A Common Fixed Point Theorem.

B. E. RHOADES (*)

In this note we establish the following fixed point theorem, which is a generalization of that of Iseki [1].

**Theorem.** Let $X$ be a metric space with two metrics $d$ and $\delta$ satisfying the following conditions:

1) $d(x, y) \leq \delta(x, y)$ for each $x, y \in X$,
2) $X$ is complete with respect to $d$,
3) $f, g: X \rightarrow X$, each continuous with respect to $d$ and satisfying the following contractive condition: there exists a real number $h$ with $0 < h < 1$ such that, for each $x, y \in X$,

$$
\delta(f(x), g(y)) \leq h \max \left\{ \delta(x, y), \delta(x, f(x)), \delta(y, g(y)), \left[ \delta(x, g(y)) + \delta(y, f(x)) \right]/2 \right\}.
$$

Then $f$ and $g$ have a unique common fixed point.

**Proof.** Let $x_0 \in X$ and define the sequence $\{x_n\}$ by

$$
x_1 = f(x_0), \quad x_2 = g(x_1), \ldots, \quad x_{2n} = g(x_{2n-1}), \quad x_{2n+1} = f(x_{2n}), \ldots.
$$

As in the proof of Theorem 14 of [2], one can show that, for $m > n$,

$$
\delta(x_m, x_n) \leq h^n \cdot r(x_0)(1 - h)^{-1}, \text{ where } r(x_0) = \max\{\delta(x_0, x_1), \delta(x_1, x_2)\}.
$$

(*) Indirizzo dell'A.: Dept. of Mathematics, Indiana University, Swain-Hall East, Bloomington, Indiana, U.S.A.
Therefore $d(x_h, x_n) \to 0$ as $n \to +\infty$ so that \( \{x_n\} \) is Cauchy, hence convergent. Call the limit $z$. Since $f$ and $g$ are continuous with respect to the metric $d$ it then follows that $z$ is a common fixed point.

Suppose $w$ is also a common fixed point. Using (3), $\delta(z, w) \leq h\delta(z, w)$, which implies $z = w$.

REFERENCES


Manoscritto pervenuto in Redazione il 19 Gennaio 1977.