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On the Algebraic Compactness of Some Complete Modules.

S. Bazzoni (*)

Introduction.

Let $R$ be a commutative ring with unit.

An $R$-module $M$ is algebraically compact if every finitely soluble family of linear equations over $R$ in $M$ has a simultaneous solution.

If $R$ is a noetherian ring and $\mathcal{Q}$ is the set of the maximal ideals of $R$, we can define over any $R$-module $M$ the $\mathcal{Q}$-adic topology, by taking as a base of neighborhoods of 0 the submodules $IM$, where $I$ is a finite intersection of powers of the maximal ideals.

If $R$ is any ring and $M$ is any $R$-module, we can define on $M$ the $R$-topology, by taking as a base of neighborhoods of 0 the submodules $rM$ with $0 \neq r \in R$.

Warfield [W.] has proved that any algebraically compact $R$-module is complete in the $\mathcal{Q}$-adic topology, if $R$ is a noetherian ring, and in the $R$-topology if $R$ is any ring.

Moreover, Warfield has raised the problem to see if any complete Hausdorff module over a noetherian ring is necessarily algebraically compact.

In this work we answer in the affirmative to the question posed by Warfield and we characterize the neotherian rings $R$ such that

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any $R$-module which is complete and Hausdorff in the $R$-topology is algebraically compact.

1. Complete modules in the $\Omega$-adic topology.

Let $R$ be a noetherian commutative ring with unit, $M$ a topological $R$-module equipped with the $\Omega$-adic topology.

We denote by $\hat{M}$ the $\Omega$-adic completion of $M$.

("Complete module" means "Hausdorff complete module").

The purpose of this section is to prove that, for any $R$-module $M$, $\hat{M}$ is an algebraic compact $R$-module.

First of all, we recall that $M$ is topologically isomorphic to the product $\prod_{m \in \Omega} M_m$ where $\hat{M}_m$ denotes the $m$-adic completion of the localization of $M$ at $m$, so since the class of algebraically compact modules is closed under direct products, we shall have to settle the problem only with respect to the $m$-adic completion of a module.

By a suitable definition of pure submodule, Warfield has proved that the class of algebraically compact modules coincides with the class of pure-injective modules. Therefore we now recall the principal definitions concerning the concept of purity and pure-injectivity.

**Definition 1.** Let $R$ be a ring, $S$ a class of $R$-modules. $N$ is an $S$-pure submodule of an $R$-module $M$, if every element of $S$ is projective for the exact sequence:

$$0 \to N \to M \to M/N \to 0.$$  

An equivalent definition to definition 1 is the following: ([W₂])

**Definition 1'.** Let $S$ be a class of $R$-modules. $N$ is an $S$-pure submodule of an $R$-module $M$ if it is a direct summand of any module $H$ such that: a) $N \leq H \leq M$, b) $H/N \in S$.

Walker ([W₂]) has introduced the notion of $S$-copure submodule by dualizing the definition 1' in the following way:

**Definition 2.** A submodule $N$ of an $R$-module $M$ is $S$-copure in $M$, if for every submodule $H$ of $N$ such that $N/H \in S$, $N/H$ is a summand of $M/H$.  

We are interesting with a particular class of modules, namely we consider the class $F$ of all finitely presented modules, so that we give the following definition:

**Definition 3.** A submodule of a module $M$ is pure (copure) in $M$ if it is $F$-pure ($F$-copure).

Moreover we say that a module is pure-injective if it is injective for any pure exact sequence.

**Remark.** If $R$ is a noetherian ring; the class $F$ is the class of all finitely generated modules.

**Lemma 1.** Let $R$ be an artinian local ring. If $N$ is a pure submodule of $M$, then it is also copure in $M$.

**Proof.** Let $H$ be a submodule of $N$ such that $N/H \in F$. We have to prove that $N/H$ is a summand of $M/H$.

Since $N$ is pure in $M$, $N/H$ is pure in $M/H$ ([W₂], Theor. 2.1); moreover, since the maximal ideal of $R$ is nilpotent, the $\Omega$-adic topology over any $R$-module is the discrete topology, so $N/H$ is a finitely generated and a complete module in the $\Omega$-adic topology.

Then, by Theor. 3 of [W₁], $N/H$ is pure-injective and therefore it is a summand of $M/H$.

**Lemma 2.** Let $R$ be a ring satisfying the hypotheses of the preceding lemma. If $N$ is a copure submodule of a module $M$, then it is a summand of $M$.

**Proof.** Let:

$$F(N) = \{N_x \triangleleft N: N/N_x \in F\}$$

$$N^F = \bigcap_{N_x \in F(N)} N_x$$

For each $0 \neq x \in N$, let $H_x$ be a submodule of $N$ maximal with respect to the property of not containing $x$.

It is easy to verify that the submodule generated by $x + H_x$ is simple and essential in $N/H_x$.

Therefore the injective envelope $E(N/H_x)$ of $N/H_x$ is isomorphic to $E(R/m) = E$. Now, by [M] Theor. 3.4 and 3.11, $E = \bigcup E_k$ where $E_k$ is an increasing sequence of finitely generate submodules of $E$ with $E_k = \{x \in E: m^k x = 0\}$. Therefore, since $m^h = 0$, for a convenient integer $h$, we have $E = E_h$; then $N/H_x$ is finitely generated since it is
a submodule of the noetherian module $E$ and then we have:

$$0 = \bigcap_{\alpha \neq \emptyset} H_{\emptyset} \cap N = N^F.$$  

Now, ([W₂], Corollary 2.9') the group $\text{Coper}(L, N)$ of the copure extensions of $N$ by a generical module $L$, is the image of the homomorphism $f: \text{Ext}(L, N^F) \to \text{Ext}(L, N)$ induced by the inclusion $N^F \to N$. Then, since $N^F = 0$, $N$ is a summand of every module in which it is a copure submodule. //

**Theorem 1.** Let $R$ be a noetherian ring, $m$ an element of $\Omega$ and $M$ an $R$-module. For every $k \in \mathbb{N}$, $M/m^k M$ is an algebraically compact $R$-module.

**Proof.** $M/m^k M$ is an $R/m^k$-module, then by lemmas 1 and 2, it is an algebraically compact $R/m^k$-module. Moreover we can easily deduce from the definition of algebraically compactness, that $M/m^k M$ is also an $R$-module algebraically compact //.

Let $M$ be an $R$-module, we denote by $B(M)$ the Bohr compactification of $M$, that is:

$$B(M) = \text{Hom}_2(\text{Hom}_2(M, K), K)$$

where $K$ is the circle group ([W₁], § 3).

Let $\omega_M$ be the natural homomorphism of $M$ in $B(M)$; then $\omega_M(M) = \hat{M}$ is canonically isomorphic to $M$.

Warfield ([W₁], § 3), has proved that $B(M)$ is a topological compact $R$-module and that $M$ is a pure (and dense) submodule of $B(M)$.

Now we have the following:

**Theorem 2.** Let $R$ be a noetherian ring, $m$ a maximal ideal of $R$. The $m$-adic completion of any $R$-module $M$ is an algebraically compact $R$-module.

**Proof.** Let $M = M/m^k M$ and $\pi^k_h$ the natural homomorphisms $\pi^k_h: M_k \to M_h$ ($k > h, h, k \in \mathbb{N}$); then we have:

$$\hat{M} \cong \lim_{\leftarrow} \{ M_k, \pi^k_h \mid k > h \}_{k \in \mathbb{N}}$$

Let $B_k$ be the Bohr compactification of $M_k$ for every $k \in \mathbb{N}$, $\hat{M}_k$ the copy of $M_k$ in $B_k$ and let $\pi^k_h$ be the homomorphisms induced by the $\pi^k_h$. 

Then $M$ is isomorphic to $\lim \{ M_k, \pi^k_h \}$ since $\omega_k$ are natural isomorphism for every $k \in \mathbb{N}$.

Therefore it will suffices to prove that $\lim \{ M_k, \pi^k_h \}$ is algebraically compact. Let's consider the following diagram:

\[
\begin{array}{cccc}
\tilde{M}_k & \xrightarrow{i_k} & B_k & \\
\downarrow{\pi^k_h} & & \uparrow{f^k_h} & \\
\tilde{M}_h & \xrightarrow{i_h} & B_h & \\
\end{array}
\]

The universal property of $B_k$, assures the existence of a unique continuous homomorphism $f^k_h$ such that the diagram commutes.

Now, by Theor. 1, $\tilde{M}_k$ is an algebraically compact $R$-module, so $B_k = \tilde{M}_k \oplus T_k$ for every $k \in \mathbb{N}$.

Let us consider the following diagram:

\[
\begin{array}{cccc}
0 & \rightarrow \tilde{M}_k & \xrightarrow{i_k} & B_k & \rightarrow T_k & \rightarrow 0 \\
\downarrow{\pi^k_h} & & \downarrow{f^k_h} & & \downarrow{g^k_h} & \\
0 & \rightarrow \tilde{M}_h & \xrightarrow{i_h} & B_h & \rightarrow T_h & \rightarrow 0 \\
\end{array}
\]

with $f^k_h \circ i_k = i_h \circ \pi^k_h$. Since the rows are exact, there is a unique homomorphisms $g^k_h$ such that the diagram commutes.

By the unicity of the $g^k_h$, the system $\{ T_k; g^k_h \ k \succ h \}$ is an inverse system. Then we have:

\[
\lim \{ M_k; \pi^k_h \} \oplus \lim \{ T_k; g^k_h \} \cong \lim \{ B_k; \pi^k_h \oplus g^k_h \}.
\]

Now, $\lim B_k$ is a compact module in the topology induced by the product topology of the $B_k$, then by [W1] Theor. 2, $\lim M_k$ is algebraically compact.
2. Complete modules in the $R$-topology.

Let $R$ be a noetherian commutative ring with unit.

The purpose of this section is to characterize the rings $R$ such that any $R$-module which is complete and $T_1$ in the $R$-topology is algebraically compact.

First of all we consider the case in which the $\Omega$-adic topology on $R$ is the discrete topology.

This hypothesis implies that every $R$-module is discrete in the $\Omega$-adic topology and so, any $R$-module is algebraically compact.

In the general case, that is, when the open ideals in the $\Omega$-adic topology on $R$ are always non zero, then the $R$-topology over any $R$-module $M$ is finer than the $\Omega$-adic topology.

Now, since $R$ is a noetherian ring, any ideal $rR$, with $r \neq 0$, contains a finite intersection of powers of prime non zero ideals of $R$.

Therefore, if $R$ has the following property:

$(P)$ every non zero prime ideal of $R$ is maximal

the $\Omega$-adic topology and the $R$-topology coincides over any $R$-module.

Then the results contained in Section 1, allow us to say that $(P)$ is a sufficient condition on $R$ to insure that any complete and $T_1$ module in the $R$-topology is algebraically compact.

(The converse has been stated by Warfield, as we have just noted).

Let us suppose that $R$ has a non zero and non maximal prime ideal $\mathfrak{p}$.

Let $m$ be a maximal ideal of $R$ containing $\mathfrak{p}$ and let $T$ be the localization of $R/\mathfrak{p}$ at $m/\mathfrak{p}$; we consider the $R$-module $A = T[x]$ where $x$ is a transcendental element over $T$.

Clearly the $R$-topology on $A$ is discrete, so $A$ is complete in such topology, but we shall prove that $A$ is not algebraically compact.

Infact, if $A$ were algebraically compact, Warfield’s results would entail the completeness of $A$ in the $\Omega$-adic topology. But now, it is easy to verify that the $\Omega$-adic topology on $A$ is the same as the $m$-adic topology, so it suffices to find a nonconvergent Cauchy sequence of elements of $A$.

We denote by $n$ the maximal ideal of $T$.

The powers of $n$ give a strictly decreasing chain of ideals of $T$, since, by Krull Theorem, $\cap n^i = 0$ and by the hypotheses on $\mathfrak{p}$ we cannot have $n^i = 0$ for any $i \in \mathbb{N}$. 
Let $a_i$ be an element of $n^i \setminus n^{i+1}$ for every $i \in \mathbb{N}$, and let us consider the following elements of $A$:

$$s_k = \sum_{i=0}^{k} a_ix_i \quad k \in \mathbb{N}.$$ 

Now it is easy to prove that $\{s_k\}_k$ is a Cauchy sequence of element of $A$ which cannot converge to any element of $A$.

**BIBLIOGRAPHY**


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