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Integral Characterization of Functionals Defined on Spaces of $BV$ Functions.

FRANCESCO FERRO (*)

SUMMARY - In [6] we extended in a suitable way a class of functionals defined on $W^{1,1}(\Omega)$ to the space $BV_b(\Omega) \oplus L^1(\partial \Omega)$. Here we give an integral characterization of the extended functional which is related with the functional defined in [10] in the one-dimensional case.

Introduction.

Many recent papers deal with the problem of defining variational functionals on spaces of functions of bounded variation. In [10] an integral functional defined on absolutely continuous functions in $[0, 1]$ is extended to the space of functions of bounded variation by means of the recession function of the integrand; the main result given in [10] is the characterization of optimal arcs in terms of a «generalized Hamiltonian condition».

In [1], [2], [3] the same integral functional is extended in an alternative method; however it is proved in [1] that under suitable hypothesis the extended functional agrees with the extension given in [10]; the same is proved in [3] by different hypothesis in the case of a non convex functional. In [1], [2] there are mainly given optimization

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theorems for the extended functional involving the boundedness of the level sets of the starting functional.

Analogous problems in the $n$-dimensional case have been studied in [4], [5], [6].

The main results are in [6] where an integral functional defined on $W^{1,1}(\Omega)$ is extended to the space $BV_b(\Omega) \oplus (C(\partial \Omega))^*$, where $BV_b(\Omega)$ is the space of functions in $L^1(\Omega)$ whose gradient is a measure with finite total variation in $\Omega$. Optimization theorems and applications to minimal surface problems are also given in [6].

The aim of this work is to give an integral characterization of the extended functional defined in [6]. In this way we emphasize the strict analogy of our results with the onedimensional case.

In Section 1 we give a survey of the functional background we developed in our preceding works and state some preliminary results of topological nature.

In Section 2 we give our main results; we remark that the hypothesis and the proof of Theorem 2.1 are quite similar to that used in [3].

1. Definitions and topological properties of some functional spaces.

Throughout this paper $\Omega$ will be an open, bounded and connected subset of $\mathbb{R}^n$, whose boundary $\partial \Omega$ verifies the local Lipschitz condition (in the sense of [7]). Let

$$BV_b(\mathbb{R}^n) = \{ u : u \in L^1_{\text{loc}}(\mathbb{R}^n), \nabla u \in (\mathcal{M}_b(\mathbb{R}^n))^* \},$$

where $\mathcal{M}_b(\mathbb{R}^n)$ is the space of all real-valued measures whose total variation is finite in $\mathbb{R}^n$. Let $C_0(\mathbb{R}^n)$ be the space of all continuous functions which have a compact support in $\mathbb{R}^n$; if we endow $C_0(\mathbb{R}^n)$ with the uniform convergence topology, $\mathcal{M}_b(\mathbb{R}^n)$ is its dual space (a Banach space) and

$$\| v \|_{\mathcal{M}_b(\mathbb{R}^n)} = \sup_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} fv : f \in C_0(\mathbb{R}^n), |f(x)| < 1 \right\}.$$ 

Then $\mathbb{R} \oplus (\mathcal{M}_b(\mathbb{R}^n))^*$ is the dual space of $\mathbb{R} \oplus (C_0(\mathbb{R}^n))^*$ and may be endowed with the weak topology of dual space (the so-called $w^*$ topology).
An element \( u \in BV_b(\mathbb{R}^n) \) may be identified with the couple
\[
\left( \int_{\Omega} u, \nabla u \right) \in \mathbb{R} \oplus (M_b(\mathbb{R}^n))^n;
\]
in this sense \( BV_b(\mathbb{R}^n) \) is a subspace of \( \mathbb{R} \oplus (M_b(\mathbb{R}^n))^n \).

As we proved in [4] \( BV_b(\mathbb{R}^n) \) is \( w^* \)-closed in \( \mathbb{R} \oplus (M_b(\mathbb{R}^n))^n \); hence it is closed also relative to the norm topology of \( \mathbb{R} \oplus (M_b(\mathbb{R}^n))^n \) (and so it is a Banach space relative to the norm topology); moreover we emphasize that the closed balls of \( BV_b(\mathbb{R}^n) \) are \( w^* \)-compact and their topology is metrizable.

A net \( \{u_\alpha\} \subset BV_b(\mathbb{R}^n) \) \( w^* \)-converges to \( u \in BV_b(\mathbb{R}^n) \) if and only if
\[
\lim_{\alpha} \int_{\Omega} u_\alpha = \int_{\Omega} u
\]
and
\[
\lim_{\alpha} \int_{\Omega} G \nabla u_\alpha = \int_{\Omega} G \nabla u, \quad \text{for every } G \in (C_0(\mathbb{R}^n))^n.
\]

Let
\[
E = \{ u \in BV_b(\mathbb{R}^n); \ u = 0 \ \text{a.e. in } \Omega \};
\]
\( E \) is \( w^* \)-closed (see [4], [5]). Let \( w_0^* \) be the quotient topology induced on \( BV_b(\mathbb{R}^n)/E \) by the \( w^* \) topology of \( BV_b(\mathbb{R}^n) \), that is the finest topology on \( BV_b(\mathbb{R}^n)/E \) such that the canonical mapping
\[
\pi: BV_b(\mathbb{R}^n) \to BV_b(\mathbb{R}^n)/E
\]
be continuous. It is well-known that \( \pi \) is an open mapping.

Now let
\[
BV_b(\Omega) = \{ u; u \in L^1(\Omega), \nabla u \in (M_b(\Omega))^n \},
\]
where \( M_b(\Omega) \) is the dual space of the space \( C_0(\Omega) \) of all continuous functions which have a compact support in \( \Omega \) (\( C_0(\Omega) \) has the uniform convergence topology).

\( BV_b(\Omega) \) is a Banach space if we put
\[
\| u \|_{BV_b(\Omega)} = \| u \|_{L^1(\Omega)} + \| \nabla u \|_{(M_b(\Omega))^n}.
\]
Let \( u \in BV_b(\mathbb{R}^n) \) and \([u]\) be its equivalence class in \( BV_b(\mathbb{R}^n)/E \); we
define
\[ i : BV_b(\mathbb{R}^n)/E \to BV_b(\Omega) \]
in the following way:
\[ i([u]) = r(u), \]
where \( r \) is the restriction operator. In [6] we proved that if \( BV_b(\mathbb{R}^n)/E \) is endowed with the strong quotient topology then \( i \) is an isomorphism between Banach spaces; so we may identify \( BV_b(\Omega) \) and \( BV_b(\mathbb{R}^n)/E \) and give the following definition (see [6]):

**DEFINITION 1.1.** A set \( D \subset BV_b(\Omega) \) is \( w_q^* \)-open if and only if \( i^{-1}(D) \) is \( w_q^* \)-open.

We remark that the closed balls of \( BV_b(\Omega) \) are \( w_q^* \)-compact and their induced topology is metrizable.

It follows by [5, Proposition 3.1] that if a sequence \( \{u_m\} \subset BV_b(\Omega) \) \( w_q^* \)-converges to \( u \in BV_b(\Omega) \) then \( \lim_{m \to +\infty} u_m = u \) in \( L^1(\Omega) \).

Now let \( f \in L^1(\partial \Omega) \); we may put
\[
\langle g, f \rangle_1 = \int_{\partial \Omega} \! g dH_{n-1}, \quad \text{for every } g \in C(\partial \Omega),
\]
where \( H_{n-1} \) is the \((n - 1)\)-dimensional Hausdorff measure on \( \partial \Omega \) and \( C(\partial \Omega) \) is the space of all continuous functions on \( \partial \Omega \); moreover we define
\[
\langle G, f \rangle_2 = \int_{\partial \Omega} \! f \nu dH_{n-1}, \quad \text{for every } G \in (C(\partial \Omega))^n,
\]
where \( \nu \) is the unit outer normal to \( \partial \Omega \).

In the sense of (1.1) \( L^1(\partial \Omega) \) is a subspace of \((C(\partial \Omega))^*\) while in the sense of (1.2) \( L^1(\partial \Omega) \) is a subspace of \(((C(\partial \Omega))^n)^*\) (a more detailed approach is in [6]). Let \( w_1^* \) and \( w_2^* \) be the weak topologies of dual space of \((C(\partial \Omega))^*\) and \(((C(\partial \Omega))^n)^*\) respectively. We proved in [6] that \( L^1(\partial \Omega) \) is \( w_1^* \)-dense in \((C(\partial \Omega))^*\) if \( \partial \Omega \) is of class \( C^1 \), while, without this hypothesis on \( \partial \Omega \), we called \( \mathcal{M}(\partial \Omega) \) the \( w_2^* \)-closure of \( L^1(\partial \Omega) \) in \(((C(\partial \Omega))^n)^*\).

If \( u \in W^{1,1}(\Omega) = \{ u : u \in L^1(\Omega), \nabla u \in (L^1(\Omega))^n \} \) and \( \gamma(u) \) is its trace in the sense of Sobolev spaces, we have \( (u, \gamma(u)) \in BV_b(\Omega) \oplus L^1(\partial \Omega) \).
In this sense we may write $W^{1,1}(\Omega) \subset BV_b(\Omega) \oplus L^1(\partial \Omega)$.

We proved (see [6]) that $W^{1,1}(\Omega)$ is $w^*_\omega \times w^*_\omega$-dense in $BV_b(\Omega) \oplus (\partial \Omega)^*$ if $\partial \Omega$ is of class $C^1$ and that, without this supplementary hypothesis, $W^{1,1}(\Omega)$ is $w^*_\omega \times w^*_\omega$-dense in $BV_b(\Omega) \oplus M(\partial \Omega)$.

In what follows the regularity hypothesis « $\partial \Omega$ of class $C^1$ » will be implicitly assumed whenever we shall deal with $w^*_\omega$-topology.

PROPOSITION 1.1. Let $\{(u_m, f_m)\} \subset BV_b(\Omega) \oplus L^1(\partial \Omega)$ be a sequence and $(u, f) \in BV_b(\Omega) \oplus L^1(\partial \Omega)$.

Then $(u_m, f_m) \xrightarrow{w^*_\omega \times w^*_\omega} (u, f)$ for $i = 1, 2$ if and only if the following conditions hold:

\begin{align*}
(i) \quad & \lim_{m \to +\infty} u_m = u \text{ in } L^1(\Omega); \\
(ii) \quad & \text{for every } u' \in BV_b(\mathbb{R}^n) \text{ such that } r(u') = u \text{ there exists a sequence } \{u'_m\} \subset BV_b(\mathbb{R}^n) \text{ such that } r(u'_m) = u_m \text{ and } \\
& \lim_{m \to +\infty} \int_{\mathbb{R}^n} G \nabla u'_m = \int_{\mathbb{R}^n} G \nabla u' \text{ for every } G \in (C_0(\mathbb{R}^n))^n; \\
(iii) \quad & \lim_{m \to +\infty} \int_{\partial \Omega} f_m G\nu dH_{n-1} = \int_{\partial \Omega} f G\nu dH_{n-1} \text{ for every } G \in (C(\partial \Omega))^n.
\end{align*}

PROOF. The sufficiency of conditions (1.3) is obvious by the continuity of the canonical mapping $\pi$. As to the necessity (1.3) (i) is proved in [5] and (1.3) (iii) follows by the definition. Afterwards there exists a constant $c > 0$ such that

$$
\|u_m\|_{BV_b(\Omega)} < c, \quad \text{for every } m,
$$

by the uniform boundedness theorem, that is $\{u_m\}$ is contained in a closed ball (which is $w^*_\omega$-compact and whose induced $w^*_\omega$ topology is metrizable) of $BV_b(\Omega)$. Since $\pi$ is an open mapping $(\pi \circ i)^{-1}(\{u_m\})$ is $w^*$-relatively compact and contained in a closed ball (which is $w^*$-compact and whose induced $w^*$ topology is metrizable) of $BV_b(\mathbb{R}^n)$.

Now (1.3) (ii) holds by standard topological arguments.

Now we recall some results about traces of $BV$ functions (see the References in [5], [6]).

If $u \in BV_b(\mathbb{R}^n)$ then there exist $\gamma^-(u), \gamma^+(u) \in L^1(\partial \Omega)$ such that

$$
\int_{\partial \Omega} G \nabla u + \int_{\partial \Omega} u \text{ div } G = \int_{\partial \Omega} \gamma^+(u) G\nu dH_{n-1} , \text{ for every } G \in (C_0(\mathbb{R}^n))^n,
$$

which is

$$
\int_{\partial \Omega} G \nabla u + \int_{\partial \Omega} u \text{ div } G = \int_{\partial \Omega} \gamma^+(u) G\nu dH_{n-1} , \text{ for every } G \in (C_0(\mathbb{R}^n))^n.
$$


and
\begin{equation}
\int_{\partial} G \nabla u + \int u \, \text{div} \, G = \int_{\partial} \gamma^-(u) G \nu \, dH_{n-1}, \quad \text{for every } G \in (C_0^1(\mathbb{R}^n))^n;
\end{equation}
\( \gamma^+(u) \) and \( \gamma^-(u) \) are called respectively the outer and inner trace of \( u \) on \( \partial \Omega \).

If \( u \in BV_b(\Omega) \) we may deal only with \( \gamma^-(u) \); moreover if \( u \in W^{1,1}(\Omega) \) we have \( \gamma^-(u) = \gamma(u) \). We shall use the following notations: if \( u \in BV_b(\Omega) \) and \( f \in L^1(\partial \Omega) \) then \( u_r \) will be any function in \( BV_b(\mathbb{R}^n) \) such that \( u_r = u \) in \( \Omega \) and \( \gamma^+(u_r) = f \).

**Theorem 1.1.** Conditions (1.3) hold if and only if the following conditions hold:

\begin{equation}
\begin{cases}
(i) \lim_{m \to +\infty} u_m = u \text{ in } L^1(\Omega); \\
(ii) \lim_{m \to +\infty} \int_{\partial} G \nabla (u_m)_r = \int_{\partial} G \nabla u_r, \quad \text{for every } G \in (C(\bar{\Omega}))^n; \\
(iii) \lim_{m \to +\infty} \int_{\partial} f_m G \nu \, dH_{n-1} = \int_{\partial} f G \nu \, dH_{n-1}, \quad \text{for every } G \in (C(\partial \Omega))^n.
\end{cases}
\end{equation}

**Proof.** Let (1.3) hold; then we must prove (1.6) (ii).

If \( G \in (C^1(\bar{\Omega}))^n \) by (1.4) we have
\[ \int_{\partial} G \nabla (u_m)_r = -\int_{\partial} u_m \, \text{div} \, G + \int_{\partial} f_m G \nu \, dH_{n-1}; \]
hence by (1.3) (i) and (1.3) (iii) we obtain
\begin{equation}
\lim_{m \to +\infty} \int_{\partial} G \nabla (u_m)_r = -\int_{\partial} u \, \text{div} \, G + \int_{\partial} f G \nu \, dH_{n-1} = \int_{\partial} G \nabla u_r,
\end{equation}
for every \( G \in (C^1(\bar{\Omega}))^n \).

We have also
\begin{equation}
\| \nabla (u_m)_r \|_{(C(\bar{\Omega}))^n} \leqslant \| u_m \|_{BV_b(\Omega)} + \int_{\partial \Omega} |f_m - \gamma^-(u_m)| \, dH_{n-1} \leqslant \| u_m \|_{BV_b(\Omega)} + \int_{\partial \Omega} |f_m| \, dH_{n-1} + \int_{\partial \Omega} |\gamma^-(u_m)| \, dH_{n-1} \leq \text{const} \| u_m \|_{BV_b(\Omega)} + \| f_m \|_{L^1(\partial \Omega)}.
\end{equation}
The right hand side of (1.8) is bounded by (1.3) (ii) and (1.3) (iii),
then (1.6) (ii) is obtained by (1.7) using standard approximation tech-
niques. Now let (1.6) hold; we must prove (1.3) (ii).

We have

\[
\int_{\Omega} G \nabla (u_m)_{f_m} = \int_{\Omega} G \nabla u_m + \int_{\partial \Omega} (f_m - \gamma^-(u_m)) Gv \, dH_{n-1}
\]

for every \( G \in (C(\overline{\Omega}))^n \)

and

\[
\int_{\Omega} G \nabla u_f = \int_{\Omega} G \nabla u + \int_{\partial \Omega} (f - \gamma^-(u)) Gv \, dH_{n-1}
\]

for every \( G \in (C(\overline{\Omega}))^n \).

Using (1.6) (ii) and (1.6) (iii) in (1.9) we obtain by (1.10):

\[
\lim_{m \to +\infty} \left( \int_{\Omega} G \nabla u_m - \int_{\partial \Omega} \gamma^-(u_m) Gv \, dH_{n-1} \right) = \int_{\Omega} G \nabla u - \int_{\partial \Omega} \gamma^-(u) Gv \, dH_{n-1},
\]

for every \( G \in (C(\overline{\Omega}))^n \).

Now we take \( u' \in BV_b(\mathbb{R}^n) \) such that \( r(u') = u \); let \( u'_m = u_m \) in \( \Omega \) and \( u'_m = u' \) in \( \mathbb{R}^n - \Omega \). If \( G \in (C_0(\mathbb{R}^n))^n \) we have

\[
\int_{\mathbb{R}^n} G \nabla u'_m = \int_{\mathbb{R}^n} G \nabla u' + \int_{\Omega} G \nabla u_m + \int_{\partial \Omega} (\gamma^+(u') - \gamma^-(u_m)) Gv \, dH_{n-1},
\]

then by (1.11)

\[
\lim_{m \to +\infty} \int_{\mathbb{R}^n} G \nabla u'_m = \int_{\mathbb{R}^n} G \nabla u' + \int_{\Omega} G \nabla u + \int_{\partial \Omega} (\gamma^+(u') - \gamma^-(u)) Gv \, dH_{n-1} = \int_{\mathbb{R}^n} G \nabla u',
\]

for every \( G \in (C_0(\mathbb{R}^n))^n \).

Remark 1.1. It is easily seen that if a net \( \{u_\alpha, f_\alpha\} \subset BV_b(\Omega) \oplus \oplus L^1(\partial \Omega) \) verifies (1.6) then it verifies (1.3) and if it verifies (1.3) then \( (u_\alpha, f_\alpha) \rightharpoonup (u, f) \); that is the «if part» in Proposition 1.1 and in Theorem 1.1 is true not only for sequences but also for nets.
Theorem 1.1 characterizes the $w^*_q \times w^*_l$-convergence of sequences in $BV_b(\Omega) \oplus L^1(\partial \Omega)$. As easy consequences we obtain

**Corollary 1.1.** Let $\{u_n\} \subset BV_b(\Omega)$ be a sequence and $u \in BV_b(\Omega)$. Then $u_n \overset{w^*_q}{\to} u$ if and only if $\lim_{m \to +\infty} u_m = u$ in $L^1(\Omega)$ and for every $\varphi \in L^1(\partial \Omega)$ we have

$$\lim_{m \to +\infty} \int_\Omega G \nabla (u_m) \varphi = \int_\Omega G \nabla u \varphi \quad \text{for every } G \in (C(\overline{\Omega}))^n.$$ 

**Corollary 1.2.** Let $u_n \overset{w^*_q}{\to} u$, then $u'_m \overset{w^*_q}{\to} u'$, where $u'_m = u_m$ in $\Omega$, $u'_m = 0$ in $\Omega^c$, $u' = u$ in $\Omega$, $u' = 0$ in $\Omega^c$.

**Proof.** Let $G \in (C_0(\mathbb{R}^n))^n$; we have

$$\lim_{m \to +\infty} \int_{\mathbb{R}^n} G \nabla u'_m = \lim_{m \to +\infty} \int_{\mathbb{R}^n} G \nabla u'_m = \int_{\mathbb{R}^n} G \nabla u' = \int_{\mathbb{R}^n} G \nabla u'.$$

In the final part of this Section we give an other characterization of the sequential $w^*_q$-convergence. However in the next Section we shall not use these results.

Let us consider the imbedding

$$j: BV_b(\Omega) \to \mathbb{R}^n \oplus (M_b(\Omega))^n,$$

where $j(u) = (\int_\Omega u, \nabla u)$. It is easily seen that $j$ is an injective continuous linear mapping between Banach spaces. Moreover $\mathbb{R} \oplus (M_b(\Omega))^n$ is the dual space of $\mathbb{R} \oplus (C_b(\Omega))^n$; then it may be endowed with the weak topology of dual space (which will be noted $\check{w}^*$ topology) and so we may define the following induced topology on $BV_b(\Omega)$: a net $\{u_\alpha\} \subset BV_b(\Omega)$ $\check{w}^*$-converges to $u \in BV_b(\Omega)$ if and only if $j(u_\alpha)$ $\check{w}^*$-converges to $j(u)$, that is if and only if

$$\lim_\alpha \int_\Omega u_\alpha = \int_\Omega u \quad \text{and} \quad \lim_\alpha \int_\Omega G \nabla u_\alpha = \int_\Omega G \nabla u \quad \text{for every } G \in (C_b(\Omega))^n.$$

It is obvious that the balls of $j(BV_b(\Omega))$ are relatively $\check{w}^*$-compact; we shall prove that they are $\check{w}^*$-compact.
If $u \in BV_b(\Omega)$ and $u_h$ are its integral averages (e.g. see [7]) we have $u_h \overset{w^*}{\to} u$; then

**PROPOSITION 1.2.** $W^{1,1}(\Omega)$, as a subset of $BV_b(\Omega)$, is $\overset{w^*}{\to}$-dense in $BV_b(\Omega)$. ■

Now we may prove:

**THEOREM 1.2.** Let $\{u_m\} \subset BV_b(\Omega)$ be a sequence which $\overset{w^*}{\to}$-converges to $(a, \mu) \in \mathbb{R} \oplus (M_b(\Omega))^n$; then there exists $u \in BV_b(\Omega)$ such that $(a, \mu) = (\int u, \nabla u)$ and $\lim_{m \to +\infty} u_m = u$ in $L^1(\Omega)$ (in particular the balls of $j(BV_b(\Omega))$ are $\overset{w^*}{\to}$-compact).

**PROOF.** By Proposition 1.2 we may suppose $c$ and by the uniform boundedness theorem there exists $\epsilon > 0$ such that

$$\left| \int_{\Omega} u_m \right| \leq \epsilon \quad \text{and} \quad \|\nabla u_m\|_{(L^1(\Omega))^n} = \|\nabla u_m\|_{(M_b(\Omega))^n} \leq \epsilon.$$

By Poincaré's inequality we have also

$$\|u_m\|_{(L^1(\Omega))^n} \leq c_1,$$

for a suitable constant $c_1 > 0$.

Then, by a well-known strong compactness criterion in $L^1(\Omega)$ we may say that, given any subsequence $\{u_{s_i}\}$ of $\{u_m\}$, there exists a subsequence $\{u_s\}$ of $\{u_{s_i}\}$ and $u \in BV_b(\Omega)$ such that $\lim_{s \to +\infty} u_s = u$ in $L^1(\Omega)$; then $\int u = a$, $\nabla u_s \to \nabla u$ in the sense of distributions and so $\nabla u = \mu$

Hence $u$ is the same for every $\{u_{s_i}\}$ and $\{u_s\}$. The proof is complete since the $\overset{w^*}{\to}$ topology is metrizable on the balls. ■

**THEOREM 1.3.** Let $\{u_m\} \subset BV_b(\Omega)$ be a sequence and $u \in BV_b(\Omega)$. Then $\{u_m\}$ $\overset{w^*}{\to}$-converges to $u$ if and only if $\{u_m\}$ $\overset{w^*}{\to}$-converges to $u$.

**PROOF.** The «only if part» is an obvious consequence of Corollary 1.1.

As to the «if part» let $G \in (C^1(\bar{\Omega}))^n$ and $\varphi \in L^1(\partial \Omega)$; we have by
Theorem 1.2:

\[
\lim_{m \to +\infty} \int_{\Omega} G \nabla(u_m) \varphi = \lim_{m \to +\infty} \left( \int_{\partial \Omega} q G \nabla u_m \cdot \nu \right) = \int_{\partial \Omega} q G \nabla u \cdot \nu = \int_{\Omega} g \nabla u \cdot \nu = \int_{\Omega} G \nabla u \varphi.
\]

Afterwards, since \( \|\nabla u_m\|_{(L^1(\Omega))^n} < \text{const} \), we have also

\[
\lim_{m \to +\infty} \int_{\Omega} G \nabla(u_m) \varphi = \int_{\Omega} G \nabla u \varphi,
\]

for every \( G \in (C(\overline{\Omega}))^n \), and the proof is complete by Theorem 1.1.

We wish to remark that Theorem 1.2 and Theorem 1.3 could also allow us to approach the problems considered in [4], [5], [6] by an alternative, and perhaps simpler, method.

2. Integral characterization.

Let

\[
L: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{ + \infty \}
\]

be a proper normal integrand, that is

\[
L(x, \cdot, \cdot) \text{ is lower semicontinuous for every } x \in \overline{\Omega};
\]

(ii) \( L(x, \cdot, \cdot) \) is not identically \( + \infty \);

(iii) \( E_L(x) = \{(u, v, \alpha): L(x, u, v) < \alpha\} \) is a measurable multifunction, i.e. \( E_L^{-1}(C) = \{x: E_L(x) \cap C \neq \emptyset\} \) is Lebesgue measurable for every \( C \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \), \( C \) closed.

We remark that \( L(x, u(x), v(x)) \) is measurable whenever \( u \) and \( v \) are measurable (see [11] for an extensive study about normal integrands).

We put

\[
I_L(u) = \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx, \quad u \in W^{1,1}(\Omega).
\]
$I_L(u)$ is well-defined if $L(x, u(x), \nabla u(x))$ is summable; otherwise we put $I_L(u) = -\infty$ if $L(x, u(x), \nabla u(x))$ is majorized by a summable function and $I_L(u) = +\infty$ in every other case.

We always suppose that there exists $u \in W^{1,1}(\Omega)$ such that $I_L(u) \in \mathbb{R}$. In [6] we defined the functionals

$$J_1(u, \mu) = \min \{\liminf s \in W^{1,1}(\Omega) : \{u_s\} \text{ is a net} \}
\quad (u_s, \gamma(u_s)) \overset{w^* \times w_2^*}{\longrightarrow} (u, \mu), \quad (u, \mu) \in BV_b(\Omega) \oplus (C(\partial \Omega))^*$$

and

$$J_2(u, \mu) = \min \{\liminf s \in W^{1,1}(\Omega) : \{u_s\} \text{ is a net} \}
\quad (u_s, \gamma(u_s)) \overset{w^* \times w_2^*}{\longrightarrow} (u, \mu), \quad (u, \mu) \in BV_b(\Omega) \oplus M(\partial \Omega).$$

We have (see [6])

$$J_1(u, f) = J_2(u, f) , \quad \text{for every } (u, f) \in BV_b(\Omega) \oplus L^1(\partial \Omega).$$

Now let $H(x, u, \cdot)$ be the Fenchel conjugate of $L(x, u, \cdot)$, i.e.

$$H(x, u, p) = \sup \{pv - L(x, u, v) : v \in \mathbb{R}^n\}$$

and

$$P_u(x) = \{p \in \mathbb{R}^n : H(x, u, p) < +\infty\}.$$

**Lemma 2.1.** Let $L$ be a proper normal integrand and $\sigma_0 : \overline{\Omega} \times \mathbb{R}_+ \to \mathbb{R}$ such that $\sigma_0(\cdot, r) \in L^1(\Omega)$ and

$$\sup \{|L(x, u, v) - L(x, u_1, v) : |u| < r, |u_1| < r, v \in \mathbb{R}^n\} < \sigma_0(x, r).$$

Then $P_u(x)$ is independent of $u$ (in this case we shall write $P_u(x) = P(x)$).

**Proof.** Let $p \in P_u(x)$, $u_1 \in \mathbb{R}$ and $r > 0$ such that $|u| < r$ and $|u_1| < r$. By (2.2) we have

$$L(x, u, v) < L(x, u_1, v) + \sigma_0(x, r),$$

$$pv - L(x, u_1, v) < pv - L(x, u, v) + \sigma_0(x, r)$$
and so

\[ H(x, u_1, p) \leq H(x, u, p) + \sigma_0(x, r) < + \infty; \]

then \( p \in P_u(x) \).

If (2.2) holds we may put (see [3])

\[ r_L(x, z) = \sup \{ pz : p \in P(x) \}. \]

**Lemma 2.2.** Let \( L \) be a proper normal integrand and (2.2) hold; if there exist \( K_1 > 0 \) and \( \theta_1 : \bar{\Omega} \to \mathbb{R} \) such that \( \theta_1 \in L^1(\Omega) \) and

\[ L(x, u, v) \geq K_1|v| - \theta_1(x), \quad \text{for every } (x, u, v) \in \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n, \]

then \( \text{int } P(x) \neq \emptyset \).

**Proof.** We have

\[ pv - L(x, u, v) \leq pv - K_1|v| + \theta_1(x) \]

and

\[ H(x, u, p) \leq \theta_1(x), \quad \text{if } |p| < K_1, \]

then

\[ \{ p : |p| < K_1 \} \subset P(x) . \]

In what follows, if \( \mu \in (C(\bar{\Omega}))^* \) we write \( \mu = \mu_a + \mu_s \), where \( \mu_a \) is absolutely continuous relative to Lebesgue measure and \( \mu_s \) is the singular part of \( \mu \) (relative to Lebesgue measure); \( d\mu_a/dx \) will be the Radon-Nykodim derivative of \( \mu_a \) relative to Lebesgue measure. If \( (u, f) \in BV_1(\Omega) \oplus L^1(\partial \Omega) \) and \( \mu = \nabla(u) \) we shall write \( \mu_a = \nabla_s u(x) \, dx \) and \( \mu_s = \nabla_s u \), where \( \nabla_s u(x) \) is the gradient of \( u \) in the elementary sense \( (\nabla_s u(x) \text{ exists a.e. in } \Omega) \); in this case we have \( d\mu_a/dx = \nabla_s u(x) \); if \( u \in W^{1,1}(\Omega) \) and \( \gamma(u) = f \) we have \( \mu_s = 0 \).

The following theorem gives a comparison between \( J_i \), \( i = 1, 2 \), and an integral functional related with the so-called recession function \( r_L \).

**Theorem 2.1.** Let \( L \) be a proper normal integrand and the fol-
lowing statements hold:

(i) there exists a summable function \( \sigma: \Omega \to \mathbb{R} \) such that
\[
\sup \{|L(x, u, v) - L(x, u_1, v)|: u, u_1 \in \mathbb{R}, v \in \mathbb{R}^n\} < \sigma(x);
\]
(ii) there exist \( K_1 > 0 \) and \( \theta_1: \mathcal{O} \to \mathbb{R} \) such that \( \theta_1 \in L^1(\Omega) \) and
\[
L(x, u, v) > K_1 |v| - \theta_1(x), \text{ for every } (x, u, v) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^n;
\]
(iii) \( G = \text{int cl } G, \) where \( G = \{(x, p): p \in \text{int } P(x)\} \);
(iv) there exists \( u_0 \in \mathbb{R} \) such that
\[
\int \frac{1}{v} |H(x, u_0, p)| \, dx < +\infty \text{ if } V \text{ is an open set and } p \in \mathbb{R}^n \text{ has a neighborhood } U \text{ contained in } P(x);
\]
(v) \( \limsup_{u \to \bar{u}} \{ |L(x, u, v) - L(x, \bar{u}, v)|: v \in \mathbb{R}^n \} = 0, \) for every \( x \in \Omega \) and \( \bar{u} \in \mathbb{R} \);
(vi) either the level sets \( \{ u: I_L(u) < z \} \) are bounded in \( W^{1,1}(\Omega) \)
or \( L = L(x, v) \) and the sets \( \left\{ u: I_L(u) < z, \int_B u = 0 \right\} \) are bounded in \( W^{1,1}(\Omega) \);
(vii) \( L(x, u, \cdot) \) is convex for every \( (x, u) \in \mathcal{O} \times \mathbb{R} \).

Then
\[
\int_{\Omega} L(x, u(x), \nabla u(x)) \, dx + \int_{\Omega} r_L \left( x, \frac{d\nabla_i(u_i)}{dq} (x) \right) q(\, dx) \leq J_i(u, f), \quad i = 1, 2,
\]
for every \( (u, f) \in BV_b(\Omega) \oplus L^1(\partial \Omega) \), where \( q \) is a non-negative measure relative to which \( \nabla u \) is absolutely continuous.

**Proof.** Since (2.4) (i) implies (2.2), by Lemma 2.1 \( P(x) \) is independent of \( u \) and \( r_L \) is well-defined by (2.3).

Afterwards let \( \bar{u} \) be a measurable function and define \( \phi_{u}(x, v) = L(x, \bar{u}(x), v) \). It is known ([11, Corollary 2P]) that \( \phi_u \) is a normal integrand on \( \mathcal{O} \times \mathbb{R}^n \).

Now we prove that there exists \( v_1 \in (L^1(\Omega))^n \) such that
\[
\int_{\Omega} \phi_{u}(x, v_1(x)) \, dx \in \mathbb{R};
\]
by the general hypothesis made on $I_L$ there exists $u_1 \in W^{1,1}(\Omega)$ such that

$$\int_{\Omega} L(x, u_1(x), \nabla u_1(x)) \, dx \in \mathbb{R}$$

and by (2.4) (i) we have

$$\left| \int_{\Omega} L(x, \bar{u}(x), \nabla u_1(x)) \, dx \right| \leq \int_{\Omega} \sigma(x) \, dx + \left| \int_{\Omega} L(x, u_1(x), \nabla u_1(x)) \, dx \right| \in \mathbb{R}.$$  

Then we may assume $v_1 = \nabla u_1$.

By [11, Proposition 2S] $\psi_u(x, v) = H(x, \bar{u}(x), v)$ is a normal integrand. Then the hypothesis of [11, Theorem 3C] are fulfilled by $\phi_u$ and $\psi_u$ and we have

(2.5) \[ \int_{\Omega} H(x, \bar{u}(x), f(x)) \, dx = \sup \left\{ \int_{\Omega} f(x) \nu(x) \, dx - \int_{\Omega} L(x, \bar{u}(x), \nu(x)) \, dx : \nu \in (L^1(\Omega))^n \right\} \]

for every $f \in (L^\infty(\Omega))^n$ and measurable function $\bar{u}$.

In this case we are interested with $\bar{u} \in BV_b(\Omega)$ and $f \in (C(\overline{\Omega}))^n$.

We put

(2.6) \[ F(\mu) = \begin{cases} \int_{\Omega} L(x, \bar{u}(x), \mu(x)) \, dx, & \mu \in (L^1(\Omega))^n, \\ +\infty, & \mu \in ((C(\overline{\Omega}))^n)^* - (L^1(\Omega))^n. \end{cases} \]

The Fenchel conjugate function of $F$ is

$$F^*(f) = \sup \left\{ \int_{\Omega} f \mu - F(\mu) : u \in ((C(\overline{\Omega}))^n)^* \right\}, \quad f \in (C(\overline{\Omega}))^n,$$

and, by (2.5), (2.6),

$$F^*(f) = \int_{\Omega} H(x, \bar{u}(x), f(x)) \, dx, \quad f \in (C(\overline{\Omega}))^n.$$
We have also

\[ F^{**}(\mu) = \sup \left\{ \int_{\Omega} f \mu - \int_{\Omega} H(x, \bar{u}(x), f(x)) \, dx ; f \in ((C(\bar{\Omega}))^*) \right\}. \]

We observe that as an easy consequence of (2.4) (i) we have

\[ |H(x, \bar{u}(x), p)| < |H(x, u_0, p)| + \sigma(x), \]

where \( u_0 \) is given in (2.4) (iv), and so \( \int_V |H(x, \bar{u}(x), p)| \, dx \) is finite whenever \( \int_V |H(x, u_0, p)| \, dx \) is so. Then by (2.4) (ii), (iii), (iv), (vii) and Lemma 2.2 we may apply \([9, \text{Theorem 5}]\); we obtain by (2.7):

\[ F^{**}(\mu) = \int_{\Omega} \left( L \left( x, \bar{u}(x), \frac{d\mu_{\alpha}}{dx} (x) \right) + q \right) \, dx, \]

for every \( \mu \in ((C(\bar{\Omega}))^*)^* \), where \( q \) is a non-negative measure relative to which \( \mu_{\alpha} \) is absolutely continuous.

Since \( F \) is a convex functional on \((C(\bar{\Omega}))^*)^*\) and \( F(\nabla u_1) \in \mathbb{R} \), by \([8]\) and (2.6) we have

\[ F^{**}(\mu) = \min \left\{ \liminf_{\alpha} F(v_{\alpha}) ; \{v_{\alpha}\} \subset (L^1(\Omega))^n, \{v_{\alpha}\} \text{ is a net,} \right\} \]

\[ \int_{\Omega} G v_{\alpha} \rightarrow \int_{\Omega} G \mu \text{ for every } G \in (C(\bar{\Omega}))^n \] .

We have also

\[ F^{**}(\mu) = \min \left\{ \liminf_{m \to +\infty} F(v_m) ; \{v_m\} \subset (L^1(\Omega))^n, \{v_m\} \text{ is a sequence,} \right\} \]

\[ \int_{\Omega} G v_m \rightarrow \int_{\Omega} G \mu \text{ for every } G \in (C(\bar{\Omega}))^n ; \]

if \( F^{**}(\mu) = +\infty \) (2.10) follows obviously by (2.9); if \( F^{**}(\mu) < M < +\infty \), \( \int_{\Omega} G v_{\alpha} \rightarrow \int_{\Omega} G \mu \) and \( \lim F(v_{\alpha}) < M \), then \( F(v_{\alpha}) < M + 1 \), whenever \( \alpha > \bar{\alpha} \), for a suitable \( \bar{\alpha} \). By (2.4) (ii) we have \( \|v_{\alpha}\|_{(L^1(\Omega))^n} < \text{const} \); then the value \( F^{**}(\mu) \) depends only on the elements of the ball whose radius is \( M + 1 \). Since the topology we consider on this ball is metrizable, (2.10) holds.
Afterwards if \((u, f) \in BV_0(\Omega) \oplus L^1(\partial \Omega)\) and if we put \(\bar{u} = u, \mu = -\nabla(u_i)\) in (2.10), then we obtain by Theorem 1.1

\[
(2.11) \quad F^{**}(\nabla(u_i)) \leq \liminf_{m \to +\infty} \int_{\Omega} L(x, u(x), \nabla u_m(x)) \, dx: \{u_m\} \subset W^{1,1}(\Omega),
\]

\(\{u_m\}\) is a sequence, \((u_m, \gamma(u_m)) \overset{w_1^s \times w_1^s}{\longrightarrow} (u, f)\).

\[
< \min \left\{ \liminf_{m \to +\infty} \int_{\Omega} L(x, u_m(x), \nabla u_m(x)) \, dx + \int_{\Omega} L(x, u(x), \nabla u_m(x)) \, dx - \int_{\Omega} L(x, u_m(x), \nabla u_m(x)) \, dx: \{u_m\} \subset W^{1,1}(\Omega), \{u_m\} \text{ is a sequence,}
\]

\[
(u_m, \gamma(u_m)) \overset{w_1^s \times w_1^s}{\longrightarrow} (u, f)\right\}.
\]

We consider the sequence

\[
a_m = \left| \int_{\Omega} \left( \frac{L(x, u(x), \nabla u_m(x)) - L(x, u_m(x), \nabla u_m(x)) \, dx - L(x, u_m(x), \nabla u_m(x)) \, dx}{\omega_1^s \times \omega_1^s} \right) (u, f) \right|
\]

let \(\{a_m\}\) be a subsequence of \(\{a_m\}\) and \(\{a_{m_s}\}\) a subsequence of \(\{a_m\}\) such that \(\lim_{s \to +\infty} u_{m_s} = u\) a.e. in \(\Omega\). By (2.4) (i) we may use Fatou's lemma and obtain

\[
\limsup_{s \to +\infty} a_{m_s} \leq \int_{\Omega} \limsup_{s \to +\infty} |L(x, u(x), \nabla u_{m_s}(x)) - L(x, u_{m_s}(x), \nabla u_{m_s}(x))| \, dx = 0
\]

by (2.4) (v).

So by a standard argument we have

\[
\lim_{m \to +\infty} a_m = 0
\]

and

\[
(2.12) \quad F^{**}(\nabla(u_i)) \leq \min_{m \to +\infty} \int_{\Omega} L(x, u_m(x), \nabla u_m(x)) \, dx: \{u_m\} \text{ as in (2.11)} = J_i(u, f),
\]

where the last equality follows by (2.4) (vi) (see [6, Lemma 4.1]).

A comparison between (2.8) and (2.12) completes the proof.
REMARK 2.1. If there exist $K > 0$ and $\theta \in L^1(\Omega)$ such that

$$L(x, u, v) \geq K(|u| + |v|) - \theta(x)$$

for every $(x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ then (2.4) (vi) holds. If $L = L(x, v)$ then (2.4) (ii) implies (2.4) (vi). □

The following lemma is similar to [12, Lemma 2] we used also in [5] and [6]. However our statement needs a completely different starting method in the proof.

LEMMA 2.3. Let $\lambda$ and $\zeta$ be non-negative, continuous functions defined on $[0, +\infty)$ such that $\lambda(0) = \zeta(0) = 0$; moreover we suppose that there exists a constant $c$ such that $\zeta(t) < ct$ for large $t$. Let

$$L: \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

satisfy

$$L \in C(\Omega \times \mathbb{R} \times \mathbb{R}^n) ,$$

$$L(x, u, \cdot) \text{ is convex for every } (x, u) \in \Omega \times \mathbb{R} ,$$

$$L(x, u, v) \geq -\varphi(x), \text{ for every } (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}^n ,$$

where $\varphi > 0$ and $|\varphi(x) - \varphi(x_1)| < \lambda(|x - x_1|)$ if $x, x_1 \in \Omega$,

$$|L(x, u, v) - L(x_1, u_1, v)| < \lambda(|x - x_1|)[1 + L^+(x, u, v)] + \zeta(|x - x_1|) ,$$

for every $x, x_1 \in \Omega$, $u, u_1 \in \mathbb{R}$ and $v \in \mathbb{R}^n$.

Let $\overline{u} \in L^1_{\text{loc}}(\Omega) \cap L^1(\Omega)$, $(u, \varphi) \in BV_\text{loc}(\Omega) \oplus L^1(\partial \Omega)$ and $\{v_m\} \subset (L^1(\Omega))^n$ be a sequence such that

$$\lim_{m \to +\infty} \int_\Omega G v_m = \int_\Omega G \nabla (u_\varphi) , \quad \text{for every } G \in (C(\overline{\Omega}))^n .$$

Then

$$\limsup_{h \to 0} \int_{\Omega(h)} L(x, \overline{u}(x), \nabla u_\alpha(x)) \, dx < \liminf_{m \to +\infty} \int_\Omega L(x, \overline{u}(x), v_m(x)) \, dx ,$$
where $u_h$ are the integral averages of $u_\varphi$ (e.g. see [7]) and

$$\Omega_{(h)} = \{x \in \Omega : d(x, \partial \Omega) > h \}.$$  

**Proof.** Let $K_h$ be mollifier functions as in [7] and $u_h(x) = \int_{\mathbb{R}^n} (x - \xi) u_\varphi(\xi) d\xi$, $x \in \mathbb{R}^n$; (indeed we need only that $u_h$ be the integral averages of any extension of $u$).

Let $v_{m_h}$ be the integral averages of any extension of $v_m$ and $x \in \overline{\Omega}_{(h)}$; we have

$$|v_{m_h}(x) - \nabla u_h(x)| = \int_{\overline{\Omega}} K_h(x - \xi) v_m(\xi) d\xi - \int_{\overline{\Omega}} \nabla_x K_h(x - \xi) u(\xi) d\xi = \delta'(m, h, x),$$

where by (2.15)

$$\lim_{m \to + \infty} \delta'(m, h, x) = 0 \quad \text{for every } h > 0 \text{ and } x \in \overline{\Omega}_{(h)}.$$

By (2.15) there exists $c > 0$ such that

$$\|v_m\|_{(L^1(\Omega))^n} < c, \quad \text{for every } m.$$

Fixed $h$, by (2.17) and (2.19) it follows that $\delta'(m, h, x)$ is a sequence of uniformly equicontinuous functions in $\overline{\Omega}_{(h)}$; moreover we have

$$|\delta'(m, h, x)| \leq \|v_m\|_{(L^1(\Omega))} + c(h) \|u\|_{L^1(\Omega)} < c + c(h) \|u\|_{L^1(\Omega)}$$

for a suitable $c(h) > 0$.

Then we may apply the Ascoli-Arzelà theorem and by (2.18) we obtain for each fixed $h$

$$\lim_{m \to + \infty} \delta(m, h) = 0,$$

where $\delta(m, h) = \sup \{\delta'(m, h, x) : x \in \Omega_{(h)}\}$.

Now we put $f = L + \varphi > 0$; if $x \in \Omega_{(h)}$ we have, by (2.17), (2.20) and the uniform continuity of $f$ on the compact subset of $\Omega \times \mathbb{R} \times \mathbb{R}^n$

$$|f(x, \bar{u}(x), \nabla u_h(x)) - f(x, \bar{u}(x), v_{m_h}(x))| < \varepsilon(m, h),$$

where $u_h$ are the integral averages of $u_\varphi$ (e.g. see [7]) and

$$\Omega_{(h)} = \{x \in \Omega : d(x, \partial \Omega) > h \}.$$  

**Proof.** Let $K_h$ be mollifier functions as in [7] and $u_h(x) = \int_{\mathbb{R}^n} (x - \xi) u_\varphi(\xi) d\xi$, $x \in \mathbb{R}^n$; (indeed we need only that $u_h$ be the integral averages of any extension of $u$).

Let $v_{m_h}$ be the integral averages of any extension of $v_m$ and $x \in \overline{\Omega}_{(h)}$; we have

$$|v_{m_h}(x) - \nabla u_h(x)| = \int_{\overline{\Omega}} K_h(x - \xi) v_m(\xi) d\xi - \int_{\overline{\Omega}} \nabla_x K_h(x - \xi) u(\xi) d\xi = \delta'(m, h, x),$$

where by (2.15)

$$\lim_{m \to + \infty} \delta'(m, h, x) = 0 \quad \text{for every } h > 0 \text{ and } x \in \overline{\Omega}_{(h)}.$$

By (2.15) there exists $c > 0$ such that

$$\|v_m\|_{(L^1(\Omega))^n} < c, \quad \text{for every } m.$$

Fixed $h$, by (2.17) and (2.19) it follows that $\delta'(m, h, x)$ is a sequence of uniformly equicontinuous functions in $\overline{\Omega}_{(h)}$; moreover we have

$$|\delta'(m, h, x)| \leq \|v_m\|_{(L^1(\Omega))} + c(h) \|u\|_{L^1(\Omega)} < c + c(h) \|u\|_{L^1(\Omega)}$$

for a suitable $c(h) > 0$.

Then we may apply the Ascoli-Arzelà theorem and by (2.18) we obtain for each fixed $h$

$$\lim_{m \to + \infty} \delta(m, h) = 0,$$

where $\delta(m, h) = \sup \{\delta'(m, h, x) : x \in \Omega_{(h)}\}$.

Now we put $f = L + \varphi > 0$; if $x \in \Omega_{(h)}$ we have, by (2.17), (2.20) and the uniform continuity of $f$ on the compact subset of $\Omega \times \mathbb{R} \times \mathbb{R}^n$

$$|f(x, \bar{u}(x), \nabla u_h(x)) - f(x, \bar{u}(x), v_{m_h}(x))| < \varepsilon(m, h),$$
where $\lim_{m \to +\infty} \epsilon(m, h) = 0$ for every $h$.

We remark that it is essential to derive (2.21) the hypothesis $\bar{u} \in L^1_{lo}(\Omega)$. By (2.21) and Jensen's inequality (which may be applied by (2.13)) we obtain

$$f(x, \bar{u}(x), \nabla u_h(x)) \leq f(x, \bar{u}(x), v_{m_h}(x)) + \epsilon(m, h) =$$

$$= f(x, \bar{u}(x), \int_{\Omega} K_h(x - \xi) v_m(\xi) \, d\xi) + \epsilon(m, h) \leq$$

$$\leq f(x, \bar{u}(x), v_m(\xi)) K_h(x - \mu) \, d\xi + \epsilon(m, h) =$$

$$= \int_{\Omega} K_h(x - \xi) f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi + \int_{\Omega} K_h(x - \xi) \cdot [f(x, \bar{u}(x), v_m(\xi)) - f(\xi, \bar{u}(\xi), v_m(\xi))] \, d\xi + \epsilon(m, h).$$

Integrating this inequality and by the use of Fubini's theorem we have

$$\int_{\Omega(x)} f(x, \bar{u}(x), \nabla u_h(x)) \, dx \leq \int_{\Omega(x)} \left( \int_{\Omega} K_h(x - \xi) f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi \right) \, dx +$$

$$+ \int_{\Omega(x)} q(m, x) \, dx + \epsilon(m, h) \, \text{mis} \, \Omega \leq$$

$$\leq \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi + \int_{\Omega(x)} q(m, x) \, dx + \epsilon(m, h) \, \text{mis} \, \Omega,$$

where

$$q(m, x) = \int_{\Omega} [f(x, \bar{u}(x), v_m(\xi)) - f(\xi, \bar{u}(\xi), v_m(\xi))] K_h(x - \xi) \, d\xi.$$

By the hypothesis on $L$ and $\varphi$ we have $L^+ \leq L + \varphi = f$ and

$$|f(x, u, v) - f(x_1, u_1, v)| \leq 2\lambda(|x - x_1|)(1 + f(x, u, v)) + \zeta(|u - u_1|),$$

for every $x, x_1 \in \Omega$ and $u, u_1 \in \mathbb{R}$.

Now we use (2.22) to evaluate $q(m, x)$ and $\int_{\Omega(x)} q(m, x) \, dx$. 
If $x \in \Omega_{(h)}$ we have

$$|q(m, x)| \leq 2 \int_{\Omega} K_h(x - \xi) \cdot \left[ 1 + \int_{\Omega} f(\xi, u(\xi), v_m(\xi)) \right] d\xi <$$

$$< 2\lambda(h) \left[ 1 + \int_{\Omega} K_h(x) f(\xi, u(\xi), v_m(\xi)) d\xi \right] + \int_{\Omega} K_h(x) \zeta(|u(x) - \bar{u}(\xi)|) d\xi.$$ 

Integrating and by the use of Fubini's theorem we obtain

$$\int_{\Omega} |q(m, x)| \, dx \leq 2\lambda(h) \cdot$$

$$\left[ \text{mis } \Omega + \int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) f(\xi, u(\xi), v_m(\xi)) d\xi \right) \, dx \right] +$$

$$+ \int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) \zeta(|u(x) - \bar{u}(\xi)|) d\xi \right) \, dx <$$

$$< 2\lambda(h) \left[ \text{mis } \Omega + \int_{\Omega} f(\xi, u(\xi), v_m(\xi)) d\xi \right] +$$

$$+ \int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) \zeta(|u(\xi) - \bar{u}(x)|) d\xi \right) \, dx.$$

Without loss of generality we may suppose that $\zeta$ is concave; moreover we remark that

$$\int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) \, dx \right) d\xi = \int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) \, d\xi \right) \, dx = \text{mis } \Omega_{(h)}.$$

Now we use Jensen's inequality:

$$\int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) \zeta(|u(\xi) - \bar{u}(x)|) \, dx \right) d\xi <$$

$$\leq \zeta \left( \frac{\int_{\Omega} \left( \int_{\Omega} K_h(x - \xi) |u(\xi) - \bar{u}(x)| \, dx \right) d\xi}{\text{mis } \Omega_{(h)}} \right) \text{mis } \Omega_{(h)}.$$
We have also
\[
\int_{\Omega} \left( \int_{\Omega(h)} K_h(x - \xi) |\bar{u}(\xi) - \bar{u}(x)| \, dx \right) \, d\xi = \\
= \int_{|z| < h} \left( \int_{\Omega} K_h(z) |\bar{u}(\xi) - \bar{u}(\xi + z)| \, d\xi \right) \, dz = \varepsilon_1(h),
\]
where \( \lim_{h \to 0} \varepsilon_1(h) = 0 \).

Finally we may write
\[
\int_{\Omega(h)} f(x, \bar{u}(x), \nabla u_h(x)) \, dx < \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi + \\
+ 2\lambda(h) \left[ \text{mis} \Omega + \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi \right] + \zeta \left( \frac{\varepsilon_1(h)}{\text{mis} \Omega(h)} \right) \text{mis} \Omega(h) + \\
+ \varepsilon(m, h) \text{mis} \Omega.
\]

Letting \( m \to +\infty \) we obtain
\[
\int_{\Omega(h)} f(x, \bar{u}(x), \nabla u_h(x)) \, dx \leq \inf_{m \to +\infty} \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi + \\
+ 2\lambda(h) \left[ \text{mis} \Omega + \inf_{m \to +\infty} \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi \right] + \zeta \left( \frac{\varepsilon_1(h)}{\text{mis} \Omega(h)} \right) \text{mis} \Omega.
\]

If \( \inf_{m \to +\infty} \int_{\Omega} f(\xi, \bar{u}(\xi), v_m(\xi)) \, d\xi = +\infty \) (2.16) holds; otherwise we obtain (2.16) letting \( h \to 0 \).

The following theorem is proved in [6]:

**Theorem 2.2.** Let (2.4) (vi) and the hypothesis of Lemma 2.3 hold; moreover we suppose that there exist \( A > 0 \) and \( g \in L^1(\Omega) \) such that

\[
L(x, u, v) \leq A(g(x) + |u| + |v|), \quad (x, u, v) \in \Omega \times \mathbb{R} \times \mathbb{R}^n.
\]

Then
\[
J_h(u, y^{-}(u)) = \lim_{h \to 0} I_h(y_h), \quad u \in BV_\varepsilon(\Omega).
\]
where \( u'_h \) are the integral averages of \( u' \) which is defined as follows: \( u' = u \) in \( \Omega \), \( u' \in BV_s(\mathbb{R}^n) \), \( \gamma^+(u') = \gamma^-(u) \).

We have also

\[
J_i(u, \gamma^-(u)) - J_i(u, f) \leq J_i(u, \gamma^-(u)) + A \int_{\partial \Omega} |f - \gamma^-(u)| \, dH_{n-1},
\]

for every \((u, f) \in BV_b(\Omega) \oplus L^1(\partial \Omega)\).

**Remark 2.2.** Condition (2.14) implies condition (2.4) (v).

Now we may prove our most important results.

**Theorem 2.3.** Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold. Then

\[
J_i(u, \gamma^-(u)) = \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx + \int_{\partial \Omega} r_L \left( x, \frac{\partial v}{\partial q}(x) \right) q(dx),
\]

for every \( u \in BV_b(\Omega) \cap L^\infty_0(\Omega) \).

**Proof.** We have

\[
(2.25) \quad \int_{\Omega} L(x, \bar{u}(x), \nabla u'_h(x)) \, dx = \int_{\Omega} L(x, \bar{u}(x), \nabla u'_h(x)) \, dx + \int_{\Omega - \Omega'} L(x, u(x), \nabla u'_h(x)) \, dx
\]

and by (2.23) and the properties of integral averages

\[
(2.26) \quad \left| \int_{\Omega - \Omega'} L(x, \bar{u}(x), \nabla u'_h(x)) \, dx \right| \leq A \left( \int_{\Omega - \Omega'} |g(x)| \, dx + \int_{\Omega - \Omega'} |\bar{u}(x)| \, dx + \int_{\Omega - \Omega'} |\nabla u'(x)| \, dx \right),
\]

where \( \Omega' = \{ x : d(x, \delta \Omega) < h \} \).

The limit of the right hand side of (2.26) is 0 since

\[
\lim_{h \to 0} \int_{\Omega - \Omega'} |\nabla u'(x)| \, dx = \int_{\partial \Omega} |\nabla u'| \, dH_{n-1} = \int_{\partial \Omega} |\gamma^+(u') - \gamma^-(u')| \, dH_{n-1} = 0.
\]
Then, if $\bar{u} \in L^\infty_{\text{loc}}(\Omega) \cap L^1(\Omega)$, by (2.25) and (2.16) we obtain

$$
\lim_{h \to 0} \sup_{\Omega} \int L(x, \bar{u}(x), \nabla u_h(x)) \, dx < \inf_{m \to +\infty} \left\{ \lim_{m \to +\infty} \int_{\Omega} L(x, \bar{u}(x), v_m(x)) \, dx : \{v_m\} \subset (L^1(\Omega))^n, \int_{\Omega} G v_m \to \int_{\Omega} G \nabla u \text{ for every } G \in (C(\bar{\Omega}))^n \right\},
$$

and so, by (2.8) and (2.10),

$$
\lim_{h \to 0} \sup_{\Omega} \int L(x, \bar{u}(x), \nabla u'_h(x)) \, dx \leq \int_{\Omega} L(x, \bar{u}(x), \nabla u(x)) \, dx + \int_{\partial \Omega} r_L \left( x, \frac{d \nabla u'}{dq} (x) \right) q(dx),
$$

for every $\bar{u} \in L^\infty_{\text{loc}}(\Omega) \cap L^1(\Omega)$ and $u \in BV_{\text{loc}}(\Omega)$.

Now we take $u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ and $\bar{u} = u$; then

$$
\lim_{h \to 0} \sup_{\Omega} \int L(x, u(x), \nabla u'_h(x)) \, dx \leq \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx + \int_{\partial \Omega} r_L \left( x, \frac{d \nabla u'}{dq} (x) \right) q(dx).
$$

By (2.4) (i) and (2.4) () as in the final part of the proof of Theorem 2.1, it is easily proved that

$$
\lim_{h \to 0} \int_{\Omega} (L(x, u(x), \nabla u'_h(x)) - L(x, u'_h(x), \nabla u'_h(x))) \, dx = 0;
$$

then by (2.28) and (2.24)

$$
J_i(u, \gamma(u)) \leq \int_{\Omega} L(x, u(x), \nabla u(x)) \, dx + \int_{\partial \Omega} r_L \left( x, \frac{d \nabla u'}{dq} (x) \right) q(dx).
$$

The proof is complete by a comparison between (2.29) and Theorem 2.1. ■
THEOREM 2.4. Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold. If \( L = L(x, v) \), then
\[
J_1(u, \gamma^{-}(u)) = \int_{\Omega} L(x, \nabla u(x))
\]
for every \( u \in BV_b(\Omega) \).

PROOF. If \( L = L(x, v) \), \( \overline{u} \) does not appear in (2.27) and there is no restriction about \( u \). □

THEOREM 2.5. Let (2.4), (2.23) and the hypothesis of Lemma 2.3 hold; if in addition there exists a non decreasing continuous function \( \eta: [0, +\infty) \rightarrow \mathbb{R} \) such that \( \eta(0) = 0 \) and
\[
(2.30) \quad |L(x, u, v) - L(x, u, v_1)| < \eta(|v - v_1|)
\]
then
\[
J_1(u, \gamma^{-}(u)) = \int_{\Omega} L(x, u(x), \nabla u(x))
\]
for every \( u \in BV_b(\Omega) \).

PROOF. It suffices to remark that in the proof of Lemma 2.3 we may derive, if (2.30) holds, inequality (2.21) for every \( \overline{u} \in L^1(\Omega) \); then in (2.28) we may take \( u \in BV_b(\Omega) \) instead of
\[
u \in BV_b(\Omega) \cap L^\infty(\Omega).
\]

REFERENCES


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