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0. Introduction.

Let \( l = (E, X, \pi) \) be a differentiable fiber bundle. It is well-known that, both in the finite and infinite-dimensional context, the set \( \tilde{E} = J^1(E) \) of 1-jets of sections of \( l \) admits a differentiable fiber bundle structure \( p: \tilde{E} \to E \). If \( l = (E, G, X, \pi) \) is a principal fiber bundle, then P. Garcia [3] proved that \( \tilde{l} = (\tilde{E}, G, \tilde{E}/G, \tilde{\pi}) \) is a principal fiber bundle with its connections being in 1-1 correspondence with the equivariant sections of \( p \).

In addition, there exists on \( \tilde{E} \) a \( g \)-valued 1-form \( \theta \), the so-called structure 1-form, which is the connection form of a canonical connection on \( \tilde{l} \) and satisfying the following universal property: for every connection \( \sigma \) of \( l \) (regarded as a section of \( p \)) and its corresponding form \( \theta^\sigma \), equality \( \theta^\sigma = \sigma^* \cdot \theta \) holds.

The aim of the present note is twofold:

I) to show that the main results of [3] are valid within the context of Banachable principal bundles. In doing so, we give in section 2 the infinite-dimensional version of the main points of [3], using methods of Banachable vector bundles.

II) to show that the above mentioned result on the universal property of \( \theta \) can be reversed, so that an «iff» condition can be stated. Roughly speaking: each connection \( \sigma \) with corresponding form \( \theta^\sigma \) in-
duces a uniquely determined connection on $\mathfrak{L}_{\sigma(G)}$, with connection form $\theta$ satisfying $\theta^\sigma = \sigma^* \cdot \theta$, for every $\sigma$ and $\theta^\sigma$. $\theta$ turns out to be precisely the structure 1-form. This is described in section 3.

The key to our approach is the notion of $(f, \varphi, h)$-related connections, briefly studied in section 1 (cf. also [6; p. 40]).

Manifolds and bundles are modelled on Banach spaces and, for the sake of simplicity, differentiability is of class $C^\infty$ (smoothness). We mainly follow the terminology and notations of [1], [5], although we try to be as close as possible to [3], which is our main source of motivation.

1. Related connections.

Let $l = (E, G, X, \pi)$ be a principal fiber bundle (p.f.b. for short) and let $g$ be the Lie algebra of $G$.

**DEFINITION 1.1.** A (smooth) connection on $l$ is a smooth splitting of the (direct) exact sequence of vector bundles (v.b. for short) that is, there exists either a $G$-v.b.-morphism $c : \pi^*(TX) \rightarrow TE$ such that $\pi \circ c = \text{id}_{\pi^*(TX)}$, or a $G$-v.b.-morphism $V : TE \rightarrow E \times g$ such that $V \circ v = \text{id}_{E \times g}$. Here $T \pi!$ is the v.b.-morphism defined by the universal property of pull-backs and $v$ is given by $v(p, A) = T_{[p, 1]} \delta(A)$, if $\delta$ denotes the (right) action of $G$ on $P$ and $g \cong T_1 G$ ($1$ is the identity element of $G$). $V$ and $c$ are $G$-v.b.-morphisms in the sense they preserve the natural action of $G$ on the corresponding bundles (cf. [7; p. 67]).

As well-known, $TE = VE \oplus HE$, where $VE = \text{Im}(v)$ and $HE = \text{Im}(c)$ are respectively the vertical and horizontal subbundles of $TE$. Hence, each vector $u \in T_E$ has the unique expression $u = u^v + u^h$, with the exponents $v$ and $h$ denoting vertical and horizontal parts respectively.

**DEFINITION 1.2.** Let $l = (E, G, X, \pi)$ and $\bar{l} = (\bar{E}, \bar{G}, \bar{X}, \bar{\pi})$ be two p.f.b’s and let $(f, \varphi, h)$ be a p.f.b-morphism of $l$ into $\bar{l}$. Two connections $c$ and $\bar{c}$ on $l$ and $\bar{l}$ respectively, are said to be $(f, \varphi, h)$-related iff $Tf \circ c = \bar{c} \circ (f \times Th)$.
If $\tilde{\varphi}$ is the Lie algebra homomorphism induced by $\varphi_\ast$, and $\omega$, $\tilde{\omega}$ the connection forms of $c$ and $\tilde{c}$ respectively, then we can prove (cf. [9], [10; p. 76]):

**Theorem 1.3.** The following conditions are equivalent:

1) $c$ and $\tilde{c}$ are $(f, \varphi, h)$-related

2) \[ \bar{V} \circ Tf = (f \times \tilde{\varphi}) \circ V \]

3) \[ Tf(u^h) = (Tf(u))^\tilde{h} \]

4) \[ Tf(u^\ell) = (Tf(u))^\tilde{\ell} \]

5) \[ \tilde{\varphi} \cdot \omega = f^* \cdot \tilde{\omega} \]

The main result of this section is to show that each connection $c$ on $\mathfrak{g}$ and each p.f.b.-morphism of $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$ induce a properly related connection $\tilde{c}$ on $\tilde{\mathfrak{g}}$. We need the following lemma proved in [7; p. 66].

**Lemma 1.4.** Let $B, E, F$ be Banach spaces and let $U$ be an open subset of $B$. If $f: U \rightarrow L(E, F)$ is a map such that, for each $e \in E$, each map $x \mapsto f(x) e$ is of class $C^0$, then $f$ is of class $C^0$.

**Proposition 1.5.** Let $\pi: E \rightarrow X$ and $\bar{\pi}: \bar{E} \rightarrow X$ be two v.b's over the same base $X$ and $(g, \text{id}_X)$ a v.b.-morphism of $\bar{\pi}$ into $\pi$. If in addition, for each $x \in X$, there exists $f_x \in L(E_x, \bar{E}_x)$ such that $g_x \circ f_x = \text{id}_{E_x}$, then the couple $(f, \text{id}_X)$, with $f: E \rightarrow \bar{E}$ given by $f|E_x := f_x$, is a v.b.-morphism of $\pi$ into $\bar{\pi}$.

**Proof.** In virtue of [5; p. 37] it is sufficient to prove $VBM$-2. Since $(g, \text{id}_X)$ is a v.b.-morphism, for every $x_0$ there is a neighborhood $U$ of $x_0$ such that

$$U \ni x \mapsto r_x \circ g_x \circ \bar{r}_x^{-1} \in L(\bar{E}, E)$$

is a smooth map (the letter $r$ is used to denote local trivializations). By the assumption, the restriction of $g_x$ on $\text{Im}(f_x)$ is a toplinear isomorphism $g_x: f_x(E_x) \rightarrow E_x$ with $f_x$ as its inverse. Hence, for each $u \in E$ there exists $\bar{u} \in \bar{E}$ with $\bar{r}_x \circ g_x \circ \bar{r}_x^{-1}(\bar{u}) = u$; thus the map

$$U \ni x \mapsto \bar{r}_x \circ f_x \circ \bar{r}_x^{-1}(u) \in \bar{E}$$

is smooth, as having constant value $\bar{u}$. The late together with the pre-
vious Lemma imply the smoothness of
\[ U \ni x \mapsto \tilde{r}_x \circ f_x \circ r_x^{-1} \in L(E, E) \]
which completes the proof. \( \blacksquare \)

**Theorem 1.6.** Let \( l = (E, G, X, \pi) \) and \( \tilde{l} = (\tilde{E}, \tilde{G}, \tilde{X}, \tilde{\pi}) \) be two p.f.b.'s and let \((f, \varphi, h)\) be a p.f.b.-morphism of \( l \) into \( \tilde{l} \) with \( h \) being local diffeomorphism. Then every connection \( c \) on \( l \) determines a unique \((f, \varphi, h)\)-related connection \( \tilde{c} \) on \( \tilde{l} \).

**Proof.** Let \( \bar{e} \) be an arbitrary element of \( \tilde{E} \) with \( \pi(\bar{e}) = \bar{x} \). We can always find \( e \in E \) and \( \bar{s} \in \tilde{G} \) such that \( f(e) = \bar{e} \cdot \bar{s} \). If \( \pi(e) = x \), then \( h(x) = \bar{x} \). Using the fact that \( T_x h : T_x X \rightarrow T_{\bar{x}} \tilde{X} \) is a topolinear isomorphism, we define the continuous linear map \( \tilde{c}(\bar{e}, \cdot) : T_{\bar{x}} \tilde{X} \rightarrow T_{\bar{x}} \tilde{E} \) given by
\[
\tilde{c}(\bar{e}, \cdot)(\tilde{v}) := (T_{\bar{x}} R_{\bar{s}} \circ T_{\bar{e}} f \circ h) \cdot (e, T_{\bar{x}} \tilde{h}(\tilde{v}))
\]
where \( R_{\bar{s}} \) denotes the right translation (by \( \bar{s} \)) on \( \tilde{E} \) and \( \tilde{h} \) is the inverse of \( h \). The above map is independent of the choice of \( e \) and \( \bar{s} \), for a given \( \bar{e} \). We define the global map \( \tilde{c} : \pi^*(T\tilde{X}) \rightarrow T\tilde{E} \) by \( \tilde{c}(\bar{e}, \tilde{v}) = \tilde{c}(\bar{e}, \cdot)(\tilde{v}) \). Since \( T\pi \circ \tilde{c} = \text{id}_{\pi^*(T\tilde{X})} \), in virtue of Prop. 1.5, we conclude that \( \tilde{c} \) is the right splitting morphism of a connection on \( \tilde{l} \).

Setting \( \bar{e} = f(e) \) (which implies that \( \bar{s} \) is the identity element of \( \tilde{G} \)) we see that (1.1) yields
\[
[\tilde{c} \circ (f \times Th)] \cdot (e, v) = \tilde{c}(f(e), T_x h(v)) = (Tf \circ h) \cdot (e, v)
\]
which proves the relatedness of \( \tilde{c} \) and \( c \).

Finally, \( \tilde{c} \) is unique, for if \( c' \) is another \((f, \varphi, h)\)-related with \( c \) connection, then Def. 1.2 implies that
\[
(1.2) \quad c' \circ (f \times Th) = \tilde{c} \circ (f \times Th)
\]
Now, for an arbitrary \((\bar{e}, \tilde{v})\) in \( \pi^*(T\tilde{X}) \), we have \( \tilde{v} \in T_{\bar{x}} \tilde{X} \), where \( \bar{x} = \pi(\bar{e}) \). As before, there are \( e \in E \) and \( \bar{s} \in \tilde{G} \) with \( e = f(e) \cdot \bar{s} \). If \( x = \pi(e) \), then \( h(x) = \bar{x} \) and the local diffeomorphism determines \( v \in T_x X \) with \( \tilde{v} = T_x h(v) \). Hence, taking into account the invariance
of $\sigma'$ and $\bar{\sigma}$, (1.2) implies

$$o'(\bar{\sigma}, \bar{v}) = o'(f(e) \cdot \bar{\sigma}, T_xh(v)) = o'(f(e), T_xh(v)) \cdot \bar{\sigma} =$$

$$= \bar{\sigma}(f(e), T_xh(v)) \cdot \bar{\sigma} = \bar{\sigma}(f(e) \cdot \bar{\sigma}, T_xh(v)) = \bar{\sigma}(\bar{e}, \bar{v})$$

and the proof is complete.  ●

As in the finite case, a connection on $l$ can be determined by an appropriate connection form $\omega$ on $P$, satisfying the well-known properties. Here we set $\omega(u) = pr_2 o V \cdot (u)$, with $e \in E$ and $u \in T_xE$. In virtue of Def. 1.1, we easily obtain the following formula:

$$u \mapsto o(c_x(u), T_x\pi(u)) = T_{(u,1)}^1 \delta \cdot (\omega(u))$$

for every $e \in E$ and $u \in T_xE$, and with $c_x$ denoting the projection of the tangent bundle $TE \to E$. The previous formulas (1.1) and (1.3), applied for the connection form $\bar{\omega}$ of the connection $\bar{\sigma}$ of Theorem 1.6, give the following useful formula:

$$\bar{\omega}_e(\bar{u}) = (T_{(u,1)}^1 \delta)^{-1} \cdot \left( \bar{u} \mapsto (T_xR_x o T_xf o o) \cdot (e, T\pi o T\pi(\bar{u})) \right)$$

where $\bar{e} = f(e) \cdot \bar{\sigma}$.

2. The 1-jet principal fiber bundle.

In this section we briefly present P. Garcia's [3] results needed for the purpose of the note, and modified according to our infinite-dimensional point of view.

Let $l = (E, G, X, \pi)$ be a Banachable p.f.b. If $s$ is a section of $l$, we denote by $\bar{s}_x$ the 1-jet of $s$ at $x$. The fiber bundle structure of the 1-jet bundle of sections of $l$, denoted by $\bar{E}$, is standard (cf. [3], [2], [8]). The fact we are dealing with principal bundles implies the following:

**Theorem 2.1.** $G$ acts on the right of $\bar{E}$ freely and differentiably by

$$\bar{E} \times G \ni (\bar{s}_x, a) \mapsto \bar{s}_x \cdot a := (s \cdot a)_x \in \bar{E}$$

and the quadruple $\bar{l} = (\bar{E}, G, \bar{E}/G, \bar{\pi})$ is a p.f.b.

**Proof.** The finite-dimensional proof of [3; p. 232-234] is easily extended to our context.  ●
If \( p : \tilde{E} \to E \) is the canonical p.f.b-morphism given by \( p(\tilde{s}_x) = s(x) \), then we have:

**Theorem 2.2.** For each p.f.b. \( l = (E, G, X, \pi) \), there exists a bijective correspondence of the set of sections of \( p : \tilde{E} \to E \) onto the set of connections of \( l \).

**Proof.** The proof of [3; p. 236] has the following infinite-dimensional version, involving v.b.-morphisms.

Let \( \sigma \) be an arbitrary section of \( p \). If \( s \) is a section such that \( \sigma(e) = \tilde{s}_e \) \((\pi(e) = x)\), we set \( c(e, \cdot) = T_x s \) and we define \( c : \pi^*(TX) \to TE \) by \( c(e, v) = c(e, \cdot) \cdot (v) \). Since \( T\pi! c = \text{id}_{\pi^*(TX)} \) we conclude that, in virtue of Def. 1.1 and Prop. 1.5, \( c \) is a connection.

Conversely, for a given connection \( c \) we define the section \( \sigma : E \to \tilde{E} \) of \( p \) as follows: for each \( e \in E \) there exists a section with prescribed differential equal to \( c(e, \cdot) \) i.e. we can find a section \( s^e \) passing through \( e \) such that \( T_x s^e = c(e, \cdot) \). We set \( \sigma(e) := \tilde{s}_e \). The smoothness of \( \sigma \) derives from the local structure of \( E \).

Finally, the desired bijectivity is checked as in [3].

**Definition 2.3.** The 1-form \( \theta \) defined by

\[
\theta : \tilde{E} \ni \tilde{e} \mapsto \theta_\tilde{e} : T_x \tilde{E} \ni \tilde{u} \mapsto \theta_\tilde{e}(\tilde{u}) := Tp(\tilde{u}) - T(s \circ \pi) \cdot (T\pi(\tilde{u})) \in V_x E
\]

is called the structure 1-form of \( \tilde{E} \).

**Notes.** I) In the above definition we have set \( \tilde{e} = \tilde{s}_e \) and, for the sake of simplicity, we have omitted the subscript of differentials.

II) Since each vertical space \( V_x E \) is identified with the Lie algebra \( g \) (cf. Def. 1.1), \( \theta \) can be considered as a \( g \)-valued form on \( E \).

The following result is also in [3; p. 238]. We sketch here a simple proof, involving v.b.-morphisms and the conventions of our framework.

**Theorem 2.4.** \( \theta \) is a differentiable 1-form which defines a connection on \( l \) and satisfies the universal property for all the connections \( \sigma \)'s of \( l \) and their connection forms \( \theta_\sigma \).

**Proof.** As in the proof of Theorem 2.2, we construct the v.b.-morphism \( c : \pi^*(TX) \to TE \) by \( c(e, v) = T\pi s(v) \cdot (T\pi(\tilde{u})) \) defines a connection on \( l \) corresponding to some \( \sigma \). Hence, the map

\[
\omega : E \ni e \mapsto \omega_e : T_x E \ni \omega_e(u) := u - c(e, T\pi(\tilde{u}))
\]
is a differential 1-form on $E$ and with values in $VE$, since it is induced by the v.b.-morphism

$$\text{id}_{TE} - co(\text{id}_{g} \times T\pi): TE \to VE$$

(cf. [1; p. 81]). Moreover, identifying $VE$ with $E \times g$, we see that $\omega$ is the connection form of $c$ (equivalently of $\sigma$), thus we can write $\omega = \theta^\sigma$. Definition 2.3 implies that $\theta = p^* \cdot \theta^\sigma$ on $\sigma(E)$; hence, $\theta$ is a $g$-valued connection form on $\tilde{l}$, by local arguments.

Also, for the given $\sigma$ (or $c$) we have that

$$\sigma^* \cdot \theta = \sigma^* \cdot (p^* \cdot \theta^\sigma) = (p \circ \sigma)^* \cdot \theta^\sigma = \theta^\sigma.$$

The above procedure is valid for each connection $\sigma$, so the proof is complete. 

Since $E$ is a p.f.b., each section $\sigma: E \to \tilde{E}$ is a p.f.b.-morphism and defines a differentiable map $\sigma^\circ: X \to \tilde{E}/G$ so that the following diagram is commutative

$$\begin{array}{ccc}
E & \xrightarrow{\sigma} & E \\
\downarrow \pi & & \downarrow \tilde{\pi} \\
X & \xrightarrow{\sigma^\circ} & \tilde{E}/G
\end{array}$$

$\sigma^\circ$ turns out to be a section of $p^\circ: \tilde{E}/G \to X$ (cf. [3; p. 234]).

Under the above notations and the terminology of section 1, we have:

**Theorem 2.4 (Restated).** The structure 1-form $\theta$ induces on $\tilde{l}$ a canonical connection which is $(\sigma, \text{id}_g, \sigma^\circ)$-related with every connection $\sigma$ of $l$.

### 3. The main result.

**Theorem 3.1.** There exists on each $\tilde{l}_{\sigma,i}$ a unique connection, which is $(\sigma, \text{id}_g, \sigma^\circ)$-related with each connection $\sigma$ on $l$. The corresponding connection form is precisely the structure 1-form $\theta$ and satisfies the universal equality $\theta^\sigma = \sigma^* \cdot \theta$, for every connection $\sigma$ on $l$.

**Proof.** Let $\sigma$ be an arbitrary connection on $l$, which corresponds to the section $\sigma$ and has connection form $\theta^\sigma$. As a consequence of
Theorem 1.6, there exists a unique connection, say $\tilde{c}$, on $\tilde{\mathcal{L}}_{(s, a)}$ which is $(\sigma, \text{id}_s, \sigma_a)$-related with $c$. Let us denote by $\theta$ the connection form of $\tilde{c}$. Under the present data, (1.4) takes the form

$$
(3.1) \quad \theta_{\tilde{u}}(\tilde{u}) = (T_{(s, a)} \delta)^{-1} \cdot \left(\tilde{u} - (T_{(s, a)}(\sigma) \circ c) \cdot (s(x), T\sigma_a^{-1} \circ T\tilde{\sigma}(\tilde{u}))\right).
$$

Indeed, this is the case, for if $\tilde{s}_a$ is an arbitrary element of $\sigma(E)$, there exists $e \in E$ with $\tilde{s}_a = \sigma(e)$. Hence, $e = s(x)$ and the element $s \in \tilde{G}$ of Theorem 1.6 is now the identity 1 of $G$. Setting

$$
T_{(s, a)} \delta = \tilde{v}_{s_a}
$$

(cf. Def. 1.1) and taking into account the commutativity of the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\pi} & E \\
\downarrow & & \downarrow \pi \\
\tilde{E}/G & \xrightarrow{\pi} & X
\end{array}
$$

we write (3.1) as

$$
(3.2) \quad \theta_{\tilde{u}}(\tilde{u}) = \tilde{v}_{s_a}^{-1} \cdot \left(\tilde{u} - (T_{(s, a)}(\sigma) \circ c) \cdot (s(x), T\pi \circ p \cdot (\tilde{u}))\right).
$$

On the other hand, we easily check that the p.f.b.-morphism $p$ satisfies $T\pi \circ \tilde{v} = \tilde{v} \circ (p \times \text{id}_p)$; thus, since $\tilde{v}$ and $v$ are toplinear isomorphisms on the fibers, we have that

$$
\tilde{v}_{s_a}^{-1} = v_{s(a)}^{-1} \circ T_{s_a} p
$$

and (3.2) yields:

$$
\theta_{\tilde{u}}(\tilde{u}) = v_{s(a)}^{-1} \left(T_{s_a} p(\tilde{u}) - (T_{s_a} p \circ T_{s(a)}(\sigma) \circ c) \cdot (s(x), T\pi \circ p \cdot (\tilde{u}))\right)
$$
or

$$
\theta_{\tilde{u}}(\tilde{u}) = v_{s(a)}^{-1} \cdot \left(T_{s_a} p(\tilde{u}) - c(s(x), T_{s(a)} \pi (T_{s_a} p(\tilde{u}))\right).$$

or, in virtue of Theorem 1.6

$$
\theta_{\tilde{u}}(\tilde{u}) = v_{s(a)}^{-1} \cdot \left(T_{s_a} p(\tilde{u}) - T s_a (T_{s_a} p(\tilde{u}))\right).
$$

Since $v_{s(a)}$ is exactly the identification $g \cong V_{s(a)}E$, we see that $\theta$ is precisely the structural 1-form.
By the \((\sigma, \text{id}_\sigma, \sigma_\sigma)\)-relatedness we have that \(\theta^\sigma = \sigma^* \cdot \theta\) if \(\theta^\sigma\) is the connection form of \(\sigma\) (or \(c\)). The same argument assures the uniqueness of \(\theta\). Finally, since the above construction of \(\theta\) is valid for all \(\sigma\)'s, we see that \(\sigma^\sigma = \sigma^* \cdot \theta\) is universally satisfied.

The above Theorem, compared with Theorem 2.4 (restated), can be considered as its converse; hence, for the sake of completeness we state the following:

**Theorem 3.2.** Let \(\theta\) be a \(g\)-valued differentiable 1-form on \(\tilde{E}\). Then the following conditions are equivalent:

I) \(\theta|_{\sigma(\tilde{E})}\) is the structural 1-form on \(\sigma(E)\).

II) For each connection \(\sigma\) of \(\tilde{l}\) (regarded as a section of \(p: \tilde{E} \rightarrow E\)), the corresponding connection form \(\theta^\sigma\) satisfies \(\theta^\sigma = \sigma^* \cdot \theta\).

III) \(\theta\) is the connection form of a uniquely determined connection on \(\tilde{l}\), which is \((\sigma, \text{id}_\sigma, \sigma_\sigma)\)-related with every connection \(\sigma\) of \(l\).

**References**


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