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A density theorem about some system

Rendiconti del Seminario Matematico della Università di Padova, tome 57 (1977), p. 167-172

<http://www.numdam.org/item?id=RSMUP_1977__57__167_0>
A density theorem about some system.

GIULIANO BRATTI (*).

Introduction.

Let $A$ be an open subset of $\mathbb{R}^n$; suppose $P = P(x, D)$, $Q = Q(x, D)$ linear partial differential operators with $C^\infty(A)$ coefficients.

**Definition 1.** We say that the system

\[ (+) \quad \{ Pu = f, \quad Qu = 0 \} \quad f \in C^\infty(A) \]

is $C^\infty(A)$-locally solvable in $A$ if for every $p \in A$ there is a neighbourhood, $V_p$, of $p$ and a function $u_p \in C^\infty(V_p)$ such that the $(+)$ is satisfied in $V_p$.

**Definition 2.** If $B$ is an open subset of $A$, we say that the above system $(+)$ is $C^\infty(B)$-globally solvable if for every $f \in C^\infty(A)$ for which $(+)$ is locally solvable in $A$, there is a function $u \in C^\infty(B)$ such that $(+)$ is satisfied in $B$.

In (2) there is the following conjecture:

let $(B_n)_{n \in \mathbb{N}}$ be a sequence of open subsets of $A$ such that: $B_n \subset B_{n+1} \subset \bigcup B_n = B$ and the $(+)$ is $C^\infty(B_n)$-globally solvable for every $n \in \mathbb{N}$. Then $(+)$ is $C^\infty(B)$-globally solvable.

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It is already known, (4), the conjecture is false in the case in which $P$ and $Q$ have constant coefficients and $Q$ is semi-elliptic, but the conjecture is still open when $Q$ is an elliptic operator.

It seems to the A. that to solve the above conjecture it is important to have some example of system like $(\dagger)$ without $C^\infty(A)$-globally solutions for $f \in C^\infty(A)$ for which $(\dagger)$ is $C^\infty(A)$-locally solvable.

First of all, by a Lojasiewicz-Malgrange's theorem, see (1), it is easy to show that: if $P$ and $Q$ are prime between them, the subspace of $C^\infty(A)$ of the functions for which the system $(\dagger)$ is $C^\infty(A)$-locally solvable is $\ker Q_A = \{ f \in C^\infty(A) : Qf = 0 \}$.

The object of this paper is that to characterize the open subset $A$ of $\mathbb{R}^n$ for which there are systems like $(\dagger)$, with $Q$ elliptic, such that:

$$P(\ker Q_A)$$

is not $C^\infty(A)$-dense in $\ker Q_A$.

1) Let $A$ be an open subset of $\mathbb{R}^n$ and let $b(A)$ be its boundary.

Let $G$ be the subset of $b(A)$ so defined:

$$G = \{ p \in b(A) : \text{the connexe component, } Z_p, \text{ of } E^n - A \text{ with } p \in Z_p \text{ is compact} \}$$

$P = P(D)$ and $Q = Q(D)$ are linear partial differential operators, with constant coefficient; $Q$ will always be elliptic.

Lemma a). If we put: $Z_A = \bigcup_{p \in G} Z_p$ and $L = A \cup Z_A$, we have: $L$ is an open set. Proof. It is sufficient to see that every compact component, $Z$, of $\mathbb{R}^n - A$, is such that: $Z \cap b(A) \neq \emptyset$. Then the proof of the Lemma a) is in (5), pag. 235.

Lemma b). Let $n$ be a distribution with compact support: $n \in E'(\mathbb{R}^n)$. If $m = Q(D)n$ has its support in $A$, then $n \in E'(L)$.

Proof. If $p \notin A$ and $Z_p$, the connexe component of $E^n - A$ with $p \in Z_p$, is not bounded then there exists a neighbourhood of $Z_p$ in which $n$ is an analytic function. Because $n$ has compact support, in such neighbourhood $n$ must be zero. This shows that: if $p \in \text{supp } (n)$ and $p \notin A$, then $p \in Z_A$.

Theorem. If $n \in E'(\mathbb{R}^n)$ and is orthogonal to all exponential solutions of the equation $Pu = 0$, then there exists $m \in E'(\mathbb{R}^n)$ such that: $n = P(-D)m$.

Proof. See Lemmas 3.4.1 and 3.4.2. of (3) pagg. 77/78.
LEMMA c). Let $g \in C_0^\infty(L)$ be a function such that: $P(-D)g \in C_0^\infty(A)$. If $P(-D)g$, with $P$ hypoelliptic, is orthogonal to $\ker P_{/A}$, then: if $p \in \text{supp}(g) \cap Z_A$, $g(p) = 0$.

Proof. If $\delta_p$ is the Dirac distribution at the point $p$, the distribution $E_p \delta_p$ is in $\ker P_{/A}$ if $E_p$ is a fundamental solution of $P$: $PE_p = \delta$. Then: $\langle (E_p \delta_p)_{/A} \cdot P(-D)g \rangle = \langle \delta_p \cdot g \rangle = g(p) = 0$.

DEFINITION 3). We say that a compact subset $K$ of $L$ disjoins $Z_A$ if there exists a partition of $G$, $G = G_1 + G_2$, $G_1 \neq \varnothing$, and an open subset $B$ of $L$ such that $\sqcup_{p \in G_1} Z_p \subset K \subset B$ and $B \setminus (\sqcup_{p \in G_2} Z_p) = \varnothing$.

DEFINITION 4). We say that an open subset $A$ of $\mathbb{R}^n$ has the $b$-propriety if (or $Z_A = \varnothing$ or) there is no compact $K$ of $L$ which disjoins $Z_A$.

THEOREM. The following two propositions, $p_1$ and $p_2$, are equivalent:

$p_1$) $A$ is an open subset of $\mathbb{R}^n$ which has the $b$-propriety;

$p_2$) for every couple, $(P, Q)$, of partial differential operators with constant coefficients, prime between them, with $Q$ elliptic, we have:

\[ P(\ker Q_{/A}) \text{ is } C^\infty(A)\text{-dense in } \ker Q_{/A}. \]

Proof.

From $p_1$ to $p_2$). Suppose there exists $P$ prime with $Q$ such that $P(\ker Q_{/A})$ is not $C^\infty(A)$-dense in $\ker Q_{/A}$; we will show that absurd.

From the Hahn-Banach theorem, we have: there exists $m \in E'(A)$ such that $m$ is not orthogonal to $\ker Q_{/A}$ but $m$ is orthogonal to $P(\ker Q_{/A})$.

By the precedent theorem, there exists, then, a distribution $n \in E'(\mathbb{R}^n)$ such that: $P(-D)m = Q(-D)n$. Because $P$ and $Q$ are prime between them, there exists $n_0 \in E'(\mathbb{R}^n)$ with: $m = Q(-D)n_0$, and, from lemma $b)$, $n_0 \in E'(L)$.

Let $K$ be the support of $n_0$; we will show that $K$ disjoins $Z_A$, so we will have the absurd.

In fact: it can't be: $K \cap Z_A = \varnothing$, because, otherwise, $n_0 \in E'(A)$ and so $m$ would be orthogonal to $\ker Q_{/A}$.

Let $G_1$ be the subset of $G$, $G_1 \neq \varnothing$ with: if $p \in G_1$, $Z_p \setminus K \neq \varnothing$, (so that $Z_p \subset K$); we will show that there exists an open subset $B$ of $L$ with: $K \subset B$ and $B \setminus (\sqcup_{p \in G_1 \setminus G_1} Z_p) = \varnothing$. 

Of course, this is the case if \( G = G_1 = \emptyset \). Otherwise, let \((B_n)_{n \in \mathbb{N}}\) a sequence of open subsets of \( L \) such that \( B_n \supset B_{n+1} \) and \( \bigcap_n B_n = K \).

Suppose that \( x_n \in B_n \cap \bigcup_{p \in G - G_1} Z_p \) for every \( n \in \mathbb{N} \); we can suppose, directly, \( \lim_n x_n = x_0 \), with, of course, \( x_0 \) in \( K \).

It is impossible that infinite terms of the sequence \((x_n)\) are in the same component \( Z_q \), \( q \in G - G_1 \); in fact if it is so, we have \( x_0 \in Z_q \cap K \); absurd.

It is easy to see that \( x_0 \in b(A) \), because every segment \((x_n, x_{n+1})\) has a point of \( A \); it comes out that \( n_0 \) must be an analytic function in a neighbourhood \( V \) of \( x_0 \). In such \( V \) there is a point \( x_n \in Z_{q_n} \) with \( q_n \in G - G_1 \). Because \( Z_{q_n} \cap K = \emptyset \), in a neighbourhood of \( x_n \), \( n_0 \) is zero; so we can suppose \( n_0 \) equal to zero in all \( V \). Absurd, because \( x_0 \) belongs to \( \text{supp}(n_0) \).

**From p.2 to p.1.** If \( K \) is a compact subset of \( L \) and \( K \) disjoints \( Z_A \), let \( g \) be a function in \( C^\infty_c(B) \), with \( g = 1 \) on \( B' \): \( K \subset B' \subset \bar{B}' \subset B \). If \( h = Q(-D)g \), \( h \in C^\infty_c(A) \) if \( Q(0) = 0 \); for the lemma c) above, \( h \) can't be orthogonal to \( \ker Q \mid_A \).

But: if \( P = P(D) \) is an operator prime with \( Q \) and \( P(0) = 0 \), \( h \) is orthogonal to \( P(\ker Q \mid_A) \) because \( P(-D)h = Q(-D)P(-D)g \) and \( P(-D)g \in C^\infty_c(A) \).

This completes the proof.

The above theorem permits the construction of system like (\(+\)) without \( C^\infty(A) \)-global solution. So, for the system.

\[(0) \quad \{ D_xu = f \ , \ D_xu + i D_yu = 0 \}\]

in the set \( A \subset \mathbb{R}^2 \) so defined: \( |x| < 1, \ |y| < 1, \ x^2 + y^2 \neq 0 \), for the reason that \( Z_A = (0,0) \), there is a function, \( f_0 \in \ker (D_x + iD_y) \mid_A \) for which there is no global solution in \( A \); on the other hand, by the Lojasiewicz-Malgrange theorem, (\(\ast\)), it is easy to show that there is a sequence, \((B_n)_{n \in \mathbb{N}}\), of subset of \( A \), such that:

\(\ast\) The theorem is the following: if \( A(D) \) is the differential matrix \( A(D) = \|a_{ij}(D)\| \), \( I \leq i \leq p \), \( I \leq j \leq q \), \( u \in E^q(A) \), \( f \in E^p(A) \), respectively \( p \) and \( q \) times product of \( E(A) \), the space of indefinitely differentiable functions over \( A \), the system \( A(D)u = f \) has a solution if and only if: for every \( v = (v_1, ..., v_p) \), \( v_i \) polinomial, for which \( v(x)A(x) = 0 \), we have \( v(D)f = 0 \), if \( A \) is convex.
a) $B_n \subseteq B_{n+1} \subseteq \bigcap_n B_n = A$; b) for every $n \in \mathbb{N}$ there is an open subset $B'_n \subseteq A$ such that: $B_n \subseteq B'_n$ and the system \((0)\) is $C^\omega_c(B'_n) -$ globally solvable.

Of course, this example is very near to show the De Giorgi’s conjecture is false also in the case: $Q$ is elliptic.

2) I like to end this paper giving an abstract condition to have $P(\ker Q \arrow{A}) = \ker Q \arrow{A}$.

We put, over $C^\omega(A)$, the following $T_p$-topology:

$V$ is a neighbourhood of zero in the $T_p$-topology if:

$V \ni W + \ker P \arrow{A}$, for some $W$ neighbourhood of zero in the usual topology of $C^\omega(A)$.

So we have: if $A$ has the $b$-propriety, $P$ and $Q$ are linear partial differential operators, prime between them, and $Q$ is elliptic,

**Theorem.** The following two proposition, $q_1)$ and $q_2)$, are equivalent:

$q_1)$ $\ker (Q \arrow{A}) = \ker Q \arrow{A}$;

$q_2)$ $\ker (Q \circ P) \arrow{A} = \ker Q \arrow{A} + \ker P \arrow{A}$; $\ker (Q \circ P) \arrow{A}$ is a complete subspace of $C^\omega(A)$ with the $T_p$-topology and $P: \ker (Q \circ P) \arrow{A} \rightarrow P(\ker (Q \circ P) \arrow{A})$ is an open mapping.

**Proof.**

$q_1) \Rightarrow q_2)$. The first part of $q_2)$ is simple. For the second part, we have: $\ker (Q \circ P) \arrow{A}$ is a closed subspace of $C^\omega(A)$ with the $T_p$-topology, so:

$(\ker Q \arrow{A}) \wedge \subseteq (\ker (Q \circ P) \arrow{A})$. On the other hand, $\ker Q \arrow{A} + \ker P \arrow{A} \subseteq (\ker Q \arrow{A}) \wedge$. Because $P: \ker Q \arrow{A} \rightarrow \ker Q \arrow{A}$ is an open mapping, (it is a surjective map between Frechet spaces), we have:

if $W$ is an usual neighbourhood of zero in $C^\omega(A)$, $P(W \wedge \ker Q \arrow{A})$ is open in $\ker Q \arrow{A}$, so: $P(W + \ker P \arrow{A}) \wedge \ker (Q \circ P) \arrow{A}$.

$q_2) \Rightarrow q_1)$. It is sufficient to see that in the diagram

$$
\begin{array}{ccc}
\ker Q \circ P \arrow{A} & \xrightarrow{P} & P(\ker Q \circ P \arrow{A}) \\
\downarrow \quad \quad \quad \downarrow P \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
But the last one is also a dense subspace of \( \ker Q_{/A} \); so:
\[
P(\ker Q_{/A}) = \ker Q_{/A}.
\]

Remark 1) It is very easy to see that: if \( A \) is \( P(-D) \) - convex the topological part of \( q_2 \) it is always true. It comes out:

If \( A \) is \( P(-D) \) - convex, (and it has the b-proriety, which is not a consequence if \( P \) is elliptic!), the necessary and sufficient condition to have:

\[
P(\ker Q_{/A}) = \ker Q_{/A}
\]
is:
\[
\ker (Q_{0}P)_{/A} = \ker P_{/A} + \ker Q_{/A}.
\]

Remark 2) The \( P(-D) \) - convexity of \( A \), is not, of course, a necessary condition to have the above result, as we can see by the system \((0)\) in \( A \) like that, without the points: \( x = o, o \leq y \).

**BIBLIOGRAPHY**


