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G-domains and pseudo-valuations

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G-Domains and Pseudo-Valuations.

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1. Introduction.

In this note we show that there is a one to one correspondence between the equivalence classes of pseudovalutations on a field K and the equivalence classes of G-domains contained in K and having K as their field of quotient (theorem 3). We also show that if a G-domain is completely integrally closed, then it gives rise to a homogeneous pseudo-valuation and conversely (theorem 5 and 6). We recall all the necessary definitions and basic results to make this note reasonably self contained.

2. Definitions.

Let R be an integral domain and K be its field of quotients. We say R is a *G-domain* if K is finitely generated as a ring over R. That is to say $R[a_1, \dots, a_n] = K$ where $a_i \in K$. It is easy to see that if R is a G-domain then $K = R[u^{-1}]$ where u belongs to R. See [2] for details. In case $K \neq R$, then we have

(i) $u^{-1} \notin R$,

(ii) $\dots u^2 \cdot R \subset u \cdot R \subset u^{-1} \cdot R \subset u^{-2} \cdot R \subset \dots$,

(iii) $K = \bigcup_{n=1}^{\infty} u^{-n} R$,

(iv) $0 = \bigcap_{n=1}^{\infty} u^n R$.

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Only the last needs to be checked as the other three are evident. Suppose $0 \neq a \in \bigcap u^n \cdot R$. Then if we take a^{-1} it cannot belong to any $u^{-n}R$, in contradiction to *(iii)*, as otherwise $a^{-1} \in u^{-n}R$, $a \in u^{n+1}R \Rightarrow \Rightarrow a^{-1}a = 1 \in uR$, in contradiction to the fact that u is not a unit.

We recall a *pseudo valuation* ω on a field K is a real valued function such that

(i) $\omega(x) \geqslant 0$ for all $x \in K$ with equality holding where $x = 0$;

(ii) $\omega(x \cdot y) \leqslant \omega(x) \cdot \omega(y)$ and

(iii) $\omega(x - y) \leqslant \omega(x) + \omega(y)$ for all x and y in K.

In case we have

(iii′) $\omega(x - y) \leqslant \operatorname{Max}\{\omega(x), \omega(y)\}$

then ω is said to be a *non-archimedean*.

If R is a G-domain with $K = R[u^{-1}]$, then by setting

$$\omega(x) = \begin{cases} \alpha^{-\nu(x)} & \text{if } x \neq 0\,,\ 1 < \alpha \leqslant 2\,, \\ 0 & \text{if } x = 0\,, \end{cases}$$

where $\nu(x) = n$ whenever $x \in u^n \cdot R \setminus u^{(n+1)} \cdot R$, we find that ω satisfies all the conditions of a pseudo valuation with

$$\omega(x - y) \leqslant \alpha \cdot \operatorname{Max}\{\omega(x), \omega(y)\}\,.$$

We have the following:

3. Results.

THEOREM 1. *Let R be a G-domain with its quotient field $K = R[u^{-1}]$. Then there exists a pseudo valuation ω_u on K such that*

$$R = \{x \in K | \omega_u(x) \leqslant 1\}\,.$$

Moreover, if t is any other element in R such that $K = R[t^{-1}]$ and ω_t is the pseudo valuation arising out of t then ω_u and ω_t are equivalent in the sense that they define the same topology.

PROOF. Setting $\omega = \omega_u$ in the discussion in **2** and using theorem 4.1 of Cohn [1], we get that ω_u is a pseudo valuation.

Since $\nu(x) \geqslant 0$ for all $x \in R$ and $\omega_u(x) = 2^{-\nu(x)} \leqslant 1$ for these x we get that

$$R = \{x \in K | \omega_u(x) \leqslant 1\} .$$

The topology arising out of ω_u has $\{u^n \cdot R\}$ as basis of neighbourhoods of 0. Similarly $\{t^n \cdot R\}$ is a basis of neighbourhoods of 0 with respect to the pseudovaluation ω_t and as these topologies are dependent on R and not on the gauge elements u and t we have the desired conclusion.

Thus to each G-domain R we have associated an equivalence class of pseudo valuations. Next we show that given a pseudovaluation ω on K, there exists a G-domain R_ω associated with ω such that the pseudovaluation arising out of R_ω is equivalent to ω.

THEOREM 2. *Let ω be a non-trivial pseudovaluation on a field K and $\mathfrak{D} = \{x \in K | \omega(x) < 1\}$. If $R_\omega = \{x \in K | x \cdot \mathfrak{D} \subset \mathfrak{D}\}$ then R_ω is a G-domain having K as its field of quotients.*

PROOF. That R_ω is a subring of K can be verified easily. As ω is non-trivial there exist elements u in K such that $0 < \omega(u) < 1$. Then as $K = \bigcup_n u^{-n} \cdot R_\omega$ it is easily seen that $K = R[u^{-1}]$. Now $\{u^n \cdot R_\omega\}$ is a basis of neighbourhoods of 0 with respect to the topology of the pseudovaluation ω. If ω_u is the pseudo valuation on K arising out of R_ω with u as a gauge element, then the topology induced by ω_u and ω are equal. Thus ω and ω_u are equivalent pseudo valuations.

Next we define two G-domains R_1 and R_2 having the same quotients field K to be equivalent if there exist non-zero element a_1 and a_2 in K such that $a_1 \cdot R_1 \subset R_2$ and $a_2 \cdot R_2 \subset R_1$.

The following theorem establishes a one-one correspondence between the equivalence class of pseudo valuations and equivalent G-domains.

THEOREM 3. *Let R_i $(i = 1, 2)$ be two G-domains having the same field of quotients K. Then R_1 and R_2 are equivalent if and only if both these give rise to the same equivalence class of pseudo valuations.*

PROOF. Suppose $K = R_i[u_i^{-1}]$ for $i = 1, 2$. If R_1 and R_2 are equivalent, then the topologies induced by the pseudovaluations are equivalent and hence the two pseudo valuations belong to the same class.

On the other hand, if ψ_1 and ψ_2 are two equivalent pseudo valuations and $R_i = R_{\psi_i}$ $(i = 1, 2)$ then R_i is a G-domain by theorem 2. If $u_1, u_2 \in K$ such that $K = R_i[u_i^{-1}]$, then a basis of neighbourhoods of 0 under the topology induced by ψ_i is given by $\{u_i^n \cdot R_i\}$ for $i = 1, 2$. As these topologies are equivalent we find that $R_2 \subset u_1^{-n_1} \cdot R_1$ and $R_1 \subset u_2^{-n_2} \cdot R_2$. Thus $u_1^{n_1} \cdot R_2 \subset R_1$ and $u_2^{n_2} \cdot R_1 \subset R_2$ so that R_1 and R_2 are equivalent.

We recall that a pseudovaluation is called *homogeneous* if $\omega(x^n) = (\omega(x))^n$ for all integers $n > 0$, and all x in K.

As examples of homogeneous pseudovaluations we cite the usual valuations and Min $\{v_i(x)\} = \omega(x)$ for any finite set of valuations on a given field.

We need the notion of complete integral closures. We begin with the definition of almost integral elements. Let $R \subset S$ be two commutative rings with the same identity. An element s in S is called *almost integral* over R if $\{s^n\}$, for all $n > 0$ belongs to a finite R-submodule of S.

If $R = R^* = \{x \in S \mid x \text{ is almost integral over } R\}$, then we say that R is *completely integrally closed* in S. If $R \subset R^*$ then R^* is called the complete integral closures of R in S. In case S is taken as the total quotient ring of R and R is completely integrally closed in S, then we say that R is completely *integrally closed.*

The complete integral closure R^* of a ring R with total quotient ring K is given by

$$R^* = \{x \in K \mid \text{there exists a regular element } r \text{ in } R \text{ such that } r \cdot x^n \text{ belongs to } R \text{ for all positive integers } n\}.$$

THEOREM 4. *If two G-domains R_1 and R_2 having the same field of quotients K are equivalent, then their complete integral closures are equal.*

PROOF. Let R_i^* be the complete integral closure of R_i for $i = 1, 2$. As R_1 is equivalent to R_2, we have an element $a_1 \neq 0$ such that $a_1 R_1 \subset R_2$. If $x \in R_i^*$ then, from the definition of complete integral closure, we have a regular element r in R_1 such that $r \cdot x^n \in R_1$ for all n. Therefore, $(a \cdot r) \cdot x^n \in R_2$ for all n. As R_1 and R_2 are both domains and $(r \cdot a_1)$ is also regular we find that $x \in R_2^*$. Thus $R_1^* \subset R_2^*$ and similarly $R_2^* \subset R_1^*$.

The next theorem connects the homogeneous pseudo valuation with completely integrally closed G-domains.

THEOREM 5. *Let ω be a homogeneous pseudo valuation on K. Then the set of ω-integers, namely*

$$R = \{x \in K | x \cdot \mathfrak{D} \subseteq \mathfrak{D}\}$$

where

$$\mathfrak{D} = \{x \in K | \omega(x) < 1\} ,$$

is a completely integrally closed G-domain in K having K as its field of quotients.

PROOF. From theorem 2, we find that R is a G-domain having K as its field of quotients. We need to show that R is completely integrally closed. For this, let $x \in K$ and $a, a \cdot x, a \cdot x^2, \ldots$ belong to R for some non-zero element a in R. As ω is homogeneous

$$\omega(x)^n = \omega(x^n) = \omega(a^{-1} \cdot a \cdot x^n) \leqslant \omega(a^{-1}) \cdot \omega(a \cdot x^n) \leqslant \omega(a^{-1})$$

since $\omega(a \cdot x^n) \leqslant 1$ as $a \cdot x^n \in R$. Thus $\omega(x) \leqslant \sqrt[n]{\omega(a^{-1})}$ and this holds for every integer n. Therefore $\omega(x) \leqslant 1$ so that $x \in R$.

The following is converse to the above.

THEOREM 6. *Let R be a completely integrally closed G-domain with K as its quotient field. Then in the equivalence class of pseudo valuations arising out of R, there is a homogeneous pseudo valuation.*

PROOF. Let $u \in R$ be such that $K = R[u^{-1}]$. This u enables us to define the integer valued function ν on K. Now set $\mu(x) = \lim_{n \to \infty} 1/n \cdot \cdot \nu(x^n)$. This limit exists since

$$\nu(x) \leqslant \frac{1}{n} \cdot \nu(x^n) \leqslant \nu(x) + 1 .$$

We can used μ to define a gauge function in the sense of Cohn [1] and use this gauge function to define a pseudovaluation by stipulating that

$$\omega(x) = 2^{-\mu(x)} .$$

This ω is homogeneous as $\mu(x^n) = n \cdot \mu(x)$.

This we see that there is a one-one correspondence between the equivalence classes of homogeneous pseudo valuations on a field K

and completely integrally closed G-domains having K as their field of quotients.

Surjit Singh, in his thesis, has shown that any pseudo valuation on an A-field (number or an algebraic function field in one variable over a finite field) can be expressed as supremum of a finite number of valuations. Now, given a valuation v on an A-field, its valuation ring is evidently a G-domain with any uniformizing parameter playing the role of u whose inverse generates the quotients field. If ω is any pseudovaluation on an A-field then the ω-integers form a G-domain which is moreover a completely integrally closed ring. On the other hand, every G-domain in an A-field gives rise to a pseudovaluation which can be realized as the supremum of a finite number of valuations. Thus we get a complete description of all completely integrally closed G-domains contained in an A-field.

REFERENCES

[1] P. M. Cohn, *An invariant characterization of pseudovaluations on a field*, Proc. Camb. Phil. Soc., **50** (1954).

[2] I. Kaplansky, *Commutative Rings*, Allyn and Bacon.

[3] S. Singh, *Ph.D. thesis*, Panjab University (1975).

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