Some remarks on an operational time dependent equation

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Some Remarks
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Introduction.

Let $E$ be a complex Hilbert space and $\{A(t)\}_{t \in [0,T]}$, $\{B(t)\}_{t \in [0,T]}$ two families of linear operators (generally not bounded) in $E$.

Consider the Cauchy problem:

$$\begin{cases}
U'(t) = A(t)U(t) + U(t)B^*(t) + f(t, U(t)), \\
U(0) = U_0,
\end{cases}$$

where $f$ is a mapping $[0, T] \times Q \to \mathcal{L}(E)$ and $Q \subset \mathcal{L}(E)$.

Problems of this kind arise in several fields as Optimal Control theory ([2], [3], [7], [8], [9]) and the Hartree-Fock time dependent problem in the case of finite Fermi system ([1]).

In this paper we generalize the results contained in [3] and we give some new regularity result for the case where $A(t)$ and $B(t)$ generate « hyperbolic » semi-groups.

1. The semi-group $T \to e^{tA}T e^{tB}$.

Let $E$ be a complex Hilbert space (norm $| |$, inner product $( , )$). We note by $\mathcal{L}(E)$ (resp. $H(E)$) the complex (resp. real) Banach space of linear bounded (resp. hermitian) operators $E \to E$ and by $H_+(E)$ the cone of positive operators.

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Let $A$ and $B$ be the infinitesimal generators of two semi-groups $e^{ta}$ and $e^{tb}$; we assume that:

\[(1.1) \quad |e^{ta}| \leq M_A \exp(w_AT), \quad |e^{tb}| \leq M_B \exp(w_BT).\]

We note finally by $L_s(E)$ (resp. $H_s(E)$) the set $L(E)$ (resp. $H(E)$) endowed by the strong topology; $L_s(E)$ is a locally convex space.

Consider the following semi-group in $L_s(E)$:

\[(1.2) \quad G(t) = e^{ta}Te^{tb}, \quad \forall T \in L(E), \quad t \geq 0,\]

$G_t$ is not strongly continuous in $L(E)$, but it is sequentially strongly continuous in $L_s(E)$, that is:

\[T_n \rightarrow T \text{ in } L_s(E) \Rightarrow G_t(T_n) \rightarrow G_t(T) \text{ in } L_s(E)\]

and the mapping:

\[\mathbb{R}_+ \rightarrow L_s(E), \quad t \mapsto G_t(T)\]

is continuous $\forall T \in L_s(E)$.

If $B = A^*$ (1) it is:

\[(1.3) \quad G_t(T) \in H(E), \quad \forall T \in H(E).\]

Put:

\[(1.4) \quad D(L) = \left\{ T \in L(E); \exists \lim_{h \to 0} \frac{1}{h} (G_h(T)x - Tx), \forall x \in E \right\},\]

\[(1.5) \quad L(T)x = \lim_{h \to 0} \frac{1}{h} (G_h(T)x - Tx), \quad \forall T \in D(L), \forall x \in E.\]

**Lemma 1.1.** If $T \in D(L)$ and $x \in D(B)$ then $Tx \in D(A)$ and it is:

\[(1.6) \quad L(T)x = ATx + TBx.\]

**Proof.** Let $T \in D(L)$, $x \in D(B)$, $y \in D(A^*)$; it is:

\[(L(T)x, y) = \frac{d}{dh} (Te^{hA^*}x, e^{hA^*}y)|_{h=0} = (TBx, y) + (Tx, A^*y).\]

(1) $A^*$ is the adjoint of $A$. 

It follows that the mapping:

\[ D(A^*) \to C, \quad y \to (Tx, A^*y) = (L(T)x, y) - (TBx, y) \]

is continuous, \(Tx \in D(A)\) and:

\[ (ATx, y) = (L(T)x, y) - (TBx, y) \neq \]

The following proposition is clear:

**Proposition 1.2.** If \(T \in D(L)\) then \(G_t(T) \in D(L)\) and it is:

\[ L(G_t(T)) = e^{tA}L(T)e^{tB}, \]

\[ \frac{d}{dt}(G_t(T)x) = e^{tA}L(T)e^{tB}x. \]

**Proposition 1.3.** \(L\) is closed in \(L_s(E)\) and in \(L(E)\).

**Proof.** Let \(T_n \in D(L), T_n \to T, S_n = L(T_n) \to S\) in \(L_s(E)\); due to (1.8) it is:

\[ e^{tA}T_n e^{tB}x - T_n x = \int_0^t e^{tA}S_n e^{sB}x\,ds \]

recalling the dominate convergence theorem we obtain:

\[ \frac{1}{t}(G_t(T)x - Tx) = \frac{1}{t} \int_0^t G_s(S)x\,ds \]

it follows \(T \in D(L)\) and \(L(T) = S\). Therefore \(L\) is closed in \(L_s(E)\) and consequently in \(L(E)\).

**Proposition 1.4.** \(D(L)\) is dense in \(L_s(E)\).

**Proof.** Put:

\[ Q_t x = \frac{1}{t} \int_0^t G_s(T)x\,ds, \quad \forall T \in L(E), \forall x \in E, \]
it is:
\[ \lim_{t \to 0^+} Q_t = I \quad \text{in } \mathcal{L}_s(E), \]
moreover
\[ \frac{1}{h} (G_h(Q_t) - Q_t) x = \frac{1}{ih} \left[ \int_t^{t+h} G_s(T) x \, ds \right] \]
it follows \( D_t \in D(L) \) and therefore \( D(L) \) is dense in \( \mathcal{L}_s(E) \).

**Proposition 1.5.** \( \rho(L) \supset ]w_A + w_B, \infty[ \) and it is \((2)\):

\[(1.9) \quad R(\lambda, L)(T)x = \int_0^\infty e^{-\lambda t} e^{itA} Te^{itB} x \, dt, \quad \forall x \in E, \ \forall \lambda > w_A + w_B, \]

\[(1.10) \quad \| R(\lambda, L) \|_{\mathcal{L}(E)} \leq M_A M_B (\lambda - w_A - w_B)^{-1}, \quad w_A + w_B < \lambda \quad (3). \]

**Proof.** Put
\[ F(T)x = \int_0^\infty e^{-\lambda t} e^{itA} Te^{itB} x \, dt, \quad \forall T \in \mathcal{L}(E). \]

For every \( T \in D(L) \) it is:
\[ F(L(T))x = \int_0^\infty e^{-\lambda t} G_t'(T)x \, dt = (\lambda F(T) - T)x \]
moreover if \( T \in \mathcal{L}(E) \) it is:
\[ \frac{1}{h} \{ G_h(F(T)) - F(T) \} x = \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} G_t(T)x \, dt - \frac{1}{h} \int_0^h e^{-\lambda t} G_t(T)x \, dt \]
it follows
\[ L(F(T))x = (\lambda F(T) - T)x. \quad \neq \]

\((2)\) If \( L \) is a linear operator, \( \rho(L) \) is the resolvent set and \( R(\lambda, L) \) the resolvent of \( L \).

\((3)\) \( \mathcal{L}(\mathcal{L}(E)) \) is the Banach space of the linear bounded operators \( \mathcal{L}(E) \rightarrow \mathcal{L}(E) \). We note \( \| \| \) the norm in \( \mathcal{L}(\mathcal{L}(E)) \).
PROPOSITION 1.6. If $T_n \to T$ in $C([0, T]; \mathcal{L}_s(E))$ (4) then

$$G_i(T_n(t)) \to G_i(T(t)) \quad \text{in} \quad C([0, T]; \mathcal{L}_s(E)).$$

PROOF. Let $x \in E$; for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$, $f_1, \ldots, f_{n_\varepsilon}$ in $C([0, T])$ and $x_1, \ldots, x_{n_\varepsilon} \in E$ such that:

$$\left| e^{tb}x - \sum_{i=1}^{n_\varepsilon} f_i(t) x_i \right| < \varepsilon, \quad \forall t \in [0, T],$$

it follows:

$$|G_i(T(t) - T_n(t))x| \leq M_B \exp\left( |w_B|T \right) |(T(t) - T_n(t)) e^{tb}x| \leq$$

$$\leq M_B \exp\left( |w_B|T \right) \varepsilon (|T(t)| + |T_n(t)|) +$$

$$+ M_B \exp\left( |w_B|T \right) \sum_{i=1}^{n_\varepsilon} |f_i(t)| |T(x)x_i - T_n(x)x_i|.$$

Choose $N$ such that $|T_n(t)| < N$, then:

$$|G_i(T(t) - T_n(t))x| \leq 2N M_B \exp\left( |w_B|T \right) \varepsilon +$$

$$+ M_B \exp\left( |w_B|T \right) \sum_{i=1}^{n_\varepsilon} |g_i(t)| |T(x)x_i - T_n(x)x_i|.$$

Choose $n'_\varepsilon$ such that:

$$|T(t)x_i - T_n(t)x_i| < \varepsilon / \left( n_\varepsilon \max \{ |g_i|; i = 1, 2, \ldots, n_\varepsilon \} \right), \quad \forall n > n_\varepsilon,$$

then

$$n > n'_\varepsilon \Rightarrow |G_i(T(t)H T_n(t))x| < (2N + 1) M_B \exp\left( |w_B|T \right) \varepsilon.$$

2. The linear problem.

Let $\mathcal{A} = \{ A(t) \}_{t \in [0, T]}, \mathcal{B} = \{ B(t) \}_{t \in [0, T]}$ be two families of linear operators in $E$.

Let $E$ be a Hilbert space (norm $\| \|$, inner product $(\cdot, \cdot)$) continuously and densely embedded in $E$.

(4) $C([0, T]; \mathcal{L}_s(E))$ is the set of the mappings $[0, T] \to \mathcal{L}_s(E)$ continuous; due to the Banach-Steinhaus theorem every $u \in C([0, T]; \mathcal{L}_s(E))$ is bounded.
Let finally $Z$ be an isometric isomorphism in $\mathcal{L}(F, E)$.

We assume:

\begin{enumerate}
\item[(a)] $\mathcal{A}$ (resp. $\mathcal{B}$) is $(M_A, w_A)$-stable and $w_A$-measurable (resp. $(M_B, w_B)$-stable and $w_B$-measurable) in $E$.
\item[(b)] It is $F \subset D(A(t))$ (resp. $D(B(t))$), $A(t)$ (resp. $B(t)$) $\in \mathcal{L}(F, E)$ and $|A(t)|$ (resp. $|B(t)|$) is bounded in $[0, T]$.
\item[(c)] The mapping $A(\cdot) \psi$ (resp. $B(\cdot) \psi$) is continuous $\forall \psi \in F$.
\item[(d)] There exists a mapping $H$ (resp. $K$): $[0, T] \rightarrow \mathcal{L}(E)$ such that:
\begin{enumerate}
\item[$d_1$] $H$ (resp. $K$) is bounded in $[0, T]$ and strongly measurable in $E$.
\item[$d_2$] It is:
\[ZA(t)Z^{-1} \psi = A(t) \psi + H(t) \psi, \quad \forall \psi \in D(A(t)),
\]
\[ZB(t)Z^{-1} \psi = B(t) \psi + K(t) \psi, \quad \forall \psi \in D(B(t)).
\]
\end{enumerate}
\end{enumerate}

If (2.1) is fulfilled it is known ([4], [6]) that there exists an evolution operator $G_A(t, s)$ (resp. $G_B(t, s)$) for the problem:

\[u' = A(t)u, \quad u(0) = x \quad \text{(resp. } u' = B(t)u, \quad u(0) = x).\]

Moreover $G_A$ (resp. $G_B$): $\Lambda = \{(t, s) \in [0, T]^2; t > s\} \rightarrow \mathcal{L}(E)$ is strongly continuous and $G(r, s) \in \mathcal{L}(F)$.

Finally it is:

\begin{equation}
\begin{cases}
\lim_{n \to \infty} G_{A,n} = G_A, \\
\lim_{n \to \infty} G_{B,n} = G_B,
\end{cases}
\quad \text{in } C(\Lambda; \mathcal{L}(E)),
\end{equation}

(\*) $\mathcal{A}$ is $w_A$-measurable in $E$ if $\sigma(A(t)) \supset \omega_A$, $+\infty$ and $R(\lambda, A(\cdot))$ is strongly measurable $\forall \lambda > \omega_A$.

$\mathcal{A}$ is $(M_A, w_A)$-stable in $E$ if $\sigma(A(t)) \supset \omega_A$, $+\infty$ and it is:

\[\left| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right| < M_A/(\lambda - w_A)^k
\]

$\forall k \in \mathbb{N}, t_1 > t_2 > \ldots > t_k$, $t_i \in [0, T], i = 1, \ldots, n$.

(\*) With the topology of $\mathcal{L}(F, E)$. 

\[\text{G. Da Prato}\]
where $G_{A_n}$ (resp. $G_{B_n}$) is the evolution operator associated to the problem:

$$u'_n = A_n(t)u_n, \quad u_n(0) = x \quad \text{(resp. } u'_n = B_n(t)u_n, \quad u_n(0) = x)$$

where $A_n(t) = n^2 R(n, A(t)) - n$ and $B_n(t) = n^2 R(n, B(t)) - n$.

Consider now the problem:

$$\begin{cases}
T'(t) = A(t)T(t) + T(t)B^*(t) + F(t), & F \in C([0, T]; \mathcal{L}(E)), \\
T(0) = T_0 \in \mathcal{L}(E).
\end{cases}$$

We define $L(t)$ as in (1.4), (1.5) and write (2.3) in the following form:

$$\begin{cases}
T'(t) = L(t)(T(t)) + F(t), \\
T(0) = T_0.
\end{cases}$$

We consider also the approximate problem:

$$\begin{cases}
T'_n(t) = L_n(t)(T(t)) + F(t), \\
T_n(0) = T_0,
\end{cases}$$

where $L_n(t)(T) = A_n(t)T + TB_n^*(t)$.

We say that $T$ is a strong solution of (2.4) if there exists:

$$\{T_k\} \subset D(L(t)) \cap C^1([0, T]; \mathcal{L}(E))$$

such that:

$$\begin{cases}
T_k' - L(T_k) \to F \quad \text{in } C([0, T]; \mathcal{L}(E)), \\
T_k(0) \to T_0 \quad \text{in } \mathcal{L}(E).
\end{cases}$$

If $T \in D(L(t)) \cap C^1([0, T]; \mathcal{L}(E))$ and (2.4) is fulfilled we say that $T$ is a classical solution of (2.4).

**Theorem 2.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be two family of linear operators in $E$ verifying (2.1). Then for every $T_0 \in \mathcal{L}(E)$ and $F \in C([0, T]; \mathcal{L}(E))$ the

$$C^1([0, T]; \mathcal{L}(E))$$

is the set of the mappings $[0, T] \to \mathcal{L}(E)$ strongly continuously differentiable.
problem (2.4) has a unique strong solution given by:

\[(2.7) \quad T(t)x = G_d(t, 0)T_0G^*_d(t, 0)x + \int_0^t G_d(t, s)F(s)G^*_d(t, s)x\,ds.\]

If \( T_0 \in \mathcal{L}(F) \) and \( F \in C([0, T]; \mathcal{L}_*(F)) \) then the solution \( T \) is classical.

**Proof.** Let first \( T_0 \in \mathcal{L}(F) \) and \( F \in C([0, T]; \mathcal{L}_*(F)) \); in this case we can easily verify that \( T \) is a classical solution.

In the general case by approximating \( T_0 \) and \( F \) we can show that \( T(t) \), given by (2.7) is a strong solution.

Assume finally that \( T \) is a strong solution of (2.4) and take \( \{T_k\} \) as in (2.6). Put \( F_k = T'_k - L(T_k) \); it is:

\[\frac{d}{ds}(G_d(t, s)T_k(s)G_B(t, s)x) = G_d(t, s)F_k(s)G_B(t, s)x, \quad \forall x \in E,\]

by integration in \([0, t]\) it follows:

\[T_k(t)x = G_d(t, 0)T_k(0)G_B(t, 0)x + \int_0^t G_d(t, s)F_k(s)G_B(t, s)x\,ds\]

and, taking the limit for \( k \to \infty \), the conclusion follows.

**3. The quasi-linear problem.**

Let \( Q \) a closed convex set in \( \mathcal{L}(E) \) and \( f \) a strongly continuous mapping

\[f: [0, T] \times Q \to \mathcal{L}(E), \quad (t, S) \to f(t, S).\]

Consider the problems:

\[(3.1) \begin{cases} U'(t) - L(t)(U(t)) + f(t, U(t)) = 0, \\ U(0) = U_0, \end{cases}\]

\[(3.2) \begin{cases} U_n'(t) - L_n(t)(U_n(t)) + f(t, U_n(t)) = 0, \\ U_n(0) = U_0. \end{cases}\]
We say that $U$ is a strong solution of (3.1) if there exists $\{U_h\} \subset D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))$ such that:

\[
\begin{cases}
    U_h' - L(U_h) + f(t, U_h) \to 0 & \text{in } C([0, T]; \mathcal{L}_s(E)), \\
    U_h(0) \to U_0 & \mathcal{L}_s(E).
\end{cases}
\]

If $U$ belongs to $D(L(t)) \cap C^1([0, T]; \mathcal{L}_s(E))$ and fulfills (3.1) we say that $U$ is a classical solution of (3.1).

The following proposition is an immediate consequence of the Theorem 2.1.

**Proposition 3.1.** $U$ is a strong solution of (3.1) if and only if it is:

\[
U(t)x = G_D(t, 0)U_0B^*_x(t, 0)x - \int_0^t G_D(t, s)f(s, U(s))G^*_D(t, s)x\,ds.
\]

We remark now that $C([0, T]; \mathcal{L}_s(E))$ is not a metric space, but we can define in it the following norm:

\[
\|U\| = \sup\{|U(t)|, t \in [0, T]\}, \quad \forall U \in C([0, T]; \mathcal{L}_s(E)),
\]

by virtue of the Banach-Steinhaus theorem.

$C([0, T]; \mathcal{L}_s(E))$ endowed by the norm (3.4) is a Banach space which we note by $B([0, T]; \mathcal{L}_s(E))$.

**Lemma 3.2.** Let $K$ be a closed subset of $B([0, T]; \mathcal{L}_s(E))$ and $\gamma_n$, $\gamma$ mappings $K \to K$. Assume that:

\[
\|\gamma_n(U) - \gamma(V)\| < \alpha \|U - V\|, \quad \alpha \in ]0, 1[, \quad U, V \in K,
\]

\[
\gamma_n(U) \to \gamma(U) \quad \text{in } C([0, T]; \mathcal{L}_s(E)), \quad \forall U \in K.
\]

Then there exists $\{U_n\}$ and $U$ unique in $K$ such that:

\[
\gamma_n(U_n) = U_n, \quad \gamma(U) = U,
\]

\[
U_n \to U \quad \text{in } C([0, T]; \mathcal{L}_s(E)).
\]

**Proof.** By virtue of the contractions principle there exists $U_n$ and $U$ such that (3.7) is fulfilled.
To prove (3.8) fix $Z$ in $K$; it is:

$$U_n = \lim_{m \to \infty} \gamma^m_n(Z), \quad U = \lim_{m \to \infty} \gamma^m(Z) \quad \text{in } B([0, T]; \mathcal{L}_*(E))$$

and

$$\|U_n - \gamma^m_n(Z)\| \leq \frac{\alpha^m}{1 - \alpha} (\|\gamma^m_n(Z)\| + \|Z\|)$$

therefore there exists $M > 0$ such that:

$$\|U_n - \gamma^m_n(Z)\| < M\alpha^m. \quad (3.9)$$

It is easy to show that:

$$\lim_{n \to \infty} \gamma^m_n(U) = \gamma^m(U) \quad \text{in } C([0, T]; \mathcal{L}_*(E)), \quad \forall U \in K, \quad m \in \mathbb{N}$$

if $x \in E$ and $t \in [0, T]$ it follows:

$$|U(t)x - U_n(t)x| \leq |U(t)x - \gamma^m(Z)(t)x| + |\gamma^m(Z)(t)x - \gamma^m_n(Z)(t)x| + |U_n(t)x - \gamma^m_n(Z)(t)x|$$

due to (3.9) it follows:

$$|U(t)x - U_n(t)x| \leq 2M\alpha^m |x| + |\gamma^m(Z)(t)x - \gamma^m_n(Z)(t)x|$$

and the conclusion follows from (3.10). \(\neq\)

We prove now the existence of the maximal solution for the problem (3.1).

We assume:

$$\begin{align*}
& a) \quad f \in C([0, T] \times Q); \quad \mathcal{L}_*(E) \cap C([0, T] \times Q; \mathcal{L}(E)), \\
& b) \quad \exists \mu: \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\
& \quad |f(t, T) - f(t, S)|Z < \mu(r)|T - S| \text{ if } |T| < r, \quad |S| < r, \\
& c) \quad \exists \alpha: \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that:} \\
& \quad r > 0, \quad |T| < r, \quad T \in Q, \quad \beta \in [0, \alpha(r)[ \Rightarrow T - \beta f(t, T) \in Q. \quad (3.11)
\end{align*}$$

We remark that $c)$ is trivial if $Q = \mathcal{L}_*(E)$ or $H(E)$.

\(Q_i\) is endowed by the topology of $\mathcal{L}_*(E)$. 

(\(\star\))
LEMMA 3.3. Assume that:

i) \( \mathcal{A} \) and \( \mathcal{B} \) verify (2.1), \( U_0 \in Q \),

ii) \( T \in Q \Rightarrow \exp (s A(t)) T \exp (s B(t)) \in Q, \forall t \in [0, T] \),

iii) \( f \) verifies (3.11).

Take \( \alpha, \beta \) such that:

\[
\begin{cases}
\alpha > M_x M_u \exp \left( (|w_A| + |w_B|) T \right) |U_0| , \\
\beta < \alpha (2a). 
\end{cases}
\]

Then there exists \( \tau > 0 \) such that the problem (3.1) has a unique strong solution in \( [0, \tau] \).

**Proof.** Put:

\[ \varphi(t, T) = T - \beta f(t, T) \]

then \( \varphi \) maps \( [0, T] \times (Q \cap P(0, 2a)) \) in \( Q \) (*) and it is:

(3.13) \( |\varphi(t, T) - \varphi(t, S)| < (1 + \beta |2a|) |T - S|, \quad \forall T, S \in Q \cap P(0, 2a) \).

Problem (3.1) is equivalent to:

(3.14)

\[
\begin{cases}
U' - L(t) U + \frac{1}{\beta} \varphi(t, U) = 0 , \\
U(0) = U_0 ,
\end{cases}
\]

put \( U = \exp(-t/\beta) V \), then it is:

(3.15) \[ V(t) x = G_A(t, 0) U_0 G_B^*(t, 0) x + \]

\[ + \frac{1}{\beta} \int_0^t e^{s/\beta} G_A(t, s) \varphi(s, U(s)) G_B^*(t, s) x \, ds \]

which is equivalent to the equation:

(3.16) \[ U(t) = G_A(t, 0) U_0 G_B^*(t, 0) e^{-t/\beta} + \frac{1}{\beta} \int_0^t e^{-(t-s)/\beta} G_A(t, s) \varphi(s, U(s)) \]

\[ G_B^*(t, s) ds = \gamma(U)(t) . \]

(*\( P(0, r) = \{ T \in \mathcal{L}(E); |T| < r \}. \)
It is:

\[ \| \gamma(U) - \gamma(V) \| \leq M_A M_B \exp \left( (|w_A| + |w_B|) T \right) (1 - e^{-\mu t}) \| U - V \|, \]

\( \forall U, V \in C([0, \tau]; (Q \cap P(0, 2a))), \)

and

\[ \| \gamma(U) \| \leq a + M_A M_B \exp \left( (|w_A| + |w_B|) T \right) (1 + \mu(2a)) 2a + \]

\[ + \sup \{ |\varphi(t, 0)|, t \in [0, T] \} (1 - e^{-\mu t}). \]

Therefore there exists \( \tau > 0 \) such that \( \gamma \) is a contraction in

\[ C([0, \tau]; (Q \cap P(0, 2a))), \]

\[ \neq \]

The following theorem is an immediate consequence of Lemma 3.3, Proposition 1.6, Lemma 3.2 and standard arguments.

**Theorem 3.4.** Assume that \( A, B, f \) verify the hypotheses of Lemma 3.3. Then there exists the maximal solution \( U \) of the problem (3.1). If \( I \) is the interval where \( U \) is defined it is:

\[ U_n \to U \quad \text{in} \quad C(I, \mathcal{L}_2(E)) \]

\( U_n \) being the solution of (3.2). Finally if \( \| U \| \) is bounded it is \( I = [0, T] \).

**Proposition 3.5.** Assume that the hypotheses of Theorem 3.4 are fulfilled. Assume moreover:

\[ \begin{align*}
  & i) \quad M_A = M_B = 1, \\
  & ii) \quad \exists \omega_1 \in \mathbb{R} \text{ such that} \\
  & |T| < |T + \alpha(f(t, T) - f(t, 0) + \omega_1 T)|, \\
  & \quad \forall \alpha > 0, t \in [0, T], T \in Q. 
\end{align*} \]

Then the maximal solution of (3.1) verifies the following inequality:

\[ |U(t)| \leq \exp \left( (w_A + w_B + \omega_1) t \right) |U_0| + \]

\[ + \int_0^t \exp \left( (w_A + w_B + \omega_1)(t - s) \right) |f(s, 0)| ds. \]
PROOF. We remember (Kato [5]) that (3.18)-ii) is equivalent to:

\begin{equation}
\langle f(t, T) - f(t, 0), \Gamma \rangle > - \omega_1 |T|, \quad \forall \Gamma \in \partial |T|,
\end{equation}

\( \partial |T| \) being the sub-differential of the norm in \( \mathcal{L}(E) \).

Due to (3.18) for every \( T \in D(L(s)) \) there exists \( \Gamma \in \partial |T| \) such that

\begin{equation}
\langle L(s)(T), \Gamma \rangle < (w_A + w_B)|T|.
\end{equation}

Suppose first that \( U \) is a classical solution of (3.1); then

\begin{equation}
\frac{d^-}{dt} |U(t)| = \inf \{ \langle U(t), \Gamma \rangle, \Gamma \in \partial |U(t)| \} < \langle L(t)(U(t)), \Gamma \rangle - \langle f(t, U(t)) - f(t, 0), \Gamma \rangle + \langle f(t, 0), \Gamma \rangle
\end{equation}

if we take \( \Gamma \) such that

\( \langle L(t)(U(t)), \Gamma \rangle < (w_A + w_B)|U(t)| \)

it is

\begin{equation}
\frac{d^-}{dt} |U(t)| < (w_A + w_B + \omega_1)|U(t)| + |f(t, 0)|
\end{equation}

which implies (3.19). If \( U \) is a strong solution the conclusion follows by approximation.

4. Regularity.

If for every \( V \in \mathcal{L}(F) \) it is \( f(t, V) \in \mathcal{L}(F) \) we put

\[ f_z(t, V) = Zf(t, Z^{-1}VZ)Z^{-1}. \]

**Theorem 4.1.** Assume that the hypotheses of Theorem 3.4 are fulfilled. Moreover assume that \( f \) maps \([0, T] \times \mathcal{L}(F) \) into \( \mathcal{L}(F) \) and that \( f_z \) verifies (3.11); then if \( U_0 \in \mathcal{L}(E) \cap \mathcal{L}(F) \) the maximal solution of (3.1) is classical and \( U(t) \in \mathcal{L}(F), \forall t \in [0, T]. \)
PROOF. Consider the problems:

\begin{equation}
\begin{cases}
V'(t) = (A(t) + H(t))V(t) + V(t)(B(t) + K(t)) + f_n(t, V), \\
B(0) = ZU_n Z^{-1}, \\
V_n'(t) = (A_n(t) + H_n(t))V_n(t) + \\
\quad + V_n(t)(B_n(t) + K_n(t)) + Zf_n(t, U_n) Z^{-1}, \\
V_n(0) = ZU_n Z^{-1},
\end{cases}
\end{equation}

where

\begin{equation}
\begin{cases}
H_n(t) = n^2 H(n, A(t)) H(t) H(n, A(t)) + H(t), \\
K_n(t) = n^2 H(n, B(t)) K(t) H(n, B(t)) + K(t).
\end{cases}
\end{equation}

By virtue of Theorem 3.4 the problems (4.1) and (4.2) have maximal solutions in \([0, \tau]\), \(\tau\) being the maximal time for \(U\); moreover

\[ V_n \to V \quad \text{in } C([0, \tau]; \mathcal{L}_s(E)). \]

It is easy to see that \(V_n = ZU_n Z^{-1}\), therefore

\[ U_n \to U \quad \text{in } C([0, \tau]; \mathcal{L}_s(E)), \quad ZU_n Z^{-1} \to V \quad \text{in } \mathcal{L}_s(E) \]

it follows \(U \in \mathcal{L}(F), V = ZU Z^{-1} \neq \).

REMARK. If \(A\) and \(B\) are independent of \(t\) we have the following result (cf. [3]).

THEOREM 4.2. Assume that the hypotheses of Theorem 3.4 are fulfilled. Suppose moreover that \(f \in C^1([0, T], \mathcal{L}_s(E))\) and \(U_0 \in D(L)\).

Then the maximal solution of (3.1) is classical.

5. Exemples.

1) Let \(f \in C^s(\mathbb{R})\), put:

\begin{equation}
\int_{-\infty}^{+\infty} f(T) = \int_{-\infty}^{+\infty} f(\lambda) dE_\lambda, \quad \forall T \in H(E),
\end{equation}

\(E_\lambda\) being the spectral projector attached to \(T\).
If we choose $Q = H(E), B = A$ then $f$ fulfills (3.11) (cf. Tartar [8]) and (3.1) has a unique maximal solution.

Assume now

(5.2) $Q = \{T \in H(E); a < T < b\}, \quad a, b \in \mathbb{R}.$

**Lemma 5.1.** If $f(a) < 0$ and $f(b) > 0$ then $\forall r > 0, \exists \beta_r > 0$ such that:

(5.3) $|x| < r, \quad x > a, \quad \beta \in ]0, \beta_r[, \Rightarrow x - \beta f(x) > a.$

**Proof.** If $f(a) < 0$ the thesis is evident. Assume $f(a) = 0$; then it is $f(x) = (x - a) \psi(x)$ and if $x > a$ it is

$$x - a - \beta f(x) = (x - a)(1 - \beta \psi(x)) > 0$$

for suitable $\beta$. ≠

The following proposition is now evident.

**Proposition 5.2.** Assume that (2.1) is fulfilled with $B = A$. Assume moreover that $f \in C^2(\mathbb{R}), f(a) < 0, f(b) > 0$. Then if $a < U_0 < b$ there exists a unique global solution $U$ such that $a < U(t) < b$.

2) Riccati equation.

Assume $Q = H_+(E), B = A, |e^{tA}| < 1$ and (2.1) fulfilled; assume $f(T) = TPT - F(t)$ where $P > 0, F(t) > 0$; then it is easy to see that $f$ verifies (3.11); therefore (3.1) has a maximal solution in $Q$. Moreover it is

$$|T| < |T + \alpha TPT|, \quad \forall \alpha > 0, \forall T > 0,$$

because

$$(T + \alpha TPT)x, x \rangle > (Tx, x)$$

therefore if $U_0 > 0$ (3.1) has a global solution.

Finally assume $P \in \mathcal{L}(E)$, put $\overline{P} = ZPZ^{-1}$ then $f_\lambda(V) = V\overline{P}V$ and the hypotheses of the Theorem 4.1 are fulfilled and the solution is classical.
REFERENCES


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