\mathcal{M}$ by a bounded assignment in $\mathcal{M}$ if and only if it is satisfied (in the usual sense) in $\cup \{ M_n : n \in \omega \}$.

\textbf{Remark VI.} If the language $L_{k,k}^2$ ($L_{k,k}^{2+}$, $L_{k,k}^{2-}$) has individual and predicate constants, the notion of $\omega$-chain of structures should be changed as follows:

An $\omega$-chain of structures for a language with constants $\mathcal{A}$ is a pair $\langle \mathcal{M}, \alpha \rangle$ where $\mathcal{M}$ is an $\omega$-chain of structures for the language without constants and $\alpha$ is a bounded assignment to the constants, i.e. a function that maps each $p$-placed predicate constant into a set of $p$-tuples of elements of $\cup \{ M_n : n \in \omega \}$ for $p \neq 0$, and for some fixed $n \in \omega$ each individual constant into some $M_n$.

The notion of $\omega$-satisfaction for languages with constants, $\langle \mathcal{M}, \alpha \rangle$, $a \models^o F$, under the bounded assignment $\alpha$ to a set of variables including the free variables of $F$, would then be obtained from the notion of $\omega$-satisfaction for languages without constants, $\mathcal{M}$, $a \models^o F$, by changing case (ii) and (vii) to read:

(ii') $F$ is $X^p(x_1, ..., x_p)$ and $\langle \delta(x_1), ..., \delta(x_p) \rangle \in \delta(X^p)$, where $X^p$ is either a $p$-placed predicate variable or a $p$-placed predicate constant, $x_i$ for $i = 1, ..., p$ is either an individual variable or an individual constant, and $\delta$ is $a \cup \alpha$.

(vii') $F$ is $x_1 = x_2$ and $\delta(x_1)$ is $\delta(x_2)$, where $x_1$, $x_2$, and $\delta$ have the same meaning as in (ii').

Hence we see that a formula with constants is $\omega$-satisfied in an $\omega$-chain of structures for languages with constants iff the same for-
mula is ω-satisfied in an ω-chain of structures for languages without
constants when the constants in the formula are considered as free
variables.

REMARK VII (Karp [7]). If $M_n = M$ for all $n \in \omega$, then, for any
formula $F$, $\mathcal{M}, a \models^\omega F$ iff $M, a \models F$. Thus standard structures may
be considered as particular ω-chains of structures, and ω-satisfiability
a generalization of the notion of satisfiability. Hence $\models^\omega F$ implies
$\models F$, but the converse is not true in general.

REMARK VIII (Karp [7]). The formula

$$\& \{ \forall v_1 - P(v_1, v_1), \forall v_1 \forall v_2 = - P(v_1, v_2), - P(v_2, v_1), v_1 = v_2 \},$$

$$\forall v_1 \forall v_2 \forall v_3 = - \& \{ \& \{ P(v_1, v_2), P(v_2, v_3) \}, - P(v_1, v_3) \},$$

$$\forall v_1 - \forall v_2 = P(v_1, v_2), \forall \{ v_n : n \in \omega \} - \& \{ P(v_n, v_{n+1}) : n \in \omega \} \}
$$

is not satisfiable in any standard model, but it is ω-satisfiable in the
ω-chain of structures $\mathcal{M}$ where $M_n$ is the set $\{0, ..., n\}$, under the
bounded assignment $a = \{(P, the strict ordering of the natural num-
bers)\}.$

REMARK IX. Suppose that the formula $F'$ is obtained from the
formula $F$ through a change of bound variables, then $F$ is ω-satisfiable
iff $F'$ is. Therefore the assumptions that each variable occurring free
in a formula does not occur bound in the same formula and that no
bound variable occur in more than one set following a quantifier,
do not cause any loss of generality from the point of view of ω-satis-
faction, and therefore from now on we will assume that either the for-
mulas satisfy these assumptions or we immediately perform a change
of bound variables that makes the formula satisfy the assumptions
and we keep the same symbol for the formula.

LEMMA. Let $\{ \forall \bar{v}_i F_i : i \in I \} \subseteq s$. If $s$, $\{ F_i(\bar{v}_i/g_i) : g_i \in \bar{v}_i \{ x : x is an
individual variable \}, i \in I \}$ is not ω-satisfiable, then also $s$ is not
ω-satisfiable.

PROOF. Indeed if there are $\mathcal{M}, a$ such that $\mathcal{M}, a \models^\omega s$, $\{ \forall \bar{v}_i F_i : i \in I \}$, then for all bounded assignment $b$ to $\cup \{ \bar{v}_i : i \in I \}$, $\mathcal{M}, a \cup b \models^\omega s$, $F_i$ for all $i \in I$. Hence, in particular, $\mathcal{M}, a \cup b \models^\omega s$, $F'_i$ for all $i \in I$, where $b'_i$ is a bounded assignment to $\cup \{ \bar{v}_i : i \in I \}$ such that if $v' \in \bar{v}_i$ and $v'' \in \bar{v}_i$ and $g_i(v') = g_i(v'')$ then $b'_i(v') = b'_i(v'')$, and if $v \in \bar{v}_i$ for
some $i \in I$ and $g_i(v) \in \text{dom } a$ then $b_x(v) = a(g_i(v))$, while if $g_i(v) \notin \text{dom } a$ then $b_x(v)$ is any fixed element $\alpha$, and $g = \bigcup \{g_i : i \in I\}$.

Let $b'_y$ be $\bigcup \{g_i^{-1} : i \in I\} \cdot b_x$ which is a function due to the definition of $b_x$. Then it is clear that $\mathcal{M}, a \cup b'_y \models \omega s$, $F_i(\overline{v}/g_i)$ for all $i \in I$, and for all $g$ and for all $i \in I \cdot \mathcal{M}_i, a \cup \{(v, \alpha) : v \in \{\text{rng } g - \text{dom } a\}: g = \bigcup \{g_i : i \in I\} \text{ for all } g_i\} \models \omega s$, $F_i(\overline{v}/g_i)$, and hence from the same $\omega$-chain of structures under the same bounded assignment $\models \omega s$, $\{F_i(\overline{v}/g_i) : i \in I, g_i \in \overline{v}_i \{x : x \text{ is an individual variable}\}\}$; a contradiction.

**CHAPTER II**

**CONSISTENCY PROPERTY**

**AND MODEL EXISTENCE THEOREM**

**II.1. - Consistency property for $L^2_{\kappa,k}$.**

Since $k$ is a strong limit cardinal of cofinality $\omega$, we may assume that $k = \bigcup \{k_n : n \in \omega\}$, where $2^{k_n} < k_{n+1}$.

Let us now define the notion of consistency property for $L^2_{\kappa,k}$.

Let $C = \bigcup \{C_n : n \in \omega\}$ be a set such that $|C_n| = k_n$ and for all $m, n \in \omega$ if $m \neq n$ then $C_m \cap C_n = \emptyset$.

Let $C^p = \bigcup \{C^p_n : n \in \omega\}$ be a set such that $|C^p_n| = k_n$ and for all $m, n, p, p' \in \omega$ if $\langle m, p \rangle \neq \langle n, p' \rangle$ then $C^p_m \cap C^p_n = \emptyset$.

Let $L_n$ be the language obtained from $L^2_{\kappa,k}$ by adding $\bigcup \{C_i : i \in n\}$ as individual variables and $\bigcup \{C^p_i : i \in n\}$ as $p$-placed predicate variables, for all $p \in \omega$, $p \neq \emptyset$.

$\Sigma$ is a **consistency property** for $L^2_{\kappa,k}$ with respect to $\{C_n : n \in \omega\}$, $\bigcup \{C^p_n : p \in \omega\}$ if $\Sigma$ is a set of sets $s$ of formulas whose free variables in $L^2_{\kappa,k}$ are all in a set $V^*$ of cardinality $< k$ (we can take $k_0 > |V^*|$) such that all of the following conditions hold:

(C0) For all $s \in \Sigma$, $|s| < k$ and there is an $n$ (depending on $s$) such that all formulas in $s$ are in $L_n$.

(C1) If $z$ is an atomic formula then either $Z$ is either $t$ or $x = x$ and $\neg Z \notin s$, or $Z$ is neither $t$ nor $x = x$ and either $Z \notin s$ or $\neg Z \notin s$. 
(C2) If \{-\text{ }F_i: i \in I\} \subseteq s \text{ and } |I| < k, \text{ then } s \cup \{F_i: i \in I\} \in \Sigma.

(C3) If \{\&\text{ }F_i: i \in I\} \subseteq s \text{ and } |I| < k \text{ and there is } m \in \omega \text{ such that for all } i \in I \text{ we have } 0 < |F_i| < k_m, \text{ then } s \cup (\cup \{\tilde{F}_i: i \in I\}) \in \Sigma.

(C4) If \{-\&\tilde{F}_i: i \in I\} \subseteq s \text{ and } |I| < k \text{ and there is } m \in \omega \text{ such that for all } i \in I \text{ we have } 0 < |\tilde{F}_i| < k_m \text{ then there is a functions } f \in \times \{\tilde{F}_i: i \in I\} \text{ such that } s \cup \{-f(i): i \in I\} \in \Sigma.

(C5) If \{\forall \tilde{v}_i F_i: i \in I\} \subseteq s \text{ and } |I| < k \text{ and for all } i \in I \text{ the variables in } \tilde{v}_i \text{ are individual and there is } m \in \omega \text{ such that for all } i \in I \text{ we have that } |\tilde{v}_i| < k_m, \text{ then for the first natural number } n \text{ such that the formulas in } s \text{ are all in } L_{m-1} \text{ we have that } s \cup \{F_i(\tilde{v}_i/f_i): \text{ for all functions } f_i \text{ from } \tilde{v}_i \text{ into } \cup \{C_j: j \leq n\} \cup V^*, \text{ and for all } i \in I \in \Sigma, \text{ where } F_i(\tilde{v}_i/f_i), \text{ as every where else in this paper, has the meaning already specified in the definition of immediate subformula in section I.2.}

(C6) If \{-\forall \tilde{V}_i F_i: i \in I\} \subseteq s \text{ and } |I| < k \text{ and there is } m \in \omega \text{ such that for all } i \in I \text{ } |\tilde{V}_i| < k_m, \text{ then for the first natural number } n \text{ such that no element of } \cup \{C^n_j: p \in \omega\} \cup C_n \text{ is in } s \text{ and for all } p \in \omega \text{ } |C^n_p| > k_m + |I| \text{ and } |C_n| > k_m + |I|, \text{ for all } 1-1 \text{ place preserving functions } f \text{ from } \cup \{\tilde{V}_i: i \in I\} \text{ into } \cup \{C^n_p: p \in \omega\} \cup C_n \text{ we have that } s \cup \{-F_i(\tilde{V}_i/f \neg \forall \tilde{V}_i): i \in I\} \in \Sigma.

[(C6')] If \{-\forall \tilde{V}_i F_i: i \in I\} \subseteq s \text{ and } |I| < k \text{ and there is } m \in \omega \text{ such that for all } i \in I \text{ } |\tilde{V}_i| < k_m, \text{ then there are a natural number } n \text{ and a place preserving function } f \text{ from } \cup \{\tilde{V}_i: i \in I\} \text{ into } \cup \{C^n_p: p \in \omega\} \cup C_n \text{ such that } s \cup \{-F_i(\tilde{V}_i/f \neg \forall \tilde{V}_i): i \in I\} \in \Sigma.

Clearly (C6) implies (C6').

(C7) (i) If \{v_{i_1} = v_{i_2}: i \in I\} \subseteq s \text{ and } |I| < k, \text{ then } s \cup \{v_{i_1} = v_{i_2}: i \in I\} \in \Sigma.

(ii) If \{Z_i(v_{i_1}), v_{i_1} = v_{i_2}: i \in I\} \subseteq s \text{ and } |I| < k \text{ and for all } i \in I \text{ } Z_i \text{ is an atomic or negated atomic formula, then } s \cup \{Z_i(v_{i_1}), v_{i_1} = v_{i_2}: i \in I\} \in \Sigma \text{ where } Z_i(v_{i_1}) \text{ is the formula obtained from } Z_i(v_{i_1}) \text{ by substituting } v_{i_1} \text{ for one occurrence of } v_{i_1}.

REMARK. We actually defined two different notions of consistency property, say CP and CP', according to whether we include (C6) or (C6'). Clearly if \Sigma \text{ is CP then it is also CP'. So once we have proved that if } s \text{ belongs to a CP' then it is } \omega \text{-satisfiable (model existence theorem) it will follow that the same will be true for the sets belonging to a CP. But there is more to it.}
Theorem. The two notions are equivalent in the sense that if $\bar{s}$ belongs to $\Sigma'$ which is a CP' then there is also $\Sigma$ which is a CP such that $\bar{s}$ belongs to it.

Proof. Let $\Sigma'_0 = \{\bar{s}\}$. Let $\Sigma'_{n+1}$ be the set of all the sets $s^* \in \Sigma'$ such that there is $s^{**} \in \Sigma_n$ and $s^* \supseteq s^{**}$ and $s^*$ is obtained from $s^{**}$ applying one of the steps (C2), (C3), (C4), (C5), (C6'), (C7). Clearly $\cup \{\Sigma'_n: n \in \omega\} = \Sigma'$.

Let us define by induction on $n$ sets $\Sigma_n$ of sets of formulas, functions $g_n$ from $\Sigma_n$ in $\Sigma'_n$ that extend $g_{n-1}$ for $n > 0$, functions $h_{n,s}$ from the set $s \in \Sigma$ in the set $s_*$ and functions $f_{n,s}$ from the free variables occurring in the formulas of $s \in \Sigma$ onto the free variables occurring in the formulas of $\{h_{n,s}(F): F \in s\}$, in such a way that for all $F \in s$ $h_{n,s}(F(\bar{v})) = F(\bar{v}/f_{n,s} \setminus \bar{v})$ where $\bar{v}$ is the set of all free variables in $F$, as follows.

$\Sigma_0 = \{\bar{s}\} = \Sigma'_0$; $g_0 = h_{0,\bar{s}}$, $f_{0,\bar{s}}$ are the identity.

Suppose that $\Sigma_n$, $g_n$, $h_{n,s}$, $f_{n,s}$ were already defined for all $s \in \Sigma_n$.

Then proceed according to the following cases, where $S$ is any subset of $s$ satisfying the conditions stated below at each step.

1) $S = \{\bar{s}: i \in I\}$ and $|I| < k$. Then let $s \cup \{F_i: i \in I\} = s' \in \Sigma_{n+1}$. Let

$h_{n+1,s'} = h_{n,s} \cup \{(F_i, F'_i): \bar{s}' = h_{n,s}(\bar{s}), i \in I\}$.

Let

$f_{n+1,s'} = f_{n,s}$ and $g_{n+1}(s') = g_n(s) \cup \{F'_i: \bar{s}' = h_{n,s}(\bar{s}), i \in I\}$.

2) $S = \{\& \bar{F}_i: i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\bar{F}_i| < k_m$. Then let $s \cup \{\bar{F}_i: i \in I\} = s' \in \Sigma_{n+1}$. Let

$h_{n+1,s'} = h_{n,s} \cup \{G_i: F'_i = F_i(\bar{v}_i/f_{n,s} \setminus \bar{v}_i)\}$ where $F_i \in \bar{F}_i$ and $\bar{v}_i$ is the set of free variables in $F_i$ and $i \in I$. Let $f_{n+1,s'} = f_{n,s}$ and

$g_{n+1}(s') = g_n(s) \cup \{F'_i: F'_i = F_i(\bar{v}_i/f_{n,s} \setminus \bar{v}_i)\}$ where $F_i \in \bar{F}_i$, $\bar{v}_i$ is the set of free variables in $F_i$ and $i \in I$.

3) $S = \{\& \bar{F}_i: i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$ $0 < |\bar{F}_i| < k_m$. Then let $f' \in \times \{G_i: F'_i = F_i(\bar{v}_i/f_{n,s} \setminus \bar{v}_i)\}$, $F_i \in \bar{F}_i$ and $\bar{v}_i$ are the free variables in $F_i$ and $i \in I$ be such that $g_n(s) \cup \{f'(i): i \in I\} \in \Sigma_{n+1}'$. Let $f \in \times \{\bar{F}_i: i \in I\} = f'(i)$ where again $\bar{v}_i$ are the free variables in $f(i)$.
Let $s \cup \{-f(i) : i \in I\} = s' \in \Sigma_{n+1}$.

Let $h_{n+1,s'} = h_{n,s} \cup \{(-f(i), -f'(i)) : i \in I\}$.

Let $f_{n+1,s'} = f_{n,s}$ and let $g_{n+1}(s') = g_n(s) \cup \{-f'(i) : i \in I\}$.

4) $S = \{\forall \bar{v}_i F_i : i \in I\}$ and $|I| < k$ and for $i \in I$ the variables in $\bar{v}_i$ are individual and there is $m \in \omega$ such that for all $i \in I$, $|\bar{v}_i| < k_m$.

Let $p$ be the first natural number such that all the formulas in $s$ are in $L_{p-1}$. Let $p'$ be the first natural number such that all the formulas in $g_n(s)$ are in $L_{p'-1}$. Let $Q_i$ be the set of all functions from $\bar{v}_i$ into $\cup \{C_j : j < p\} \cup V^*$. Let

$$s' = s \cup \{F_i(\bar{v}_i/f_i) : f_i \in Q_i, i \in I\} \in \Sigma_{n+1}.$$

Let $h_{n+1,s'} = h_{n,s} \cup \{F_i(\bar{v}_i/f_i), F_i(\bar{w}_i/(f_{n,s} \cup g_i)) : i \in I, f_i \in Q_i, \bar{w}_i \text{ are the free variables in } F_i, g_i \text{ is the function from } \bar{v}_i \text{ into } \cup \{C_j : j < p'\} \cup V^* \text{ such that for all } v \in \bar{v}_i \text{ if } f_i(v) \text{ is in Domain of } f_{n,s}, \text{ then } g_i(v) = f_i(v), \text{ while } g_i(v) = v_o, \text{ a fixed variable in } V^* \},$$. Let

$$f_{n+1,s'} = f_{n,s} \cup \{(f_i(v), g_i(v)) : v \in \bar{v}_i, f_i \text{ and } g_i \text{ are defined as before, } i \in I\}.$$

Let $g_{n+1}(s') = g_{n}(s) \cup \{h_{n+1,s'}(F_i(\bar{v}_i/f_i)) : i \in I, f_i \in Q_i\}$.

5) $S = \{- \forall \bar{V}_i F_i : i \in I\}$ and $|I| < k$ and there is $m \in \omega$ such that for all $i \in I$, $|\bar{V}_i| < k_m$.

Let $n'$ be the first natural number such that no element of $\cup \{C^P_n : p \in \omega \} \cup C_{n'}$ is in $s$ and for all $p \in \omega$, $|C^P_n| > k_m + |I|$ and $|C_{n'}| > k_m + |I|$. For all $1-1$ place preserving functions from $\cup \{\bar{V}_i : i \in I\}$ into $\cup \{C^P_n : p \in \omega \} \cup C_{n'}$, let $s \cup \{-F_i(\bar{V}_i / g \bar{V}_i) : i \in I\} = s' \in \Sigma_{n+1}$. Let $s^* = g_{n}(s) \in \Sigma_{n}; s^* = h_{n,s}(s)$. Clearly $s^* \subset s^*$ and $s^* = \{- \forall \bar{V}_i F_i(\bar{w}_i/f_{n,s} \bar{w}_i) : \bar{w}_i \text{ is the set of the free variables in } \forall \bar{V}_i F_i, i \in I\}$. Therefore there is a natural number $n''$ and a place preserving function $g$ from $\cup \{\bar{V}_i : i \in I\}$ into $\cup \{C^P_n : p \in \omega \} \cup C_{n''}$ such that

$$s \cup \{-F_i(\bar{V}_i / g \bar{V}_i) : i \in I\} \in \Sigma_{n''+1}.$$
Let

\[ h_{n+1,s'} = h_n,s \cup \left\{ (-F_i(\overline{v}_i/f \setminus \overline{v}_i), -F_i(\overline{w}_i \cup \overline{v}_i/(f_{n,s} \cup g) \setminus (\overline{w}_i \cup \overline{v}_i))): \right. \]

\[ f, g, \overline{w}_i \text{ are defined as above, } i \in I \left\} \right. \]

Let

\[ f_{n+1,s'} = f_{n,s} \cup \{ (f(v), g(v)): v \in \overline{v}_i: i \in I \} \]

note that this definition is correct since \( f \) is a 1-1 function. Let

\[ g_{n+1}(s') = g_n(s) \cup \{ h_{n+1,s'}(-F_i(\overline{v}_i/f \setminus \overline{v}_i)): i \in I \} \]

6) Let \( S = \{ v_i = v_{i'}: i \in I \} \) and \( |I| < k \). Let \( s \cup \{ v_i = v_{i'}: i \in I \} = s' \in \Sigma_{n+1} \). Let

\[ h_{n+1,s'} = h_n,s \cup \{ (v_i = v_{i'}, f_{n,s}(v_i) = f_{n,s}(v_{i'})): i \in I \} \]

Let \( f_{n+1,s'} = f_{n,s} \). Let

\[ g_{n+1}(s') = g_n(s) \cup \{ h_{n+1,s'}(v_i = v_{i'}): i \in I \} \]

7) Let \( S = \{ Z_i(v_{i'}), v_i = v_{i'}: i \in I \} \) and \( |I| < k \) and for all \( i \in I \), \( Z_i(v_i) \) is an atomic or negated atomic formula. Let \( Z_i(v_{i'}) \) be the formula obtained from \( Z_i(v_i) \) by substituting \( v_{i'} \) for one occurrence of \( v_i \). Let \( s \cup \{ Z_i(v_{i'}): i \in I \} = s' \in \Sigma^+_{n+1} \). Let \( h_{n+1,s'} = h_n,s \cup \{ Z_i(v_{i'}), Z_i'(v_{i'})\}: i \in I \) and \( Z_i' \) is \( Z_i(\overline{w}_i/f_{n,s} \setminus \overline{w}_i) \) where \( \overline{w}_i \) are the free variables in \( Z_i \), \( \overline{v}'_{i'} = f_{n,s}(v_{i'}) \) and \( Z_i'(v_{i'}) \) is obtained from \( Z_i' \) substituting \( \overline{v}_{i'} \) for the occurrence of \( f_{n,s}(v_{i'}) \) that corresponds to the occurrence of \( v_{i'} \) in \( Z_i(v_{i'}) \) that is changed in \( v_{i'} \). Let \( f_{n+1,s'} = f_{n,s} \). Let

\[ g_{n+1}(s') = g_n(s) \cup \{ h_{n+1,s'}(Z_i(v_{i'})): i \in I \} \]

Let \( \Sigma_{n+1} \) be the least set that satisfies the previously stated conditions. Hence \( g_n: \Sigma_n \rightarrow \Sigma_n \) is completely defined for all \( n \in \omega \).

Let \( \Sigma = \cup \{ \Sigma_n: n \in \omega \} \).

At this point it is clear that to show that \( \Sigma \) is a CP it is enough to show that \( \Sigma \) satisfies (C1).

Indeed suppose that there is \( s \in \Sigma \), say \( s \), such that either \( -t \in s \)
or $x \neq x \in s$ or $Z \in s$ where $Z$ is an atomic formula. Since $s \in \Sigma$ there is $n \in \omega$ such that $s \in \Sigma_n$. Then $g_n(s)$ belongs to $\Sigma_n' \subseteq \Sigma'$.

If $-t \in s$ then $h_{n,s}(-t) = -t \in g_n(s)$, a contradiction since $\Sigma'$ is a CP'. If $x \neq x \in s$ then $h_{n,s}(x \neq x) = f_{n,s}(x) \neq f_{n,s}(x) \in g_n(s)$, a contradiction since $f_{n,s}$ is a function and $\Sigma'$ is a CP'. If $Z \in s$ and $-Z \in s$ where $Z$ is an atomic formula and $v$ is the set of free variables occurring in $Z$, then $h_{n,s}(Z) = Z(v/f_{n,s} \bar{v}) \in g_n(s)$ and $h_{n,s}(-Z) = -Z(v/f_{n,s} \bar{v}) \in g_n(s)$, a contradiction since again $f_{n,s}$ is a function and $\Sigma'$ is a CP'.

Therefore also $\Sigma$ satisfies (C1) and it is a CP.

II.2. - Model existence theorem.

MODEL EXISTENCE Theorem. If $s$ is a set of formulas of $L_{k,k}^2$, $|s| = k_0 < k$, and $s$ belongs to $S$, a consistency property $\text{CP'}$ with respect to $\{C_n : n \in \omega\}$, $\{\{\text{CP}_n : p \in \omega\} : n \in \omega\}$, then $s$ is $\omega$-satisfied in an $\omega$-chain of structures by a bounded assignment.

Moreover the $n$-th set in the chain has cardinality less or equal to $k_m$.

PROOF. By a good split of a set of at most $k$ formulas we shall mean a sequence $\langle s_m : m \in \omega \rangle$ such that $|s_m| < k_m$, every formula of the type either $\& F$ or $-F$ in $s_m$ has $|F| < k_m$, every formula of the type $-\forall \bar{V}F$ in $s_m$ has $|\bar{V}| < k_m$, every formula of the type $\forall \bar{v}F$ in $s_m$ has $|\bar{v}| < k_m$ and the variables in $\bar{v}$ are all individual.

Let us define, by induction on $n$, sets $s_n \in S$, and good splits $\langle s_n,m : m \in \omega \rangle$ of each $s_n$ as follows.

$$s_0 = s ; \langle s_0,m : m \in \omega \rangle \text{ is any good split of } s_0 .$$

Suppose that $s_h$, $s_{h,m} : m \in \omega$ have been defined for all $k < n$.

Let

$$s'_{n} = s_{n,n} \cup \{ \forall \bar{v}F : \forall \bar{v}F \in \cup \{ s_{n,i} : i < n \} \} \cup \{ c = d : c = d \in \cup \{ s_{n,i} : i < n \} \} .$$

Clearly $s'_{n} \subseteq s_{n}$, $|s'_{n}| < k_n$ and all conjunction sets and quantification sets in $s'_{n}$ have cardinality $< k_n$.

Define

$$s^{(1)}_{n} = s_n \cup \{ F : F \in s'_{n} \} ,$$

$$s^{(2)}_{n} = s^{(1)}_{n} \cup ( \cup \{ F : \& \bar{F} \in s'_{n} \}) ,$$
\[ s^{(2)}_n = s^{(2)}_n \cup \{ -f(\overline{F}) : -\& \overline{F} \in s'_n \} \text{ where } f \in \{ \overline{F} : -\& \overline{F} \in s'_n \} \text{ and } f \text{ is such that if } s^{(2)}_n \in S \text{ so does } s^{(3)}_n \text{ (such an } f \text{ exists by (C6'))}, \]

\[ s^{(4)}_n = s^{(3)}_n \cup \{ F(\overline{v}_p/f_p) : \forall \overline{v}_p \in s'_n \text{ for all } f_p : \overline{v}_p \rightarrow (\bigcup \{ C_i : i < m \} \cup V^*) \}, \]

where the variables in \( \overline{v}_p \) are all individual and \( m \) is the least natural number such that the formulas of \( s^{(3)}_n \) are in \( L_{m-1} \).

\[ s^{(5)}_n = s^{(4)}_n \cup \{ -F(\overline{v}_p/f) : -\forall \overline{v}_p F \in s'_n \} \text{ where } f \text{ is a place preserving function from } \bigcup \{ \overline{v}_p : \forall \overline{v}_p F \in s'_n \} \text{ to } \bigcup \{ C_\alpha^p : p \in \omega \} \cup C_n \text{ and } f \text{ and } m \text{ are such that if } s^{(4)}_n \in S \text{ so does } s^{(5)}_n \text{ (such function } f \text{ and natural number } m \text{ exist by (C6'))}, \]

\[ s^{(6)}_n = s^{(5)}_n \cup \{ d = c : c = d \in s'_n \}, \]

\[ s^{(7)}_n = s^{(6)}_n \cup \{ Z(d) : Z(c) \in s'_n \} \text{ is an atomic or negated atomic formula} \}

Notice that for all natural numbers \( n \) and for all \( i = 1, \ldots, 7 \), \( s^{(i)}_n \in S \) due to the conditions (C2), (C3), (C4), (C5), (C6'), (C7).

Define \( s_{n+1,m} = s^{(7)}_n \).

Define \( \langle s_{n+1,m} : m \in \omega \rangle \) as a good split of \( s_{n+1} \) such that \( s_{n+1,m} = s_{n,m} \) for all \( m < n \) and \( s_{n+1,m} s_{n,m} \) for all \( m > n \).

Let \( s_\omega = \bigcup \{ s_n : n \in \omega \} \). The set \( s_\omega \) can be used to define an \( \omega \)-chain of structures using \( \{ C_i : n \in \omega \} \). The closure conditions \( s^{(6)}_n \) and \( s^{(7)}_n \) can be used to show that the relation \( \sim \) defined as \( c \sim d \) if either \( c = d \in s_\omega \) or \( c \sim d \) is an equivalence relation on \( \bigcup \{ C_i : i \in \omega \} \cup v^* \) where \( v^* \) is the set of the individual variables in \( V^* \), and

\[ \sim = \sim \cap \left( \bigcup \{ C_i : i \in m \} \cup v^* \right) \times \left( \bigcup \{ C_i : i \in m \} \cup v^* \right) \]

is an equivalence relation on \( \bigcup \{ C_i : i \in m \} \cup v^* \) such that

\[ \sim \cap \left( \bigcup \{ C_i : i \in m' \} \cup v^* \right) \times \left( \bigcup \{ C_i : i \in m' \} \cup v^* \right) = \sim \]

for all \( m' < m < \omega \).

Let \( \{ c_i/\sim : c \in \bigcup \{ C_i : i \in \omega \} \cup v^* \} \) be \( M_\omega \). Consider the \( \omega \)-chain of structures \( \langle M_n : n \in \omega \rangle \). Consider the following bounded assignment \( a_\omega \) for all \( c, e \in \bigcup \{ C_i : i \in \omega \} \cup v^* \), \( a_\omega(c) = c/\sim \), and for all predicate variables \( V^p \in \bigcup \{ C^p_i : i \in \omega \} \cup (V^* - v^*) \),

\[ a_\omega(V^p) = \{ (c_1/\sim, \ldots, c_p/\sim) : V^p(c_1, \ldots, c_p) \in s_\omega, e_1, \ldots, e_p \in \bigcup \{ C_i : i \in \omega \} \cup v^* \}. \]
This is well defined since if \( c_1 = d_1, \ldots, c_p = d_p \in s_\omega \) and \( V^p(c_1, \ldots, c_p) \in s_\omega \) and \( c_1, d_1, \ldots, c_p, d_p \in \text{dom} \ a_n \), then also \( V^p(d_1, \ldots, d_p) \in s_\omega \).

Then an induction on the rank of any formula in \( s_\omega \) shows that it is \( \omega \)-satisfied in the \( \omega \)-chain of structures \( \langle M_n : n \in \omega \rangle \) under the assignment \( a_n \) described above, where \( n \) is such that the formula is in \( L_n \), once the following properties of \( s_\omega \) are established:

- not both an atomic formula and its negation occur in \( s_\omega \),
- if \( \neg F \in s_\omega \) then \( F \in s_\omega \),
- if \( \& F \in s_\omega \) and \( F \in F \) then \( F \in s_\omega \),
- if \( \neg \& F \in s_\omega \) then there is \( F \in F \) such that \( \neg F \in s_\omega \),
- if \( \forall \bar{v} F \in s_\omega \) then the variables in \( \bar{v} \) are all individual and for all functions \( f \) from \( \bar{v} \) into \( \cup \{ C_i : i < n \} \cup v^* \) we have that \( F(\bar{v}/f) \in s_\omega \),
- if \( \neg \forall \bar{V} F \in s_\omega \) then there is a place preserving function \( f \) from \( \bar{V} \) into \( \cup \{ \cup \{ C_i^p : i \in \omega \} : p \in \omega \} \cup \{ C_i : i < n \} \} \cup V^* \)

such that \( \neg F(\bar{V}/f) \in s_\omega \),

- if \( c = d \in s_\omega \) then also \( d = c \in s_\omega \),
- if \( Z(c) \), \( c = d \in s_\omega \) where \( Z \) is an atomic or negated atomic formula, then also \( Z(d) \in s_\omega \).

A set for which these properties hold is called a Hintikka set, and therefore \( s_\omega \) is a Hintikka set.

Indeed if \( F \) is an atomic formula or a negated atomic formula in \( s_\omega \) it will be in some \( L_n \) and then \( \mathcal{M}, a_n \models ^\omega F \), due to the construction of \( \mathcal{M} \) and of \( a_n \).

Suppose that the claim has been verified for all formulas in \( s_\omega \) of rank less then the cardinal \( o \). Let \( F \) be a formula in \( s_\omega \) of rank \( o \).

If \( F \) is in \( L_n \) and is \( \neg F' \) then the rank of \( F' \) is \( < o \) and \( F' \) is in \( s_\omega \) and \( F' \) is in \( L_n \), hence \( \mathcal{M}, a_n \models ^\omega F' \), whence \( \mathcal{M}, a_n \models ^\omega F \).

If \( F \) is in \( L_n \) and is \( \& F \) then the rank of each \( F' \in F \) is \( < o \) and each \( F' \in s_\omega \) and is in \( L_n \), hence \( \mathcal{M}, a_n \models ^\omega F' \) for all \( F' \in F \), whence \( \mathcal{M}, a_n \models ^\omega F \).

If \( F \) is in \( L_n \) and is \( \neg \& F \) then there is \( F' \in F \) which is in \( s_\omega \) and in \( L_n \) and rank \( \neg F' < o \), hence \( \mathcal{M}, a_n \models ^\omega \neg F' \), whence \( \mathcal{M}, a_n \models ^\omega G \).
If \( F \) is in \( L_n \) and is \( \forall \bar{v}F' \) then for all \( m \) and for all functions \( f: \bar{v} \to \cup \{ C_i: i < m \} \cup \bar{v}^* \), \( F'(\bar{v}/f) \in s_\omega \) and is in \( L_{\text{Max}(n,m)} \) and the rank of \( F'(\bar{v}/f) \) is the same of the rank of \( F' \) which is smaller than \( \omega \), hence for all \( m \) and \( f \) we have \( \mathcal{M}, a_{\text{Max}(n,m)} \models \omega F'(\bar{v}/f) \), and for all bounded assignments \( b \) to the variables in \( \bar{v} \), \( \mathcal{M}, (a_{\text{Max}(n,m)} - a_{\text{Max}(n,m)} \cap \bar{v}) \cup b \models \omega F' \) whence \( \mathcal{M}, a_{\text{Max}(n,m)} \models \omega F \), but the free variables in \( F \) are in \( L_n \) and therefore \( \mathcal{M}, a_n \models \omega F \).

If \( F \) is in \( L_n \) and is \( \neg \forall \bar{V}F' \) then there is \( m \in \omega \) and there is a place preserving function \( f \) from \( \bar{V} \) into \( \bigcup \{ \cup \{ C_i^p: i \in \omega \}: p \in \omega \} \cup \cup \{ \{ C_i: i < m \} \} \cup \bar{V}^* \) such that \( -F'(\bar{V}/f) \in s_\omega \) and is in \( L_{\text{Max}(n,m)} \) and the rank of \( -F'(\bar{V}/f) \) is the same as the rank of \( -F' \) which is smaller than \( \omega \), hence \( \mathcal{M}, a_{\text{Max}(n,m)} \models \omega -F'(\bar{V}/f) \) and there is a bounded assignment \( b \) to the variables in \( \bar{V} \) such that \( \mathcal{M}, (a_{\text{Max}(n,m)} - a_{\text{Max}(n,m)} \cap \bar{V}) \cup \cup b \models \omega -F' \), whence \( \mathcal{M}, a_{\text{Max}(n,m)} \models \omega F \), but the free variables in \( F \) are in \( L_n \) and therefore \( \mathcal{M}, a_n \models \omega F \).

Since there are no other type of formulas in \( s_\omega \) we can conclude that a formula of \( s_\omega \) which is in \( L_n \) is \( \omega \)-satisfied in the \( \omega \)-chain of structures \( \mathcal{M} \) under the bounded assignment \( a_n \). Hence all the formulas in \( s \in S \) which are all in the same \( L_n \) are \( \omega \)-satisfied in the \( \omega \)-chain of structures \( \mathcal{M} \) under the bounded assignment \( a_n \).

**BIBLIOGRAPHY**


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