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GIULIANO BRATTI (*).

Introduction.

Let A be an open subset of R^n; suppose $P = P(x, D)$, $Q = Q(x, D)$ linear partial differential operators with $C^\infty(A)$ coefficients.

DEFINITION 1). *We say that the system*

$$(+) \qquad \{ Pu = f, \quad Qu = o \}, \quad f \in C^\infty(A)$$

is $C^\infty(A)$-locally solvable in A if for every $p \in A$ there is a neighbourhood, V_p, of p and a function $u_p \in C^\infty(V_p)$ such that the $(+)$ is satisfied in V_p.

DEFINITION 2). *If B is an open subset of A, we say that the above system $(+)$ is $C^\infty(B)$-globally solvable if for every $f \in C^\infty(A)$ for which $(+)$ is locally solvable in A, there is a function $u \in C^\infty(B)$ such that $(+)$ is satisfied in B.*

In (2) there is the following conjecture:

let $(B_n)_{n \in N}$ be a sequence of open subsets of A such that: $B_n \subset B_{n+1} \subset \sqcup B_n = B$ and the $(+)$ is $C^\infty(B_n)$-globally solvable for every $n \in N$. Then $(+)$ is $C^\infty(B)$-globally solvable.

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It is already known, (4), *the conjecture is false* in the case in which P and Q have constat coefficients and Q is semi-elliptic, but the conjecture is still open when Q is an elliptic operator.

It seems to the A. that to solve the above conjecture it is important to have some example of system like $(+)$ without $C^\infty(A)$-globally solutions for $f \in C^\infty(A)$ for which $(+)$ is $C^\infty(A)$-locally solvable.

Fisrt of all, *by a Lojasiewicz-Malgrange's theorem*, see (1), it is easy to show that: if P and Q are prime between them, the subspace of $C^\infty(A)$ of the functions for which the system $(+)$ is $C^\infty(A)$-locally solvable is $\ker Q_{/A} = \{ f \in C^\infty(A) : Qf = o \}$.

The object of this paper is that to characterize the open subset A of R^n for which there are systems like $(+)$, with Q elliptic, such that:

$$P(\ker Q_{/A}) \text{ is not } C^\infty(A)\text{-dense in } \ker Q_{/A}$$

1) Let A be an open subset of R^n and let $b(A)$ be its boundary.

Let G be the subset of $b(A)$ so defined:

$G = \{ p \in b(A)$: the connexe component, Z_p, of $R^n - A$ with $p \in Z_p$ is compact $\}$

$P = P(D)$ and $Q = Q(D)$ are linear partial differential operators, with constant coefficient; Q will always be elliptic.

LEMMA *a*). *If we put*: $Z_A = \sqcup_{p \in G} Z_p$ *and* $L = A \sqcup Z_A$, *we have*: L *is an open set*. Proof. It is sufficient to see that every compact component, Z, of $R^n - A$, is such that: $Z \wedge b(A) \neq \varnothing$. Then the proof. of the Lemma *a*) is in (5), pag. 235.

LEMMA *b*). *Let* n *be a distribution with compact support*: $n \in E'(\mathbf{R}^n)$. *If* $m = Q(D)\, n$ *has its support in* A, *then* $n \in E'(L)$.

Proof. If $p \notin A$ and Z_p, the connexe component of $R^n - A$ with $p \in Z_p$, is not bounded then there exists a neighbourhood of Z_p in which n is an analytic function. Because n has compact support, in such neighbourhood n must be zero. This shows that: if $p \in \operatorname{supp}(n)$ and $p \notin A$, then $p \in Z_A$.

THEOREM. *If* $n \in E'(R^n)$ *and is orthogonal to all exponential solutions of the equation* $Pu = o$, *then there exists* $m \in E'(\mathbf{R}^n)$ *such that*: $n = P(-D)\, m$.

Proof. See Lemmas 3.4.1 and 3.4.2. of (3) pagg. 77/78.

LEMMA *c*). *Let* $g \in C_c^\infty(L)$ *be a function such that*: $P(-D)\, g \in C_c^\infty(A)$. *If* $P(-D)\, g$, *with* P *hypoelliptic, is orthogonal to* ker $P_{/A}$, *then*: *if* $p \in supp(g) \wedge Z_A$, $g(p) = 0$.

Proof. If δ_p is the Dirac distribution at the point p, the distribution $E_p * \delta_p$ is in ker $P_{/A}$ if E_P is a foundamental solution of P: $PE_P = \delta$. Then: $\langle (E_P * \delta_p)_{/A} \cdot P(-D)\, g \rangle = \langle \delta_p \cdot g \rangle = g\,(p) = 0$.

DEFINITION 3). *We say that a compact subset* K *of* L *disjoins* Z_A *if there exists a partition of* G, $G = G_1 + G_2$, $G_1 \neq \varnothing$, *and an open subset* B *of* L *such that* $\sqcup_{p \in G_1} Z_p \subset K \subset B$ *and* $B \wedge (\sqcup_{p \in G_2} Z_p) = \varnothing$.

DEFINITION 4). *We say that an open subset* A *of* R^n *has the b-propriety if* (*or* $Z_A = \varnothing$ *or*) *there is no compact* K *of* L *which disjoins* Z_A.

THEOREM. *The following two propositions*, p_1 *and* p_2, *are equivalent*:

p_1) *A is an open subset of* R^n *which has the b-propriety*;

p_2) *for every couple*, (P, Q), *of partial differential operators with constant coefficients, prime between them, with* Q *elliptic, we have*:

$$P(ker\, Q_{/A}) \text{ is } C^\infty(A)\text{-dense in } ker\ Q_{/A}.$$

Proof.

FROM p_1) *to* p_2). Suppose there exists P prime with Q such that $P(\text{ker } Q_{/A})$ is not $C^\infty(A)$-dense in ker $Q_{/A}$; we will show that absurd.

From the Hahn-Banach theorem, we have: there exists $m \in E'(A)$ such that m is not orthogonal to ker $Q_{/A}$ but m is orthogonal to $P(\text{ker } Q_{/A})$.

By the precedent theorem, there exists, then, a distribution $n \in E'(R^n)$ such that: $P(-D)\, m = Q(-D)\, n$. Because P and Q are prime between them, there exists $n_0 \in E'(R^n)$ with: $m = Q(-D)\, n_0$, and, from lemma *b*), $n_0 \in E'(L)$.

Let K be the support of n_0; we will show that K disjoins Z_A, so we will have the absurd.

In fact: it can't be: $K \wedge Z_A = \varnothing$, because, otherwise, $n_0 \in E'(A)$ and so m would be orthogonal to ker $Q_{/A}$.

Let G_1 be the subset of G, $G_1 \neq \varnothing$ with: if $p \in G_1$, $Z_p \wedge K \neq \varnothing$, (so that $Z_p \subset K$); we will show that there exists an open subset B of L with: $K \subset B$ and $B \wedge (\sqcup_{p \in G - G_1} Z_p) = \varnothing$.

Of course, this is the case if $G - G_1 = \varnothing$. Otherwise, let $(B_n)_{n \in N}$ a sequence of open subsets ol L such that $B_n \supset B_{n+1}$ and $\wedge_n B_n = K$.

Suppose that $x_n \in B_n \wedge \sqcup_{p \in G - G_1} Z_p$ for every $n \in N$; we can suppose, directly, $\lim_n x_n = x_0$, with, of course, x_0 in K.

It is ipossible that infinite terms of the sequence (x_n) are in the same component Z_q, $q \in G - G_1$; in fact if it is so, we have $x_0 \in Z_q \wedge K$; absurd.

It is easy to see that $x_0 \in b(A)$, because every segment (x_n, x_{n+1}) has a point of A; it comes out that n_0 must be an analytic function in a neighbourhood V of x_0. In such V there is a point $x_n \in Z_{q_n}$ with $q_n \in G - G_1$. Because $Z_{q_n} \wedge K = \varnothing$, in a neighbourhood of x_n, n_0 is zero; so we can suppose n_0 equal to zero in all V. Absurd, because x_0 belongs to supp (n_0).

From p_2) *to* p_1). If K is a compact subset of L and K disjoints Z_A, let g be a function in $C_c^\infty(B)$, with $g = 1$ on B' with: $K \subset B' \subset \bar{B}' \subset B$. If $h = Q(-D)\, g$, $h \in C_c^\infty(A)$ if $Q(0) = 0$; for the lemma c) above, h can't be orthogonal to ker $Q_{/A}$.

But: if $P = P(D)$ is an operator prime with Q and $P(0) = 0$, h is orthogonal to $P(\ker Q_{/A})$ because $P(-D)\, h = Q(-D)\, P(-D)\, g$ and $P(-D)\, g \in C_c^\infty(A)$.

This completes the proof.

The above theorem permits the construction of system like $(+)$ without $C^\infty(A)$-global solution. So, for the system.

$$(^\circ) \quad \{ D_x u = f \quad , \quad D_x u + i\, D_y u = 0 \}$$

in the set $A \subset R^2$ so defined: $|x| < 1$, $|y| < 1$, $x^2 + y^2 \neq 0$, for the reason that $Z_A = (0, 0)$, there is a function, $f_0 \in \ker (D_x + i D_y)_{/A}$ for which there is no global solution in A; on the other hand, by the Lojasiewicz-Malgrange theorem, (*), it is easy to show that there is a sequence, $(B_n)_{n \in N}$, of subset of A, such that:

(*) The theorem is the following: if $A(D)$ is the differential matrix $A(D) = \|a_{i,j}(D)\|$, $I \leq i \leq p$, $I \leq j \leq q$, $u \in E^q(A)$, $f \in E^p(A)$, rispectively p and q times product of $E(A)$, the space of indefinitely differentiable functions over A, the system $A(D)u = f$ has a solution if and only if: for every $v = (v_1, \ldots, v_p)$, v_i polinomial, for which $v(x)\, A(x) = 0$, we have $v(D)f = 0$, if A is convex.

a) $B_n \subset B_{n+1} \subset \sqcup_n B_n = A$; *b*) for every $n \in N$ there is an open subset $B'_n \subset A$ such that: $B_n \subset B'_n$ and the system (°) is $C_c^\infty(B'_n)$ — globally solvable.

Of course, *this example is very near to show the De Giorgi's conjecture is false also in the case* : *Q is elliptic.*

2) I like to end this paper giving *an abstract condition* to have $P(\ker Q_{/A}) = \ker Q_{/A}$.

We put, over $C^\infty(A)$, the following T_P - topology :

V is a neighbourhood of zero in the T_P - topology if : $V \supset W + \ker P_{/A}$, for some W neighbourhood of zero in the usual topology of $C^\infty(A)$.

So we have : if A has the *b*-propriety, P and Q are linear partial differential operators, prime between them, and Q is elliptic,

THEOREM. *The following two proposition, q_1) and q_2), are equivalent* :

q_1) $P(\ker\ Q_{/A}) = \ker\ Q_{/A}$;

q_2) *ker* $(Q \circ P)_{/A} = \ker Q_{/A} + \ker P_{/A}$; *ker* $(Q \circ P)_{/A}$ *is a complete subspace of* $C^\infty(A)$ *with the* T_p *- topology and* $P : \ker\ (Q \circ P)_{/A} \to P\big(\ker\ (Q \circ P)_{/A}\big)$ *is an open mapping.*

Proof.

$q_1) \Rightarrow q_2)$. The first part of q_2) is simple. For the second part, we have: $\ker\ (Q \circ P)_{/A}$ is a closed subspace of $C^\infty(A)$ with the T_p - topology, so :

$(\ker\ Q_{/A})^\wedge \subset \ker\ (Q_0 P)_{/A}$. On the other hand, $\ker\ Q_{/A} + \ker\ P_{/A} \subset (\ker Q_{/A})^\wedge$. Because $P : \ker Q_{/A} \to \ker Q_{/A}$ is an open mapping, (it is a surjective map between Frechet spaces), we have :

if W is an usual neighbourhood of zero in $C^\infty(A)$, $P(W \wedge \ker Q_{/A})$ is open in $\ker Q_{/A}$, so: $P(W + \ker\ P_{/A}) \wedge \ker\ (Q \circ P)_{/A} \supset P(W \wedge \ker Q_{/A})$.

$q_2) \Rightarrow q_1)$. It is sufficient to see that in the diagram

$$\begin{tikzcd}
\ker Q \circ P_{/A} \arrow[rr, "P"] \arrow[d, "p"] & & P\,(\ker Q \circ P_{/A}) \\
\dfrac{\ker Q \circ P_{/A}}{\ker P_{/A}} \arrow[urr, "\tilde{P}"'] & &
\end{tikzcd}$$

the quotient is a Frechet space, so $P\,\big(\ker\ (Q_0 P)_{/A}\big)$ is a Frchet space.

But the last one is also a dense subspace of $\ker Q_{/A}$; so: $P(\ker Q_{/A}) = \ker Q_{/A}$.

Remark 1) It is very easy to see that: if A is $P(-D)$ — convex the topological part of q_2) it is always true. It comes out:

If A is $P(-D)$ — convex, (and it has the b-proriety, which is not a consequence if P is elliptic !), *the necessary and sufficient condition to have*:

$$P(\ker Q_{/A}) = \ker Q_{/A}$$

is: $\ker (Q \circ P)_{/A} = \ker P_{/A} + \ker Q_{/A}$.

Remark 2) The $P(-D)$ — convexity of A, is not, of course, a necessary condition to have the above result, as we can see by the system (^{o}) in A like that, without the points: $x = o$, $o \leq y$.

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