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Periodic Solutions of a Differential Delay Equation of Rayleigh Type.

S. INVERNIZZI - F. ZANOLIN (*)

1. Introduction.

It is well-known that the ordinary differential equation of Rayleigh type

$$(R) \quad x''(t) + f(x'(t)) + g(x(t)) = h(t)$$

is physically significant. For instance, in the problem of vibrations of a suspended wire subjected to disturbances as wind (like an electrical transmission line), the periodic solutions of

$$x'' + |x'|x' + qx' + x - P^2x^3 = r \sin \omega t$$

are of interest (see Cecconi [1]). This suggests to study the existence of p -periodic solutions of the differential delay equation

$$(D) \quad x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = h(t, x(t + r(t)), x'(t + s(t)))$$

where the deviations σ , τ , r , s are p -periodic, and h is a bounded function, p -periodic in t . We assume that g is differentiable and we allow g' to change sign: hence we need some « Lyapunov-Schmidt »

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technique. In particular, we shall use a theorem from the coincidence degree theory (see Mawhin [3]). A particular feature of our existence result for (D) (Theorem 1) is that we require only the continuity of f , according to the fact that the differentiability of a damping term is not a reasonable physical requirement (see Utz [6]).

As a corollary of Theorem 1, we have an existence theorem of periodic solutions of ordinary differential equations (Corollary 1), which contains a result due to Reissig (see [5]).

At the end of the paper, we get an existence-uniqueness theorem (Theorem 2) for periodic solutions of (R) under a monotonicity condition for g and a regularity condition for f .

2. Preliminaries.

We call $x: R \rightarrow R$ a p -periodic function ($p > 0$) if, for every $t \in R$, $x(t + p) = x(t)$. We denote $C^i(p, R)$ ($i = 0, 1, 2$) the Banach space of all p -periodic functions $x: R \rightarrow R$ of class C^i , with the norm $x \rightarrow \sum_{k=0}^i |x^{(k)}|_\infty$, where $|\cdot|_\infty$ denotes the supremum norm. Moreover, if $x \in C^0(p, R)$, the symbol $|x|_2$ denotes the $L^2(0, p)$ -norm of x , i.e. $|x|_2 = \left(\int_0^p |x(t)|^2 dt \right)^{\frac{1}{2}}$, and the symbol $\delta(x)$ denotes the diameter of the set $x(R) \cup \{0\}$. Observe that δ is an equivalent norm for $C^0(p, R)$.

In [2] the following technical lemma is proved:

LEMMA 1. *Let $\tau \in C^0(p, R)$. Then the formula*

$$x(\cdot) \rightarrow \int_0^{\tau(\cdot)} x(\cdot + s) ds$$

defines a linear operator $G(\tau): C^0(p, R) \rightarrow C^0(p, R)$ such that for every x

$$|G(\tau)x|_2 \leq \delta(\tau)|x|_2.$$

3. Main results.

We denote

$$L_f = \sup_{\xi, \eta \in R; \xi \neq \eta} \left| \frac{f(\xi) - f(\eta)}{\xi - \eta} \right| \quad (\text{possibly } L_f = +\infty),$$

and we define similarly L_g . We assume the convention that

$$0 \cdot (+\infty) = 0.$$

THEOREM 1. *Let us consider the following equation*

$$(1) \quad x''(t) + f(x'(t + \sigma(t))) + g(x(t + \tau(t))) = \\ = h(t, x(t + r(t)), x'(t + s(t)))$$

where $f \in C^0(\mathbb{R}, \mathbb{R})$, $g \in C^1(\mathbb{R}, \mathbb{R})$, $h \in C^0(\mathbb{R}^3, \mathbb{R})$ and it is p -periodic in the first variable, and the delays $\sigma, \tau, r, s \in C^0(p, \mathbb{R})$. Assume that

- (i) h is bounded, $|h(t, x, x')| \leq M$,
- (ii) the derivative g' is bounded above, and the frequency $\omega = 2\pi/p$ satisfies $g'(\cdot) \leq K < \omega^2$ for some $K \in \mathbb{R}$.

If the norms $\delta(\sigma)$ and $\delta(\tau)$ are so small that

$$(iii) \quad \omega^2 L_f \delta(\sigma) + \omega L_g \delta(\tau) + K < \omega^2,$$

and if

$$(iv) \quad \lim_{|x| \rightarrow +\infty} g(x) \operatorname{sign} x = +\infty \text{ (or } -\infty)$$

then (1) has a least one p -periodic solution.

REMARK 1. In the ordinary case, i.e. when $\sigma = \tau = 0$, we do not require any Lipschitz condition on f or on g , since in this case the hypothesis (iii) means simply $K < \omega^2$. For instance, if $\sigma = \tau = 0$, we can assume $g(x) =$ a polynomial in x of odd order with negative leading coefficient, as in the classical Rayleigh equation where $g(x) = x - P^2 x^3$. In fact for a polynomial of this kind, the hypothesis (ii) and the hypothesis (iv) with the limit equal to $-\infty$, are always satisfied, for suitable p .

COROLLARY 1. *If $g \in C^1(\mathbb{R}, \mathbb{R})$ has its derivative bounded above by a constant $K < \omega^2$ ($\omega = 2\pi/p$), if $h \in C^0(\mathbb{R}^3, \mathbb{R})$ is a bounded function, p -periodic in the first variable, and if $\lim_{|x| \rightarrow +\infty} g(x) \operatorname{sign} x = +\infty$ (or $-\infty$), then the ordinary equation*

$$x'' + f(x') + g(x) = h(t, x, x')$$

has at least one p -periodic solution, whatever the function $f \in C^0(\mathbb{R}, \mathbb{R})$ may be.

PROOF. Put $\sigma = \tau = r = s = 0$ in Theorem 1, and use the convention $0 \cdot (+\infty) = 0$.

COROLLARY 2 (Reissig [5], Theorem 5). *The ordinary equation*

$$x'' + f(x') + Kx + \gamma(x) = e(t),$$

where f, γ, e are continuous and e is p -periodic, has at least one p -periodic solution when $0 < K < \omega^2$, $|\gamma(x)| \leq P$.

PROOF. Put $Kx = g(x)$, $e(t) - \gamma(x) = h(t, x)$, and use Corollary 1.

PROOF OF THEOREM 1. We use a result of coincidence degree theory. Let X_i ($i = 0, 1, 2$) be Banach spaces, $X_2 \subseteq X_1 \subseteq X_0$ with completely continuous embeddings. Let $L: X_2 \rightarrow X_0$ be a continuous linear Fredholm map of index zero. This means that $\text{im } L$ is closed and $\dim \ker L = \dim \text{coker } L < \infty$. As a consequence, we can find two continuous projections $P: X_1 \rightarrow \ker L$, $(I - Q): X_0 \rightarrow \text{im } L$. The restriction $L: X_2 \cap \ker P \rightarrow \text{im } L$ is bijective: we call K its inverse. Let $N: X_1 \rightarrow X_0$ be an L -completely continuous map: this means that $QN: X_1 \rightarrow X_0$ is continuous and maps bounded sets into bounded sets, and that $K(I - Q)N: X_1 \rightarrow X_1$ is completely continuous. Actually the map $A: X_1 \rightarrow X_0$, $Ax = Px$, is L -completely continuous. In fact, $QA: X_1 \rightarrow X_0$ and $K(I - Q)A: X_1 \rightarrow X_2$ are linear bounded (and the embedding $X_2 \rightarrow X_1$ is completely continuous). Moreover,

$$\ker(L - A) = \{0\}.$$

Then it follows directly from a theorem by Mawhin (see [3]) that if there exists $\varrho > 0$ such that $|x|_{X_1} < \varrho$ whenever $(\lambda, x) \in]0, 1[\times X_2$ satisfies

$$Lx = (1 - \lambda)Ax + \lambda Nx,$$

then the equation $Lx = Nx$ has at least one solution $x \in X_2$.

We shall apply this result with $X_i = C^i(p, R)$ ($i = 0, 1, 2$). We define $L: C^2(p, R) \rightarrow C^0(p, R)$, $(Lx)(t) = -x''(t)$. It is well known that L is a continuous linear Fredholm map of index zero. Moreover the projections

$$P: C^1(p, R) \rightarrow \ker L = \{\text{constants maps } R \rightarrow R\}$$

and

$$Q: C^0(p, R) \rightarrow \{\text{constants maps } R \rightarrow R\}$$

can be chosen as follows:

$$(Px)(t) = (1/p) \int_0^p x(\xi) d\xi, \quad (Qx)(t) = (1/p) \int_0^p x(\xi) d\xi.$$

We define $N: C^1(p, R) \rightarrow C^0(p, R)$

$$(Nx)(t) = f(x'(t + \sigma(t))) + g(x(t + \tau(t))) - h(t, x(t + r(t)), x'(t + s(t))).$$

Since f, g, h are continuous, and Q is linear bounded, we have easily that the composite map $QN: C^1(p, R) \rightarrow C^0(p, R)$ is continuous and maps bounded sets into bounded sets. Moreover $K(I - Q): C^0(p, R) \rightarrow C^2(p, R)$ is linear bounded; hence $K(I - Q)N: C^1(p, R) \rightarrow C^1(p, R)$ is completely continuous. It follows that N is L -completely continuous.

Now equation (1) has a p -periodic solution x if and only if the coincidence equation $Lx = Nx$ has a solution $x \in C^2(p, R)$. So, to prove the existence of a p -periodic solution of (1), in virtue of the Mawhin's theorem, we need only to show that there exists a constant $\varrho > 0$ such that, if $\lambda \in]0, 1[$ and $x \in C^2(p, R)$ verify

$$(2) \quad Lx = (1 - \lambda)Ax + \lambda Nx$$

(where $Ax = (1/p) \int_0^p x(\xi) d\xi$), then we have $|x'|_\infty + |x|_\infty < \varrho$.

First we prove the existence of a bound for $|x'|^\infty$. If we multiply (2) by $-x''$ and we integrate on $[0, p]$, we have easily

$$|x''|_2^2 = -\lambda \int_0^p (Nx)x'' dt.$$

We shall use now the definition of N , the boundedness of h (condition (i)), the upper bound of g' (condition (ii)), and, possibly, the

Lipschitz constants of f and g :

$$\begin{aligned} -\int_0^p (Nx) x'' dt &= -\int_0^p f(x'(t)) x''(t) dt - \int_0^p (f(x'(t + \sigma(t))) - f(x'(t))) x''(t) dt - \\ &\quad - \int_0^p g(x(t)) x''(t) dt - \int_0^p (g(x(t + \tau(t))) - g(x(t))) x''(t) dt + \\ &\quad + \int_0^p h(t, x(t + r(t)), x'(t + s(t))) x''(t) dt \leq \\ &\leq 0 + L_r |x'(\cdot + \sigma) - x'|_2 |x''|_2 + K |x'|_2^2 + L_\sigma |x(\cdot + \tau) - x|_2 |x''|_2 + Mp^\dagger |x''|_2. \end{aligned}$$

It follows from Lemma 1 that

$$\begin{aligned} |x'(\cdot + \sigma) - x'|_2 &= \left| \int_0^{\sigma(t)} x''(t + \xi) d\xi \right|_2 \leq \delta(\sigma) |x''|_2, \\ |x(\cdot + \tau) - x|_2 &= \left| \int_0^{\tau(t)} x'(t + \xi) d\xi \right|_2 \leq \delta(\tau) |x'|_2. \end{aligned}$$

Using the Wirtinger inequality $\omega |x'|_2 \leq |x''|_2$ we obtain, since $0 < \lambda < 1$,

$$|x''|_2^2 \leq -\int_0^p (Nx) x'' dt \leq \left(L_r \delta(\sigma) + \frac{1}{\omega} L_\sigma \delta(\tau) + \frac{1}{\omega^2} K \right) |x''|_2^2 + Mp^\dagger |x''|_2.$$

It follows from condition (iii) that $|x''|_2 \leq \text{const}$, and this implies, by an elementary argument, that there exists a constant $\alpha > 0$ such that

$$|x'|_\infty \leq \alpha.$$

In order to show the existence of a bound for $|x|_\infty$, we shall use the condition (iv). There is no loss of generality if we assume that $g(x)$ sign $x \rightarrow +\infty$ (as $|x| \rightarrow +\infty$). In fact, if $g(x)$ sign $x \rightarrow -\infty$, we have only to define the map $A: X_1 \rightarrow X_0$ in the « abstract » part by $Ax = -Px$ instead of $Ax = Px$, that is, for the « concrete » case, $(Ax)(t) = -(1/p) \int_0^p x(\xi) d\xi$. It is easy to see that, with this sign modification,

the a priori bound $|x'|_\infty \leq \alpha$ is still true, and that the a priori bound for $|x|_\infty$ we shall prove for the case $g(x) \text{ sign } x \rightarrow +\infty$ can be obtained, in the case $g(x) \text{ sign } x \rightarrow -\infty$, with the same argument.

We compute the average for both terms of (2): we have $-Qx' = (1 - \lambda)QAx + \lambda QNx$, that is

$$(3) \quad 0 = (1 - \lambda)Ax + \lambda QNx.$$

Claim. There exists $\beta > 0$ such that, for any $x \in C^2(p, R)$ which satisfies (3) with some $\lambda \in]0, 1[$,

$$|Ax| \leq \beta.$$

This statement guarantees the existence of a bound for $|x|_\infty$. In fact, for each $x \in C^1(p, R)$, for every $t \in [0, p]$, there exist two points ξ, η such that $x(t) = Ax + x'(\xi)(t - \eta)$. It follows that if x is a solution of (2) then $|x - Ax|_\infty \leq \alpha p$, and so, if the claim is true, we obtain $|x|_\infty \leq \alpha p + \beta$.

Let us assume our claim is false. We can find a suitable sequence of pairs $(\lambda_n, x_n) \in]0, 1[\times C^2(p, R)$ such that

- (j) for every n , $0 = (1 - \lambda_n)Ax_n + \lambda_n QNx_n$,
- (jj) the sequence λ_n is convergent to some point of the closed interval $[0, 1]$,
- (jjj) $Ax_n \rightarrow +\infty$ or $Ax_n \rightarrow -\infty$.

By definition, QNx_n is equal to the sum of the sequence

$$a_n = (1/p) \int_0^p g(x_n(t + \tau(t))) dt$$

and of another sequence of the form

$$b_n = (1/p) \int_0^p (f(x'(\dots)) - h(\dots)) dt.$$

Clearly b_n is bounded (by $\sup_{|x'| \leq \alpha} |f(x')| + M$). Let us consider a_n . We assume that the function g reaches its minimum, on the interval

$[Ax_n - \alpha p, Ax_n + \alpha p]$, at the point u_n , and its maximum on the same interval at the point v_n . Since

$$a_n = (1/p) \int_0^p g(x_n(t + \tau(t)) - Ax_n + Ax_n) dt,$$

and since

$$\sup_{t \in [0, p]} |x_n(t + \tau(t)) - Ax_n| \leq \sup_{t \in [0, p]} |x_n(t) - Ax_n| \leq \alpha p,$$

we obtain easily that $g(u_n) \leq a_n \leq g(v_n)$. Thus, if $Ax_n \rightarrow +\infty$, we must have $u_n \rightarrow +\infty$. It follows from condition (iv) that $g(u_n) \rightarrow +\infty$ and hence $a_n \rightarrow +\infty$. This is a contradiction with (j), since we have simultaneously $Ax_n \rightarrow +\infty$ and $QNx_n \rightarrow +\infty$. On the other hand, if $Ax_n \rightarrow -\infty$, we obtain $g(v_n) \rightarrow -\infty$ and $a_n \rightarrow -\infty$, which is again a contradiction with (j).

This proves our claim and completes the proof of the theorem.

As a consequence of Corollary 1 we obtain the result that the non-linear ordinary differential equation

$$(4) \quad x'' + f(x') + g(x) = h(t),$$

where $f \in C^0(R, R)$, $g \in C^1(R, R)$, $h \in C^0(p, R)$ has at least one p -periodic solution if $g'(\cdot) \leq K < 0$. In fact this condition implies that (ii) and (iv) hold.

A natural question arises: do the monotonicity condition $g'(\cdot) \leq K < 0$ imply the uniqueness of the periodic solution of (4)? We are able to give an affirmative answer provided that f satisfies only a regularity condition: f is of class C^1 . For instance, all the viscous dampings $f(x') = \beta|x'|^q \text{sign}(x')$, with $\beta > 0$, $q \geq 1$, can be considered.

THEOREM 2. *The ordinary differential equation*

$$(5) \quad x'' + f(x') + g(x) = h(t)$$

where h is continuous and p -periodic, $g \in C^1(R, R)$, and $g'(\cdot) \leq K < 0$, has exactly one p -periodic solution whatever $f \in C^1(R, R)$ may be.

PROOF. The existence follows from Corollary 1. Let us assume that x, y are p -periodic solutions of (5). Then the difference $z = x - y$ is a p -periodic function which satisfies the linear homogeneous equa-

tion

$$z''(t) + a(t)z'(t) + b(t)z(t) = 0,$$

where

$$a(t) = \int_0^1 f'(sx'(t) + (1-s)y'(t)) ds, \quad b(t) = \int_0^1 g'(sx(t) + (1-s)y(t)) ds,$$

are continuous coefficients with $b(\cdot) < 0$. Let us define the auxiliary function $w = e^A(z^2)'$, where $A(t) = \int_0^t a(s) ds$. We have

$$w' = 2e^A(z'^2 + z(z'' + az')) = 2e^A(z'^2 - bz^2) \geq 0,$$

hence w is increasing. We consider the set $N = \{t \in R: z'(t) = 0\}$. Since z is periodic, N is not empty, and $\inf N = -\infty, \sup N = +\infty$. But clearly $w(N) = \{0\}$, and thus the monotonicity of w implies that $w(t) = 0$ for every t . From the definition of w , it follows that $z = a$ constant. Now the condition $b < 0$ implies that if z is a constant solution of $z'' + az' + bz = 0$, then we must have $z = 0$.

REMARK 2. Theorem 2 can be proved using the Caccioppoli global inversion method (see [4]). In fact we can define a map

$$T: x \in C^2(p, R) \rightarrow x'' + f(x') + g(x) \in C^0(p, R)$$

and we need only to prove that T is proper and that at each point x the differential $DT(x)$ is bijective. The differential $DT(x)$ is a linear map defined by $DT(x)[v] = v'' + f'(x')v' + g'(x)v$. Since $g'(x) < 0$, the argument of Theorem 2 shows that $DT(x)$ is one-to-one, hence it is onto by the Fredholm Alternative. To prove the properness of T , we take the L^2 -inner product of $Tx = h$ with x'' : we have

$$\|x''\|_2^2 - \int_0^p g'(x(t))x'^2(t) dt = \int_0^p h(t)x''(t) dt.$$

It follows $\|x''\|_2^2 \leq K\|x'\|_2^2 + \|h\|_2\|x''\| \leq p^{\frac{1}{2}}\|h\|_\infty\|x''\|_2$. The usual technique yields that $\|x'\|_\infty$ and consequently $\|f(x')\|_\infty$ is bounded in terms of $\|h\|_\infty$. using $Tx = h$, we deduce that $\|g(x)\|_2$ is bounded. Now it is

easy to see that $|x|_2$ is bounded: in fact, for $s \neq 0$, we have $(g(s) - g(0))/s \leq K < 0$, and so $(g(s) - g(0))^2/s^2 \geq K^2 > 0$, that is $s^2 \leq (1/K)^2 \cdot (g(s) - g(0))^2$, or $s^2 \leq c_1|g(s)|^2 + c_2|g(s)| + c_3$, with $c_1 > 0$, $c_2, c_3 \geq 0$. This last inequality holds for every s . In particular, for $s = x(t)$, we can deduce that $|x|_2$ is bounded. An elementary argument shows that $|x|_\infty$ is bounded in terms of $|h|_\infty$. This implies that T is a proper map.

In this way we obtain the further result that *the unique p -periodic solution x of the equation (5) C^1 -depends upon the forcing term h .*

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