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A link between global solvability and solvability over compacts for systems like : $(P(D_x, D_y)u = f,\ Qu = 0)$

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0. Introduction.

Let A be an open subset of R^3 such that:

if A_0 is its intersection with plane xy, every point $p \in A$ can be connected by an « orthogonal segment », (in A), with some point $p_0 \in A_0$. Then we can show the following:

THEOREM 1. Let $P = P(D_x, D_y)$ be a partial differential operator with constant coefficients; Q_2 e Q_3 will be, respectively, the Laplace's operators in two and three variables.
Then:

i) if A_0 is P-convex,
we have: the global solvability of the overdeterminated system: $(Pu = f,\, Q_3 u = 0)$ is equivalent to the solvability of the same system over compact subsets of A.

REMARK 1. Without loss generality, we can always think P and Q_2 are prime between them; it depends upon the global solvability, in A, of the system: $(Q_2 u = f,\, Q_3 u = 0)$.

REMARK 2. Connections of the above theorem with some E. De Giorgi's conjecture [2], are evident.

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1. Better then everything it's to make directly the proof of Theorem 1; for symbols and terminology, look at [3].

PROOF.

a) In A, the system: $(Pu = f,\ D_t u = 0)$ is globally solvable, with C^∞-solutions (of course for every f such that: $D_t f = 0$). *a*) depends upon the i) hypothesis.

As a consequence: if $E'(A)$ is the space of distribution with compact support in A, ${}^tPE'(A) + {}^tD_tE'(A)$ is closed in $E'(A)$. In fact: if we pose: $\ker D_{t/A} = \big(f \in C^\infty(A)\colon D_t f = 0\big)$, the solvability of the above system means that: $P(\ker D_{t/A}) = \ker D_{t/A}$. By the theorem about the « *surjections between Fréchet's spaces* » we have:

if: ${}^tPm_j + {}^tD_t n_j$ is convergent, in $E'(A)$, to m_0, there exists $m_1 \in E'(A)$ such that: $m_0 = {}^tPm_1$ over functions of $\ker D_{t/A}$.

By [3], pagg. 77-78, we have: $m_0 = {}^tPm_1 + {}^tD_t n_1$, with $n_1 \in E'(R^3)$; for the fact that: $\operatorname{supp}(m_0 - {}^tPm_1) \subseteq A$, also $\operatorname{supp}(n_1)$ is in A.

b) In A, the system: $(Pu = f,\ Q_3 u = 0,\ D_t u = 0)$ is globally solvable; with the same arguments, like in *a*), it comes out:

$$ {}^tPE'(A) + Q_3E'(A) + {}^tD_tE'(A) \qquad \text{is closed in } E'(A)\,. $$

b') In A, we'll consider the system: $(Pu = f_1,\ Q_3 u = f_2,\ D_t u = f_3)$; we suppose the data: (f_1, f_2, f_3) compatible and in $C^\infty(A)$.

We like to show the above system is solvable in $C^\infty(A)$.

Call D_3 the subspace, of $C^\infty(A)^3$, of the compatible data; call: $(P, Q_3, D_t)\colon C^\infty(A) \to D_3$, the (continuous and linear) map:

$$ (P, Q_3, Q_t)u = (Pu, Q_3u, D_tu)\ ; $$

let ${}^t(P, Q_3, D_t)$ be its transposed between the dual space.

Of course: if (m_1, m_2, m_3) is a functional over D_3 and:

$$ {}^t(P, Q_3, D_t)(m_1, m_2, m_3) = 0\ , $$

which means: $-Q_3m_2 = {}^tPm_1 + {}^tD_tm_3$, we have:

1) Q_3m_2 is orthogonal to the space: $\ker P_{/A} \wedge \ker D_{t/A}$; then:

2) m_2 is orthogonal to

$$ Q_3(\ker P_{/A} \wedge \ker D_{t/A}) = (\ker P_{/A} \wedge \ker D_{t/A})\ . $$

Last equality comes out from the solvability, over A_0, of the system: $(Pu = f,\ Q_2 u = 0)$ over simply connected open subset of R^2.

Because a) above, $m_2 = {}^tPh + {}^tD_t k$, with $(h, k) \in E'(A)^2$. Now, it's easy to see that: $m_1 = -Q_3 h + {}^tD_t p$, and $m_3 = -Q_3 k - {}^tPp$; again: $p \in E'(A)$.

This show that: ${}^t(P, Q_3, D_t)$ is injective; by b), its imagine, in $E'(A)$, is closed; then: the system: $(Pu = f_1,\ Q_3 u = f_2,\ D_t u = f_3)$ is solvable in $C^\infty(A)$ for every data, in $C^\infty(A)$, compatible.

c) From b') and from the theorem about « *surjections between Fréchet's spaces* » we have:

1) if $\operatorname{supp}({}^tPm_1 + Q_3 m_2 + {}^tD_t m_3) \subseteq K \Subset A$, and if: $\operatorname{ord.}({}^tPm_1 + Q_3 m_2 + {}^tD^t m_3) \leqslant n$, $n \in N$, there exist:

2) a compact subset of A, $K(n)$, and three distributions, $(h, k, p) \in E'(A)^3$, such that:

$$\operatorname{supp}(m_1 - Q_3 h + {}^tD_t p,\ m_2 + {}^tPh + {}^tD_t k,\ m_3 - Q_3 k - {}^tPp) \subseteq K(n)^3 .$$

d) Suppose: ${}^tPm_1 + Q_3 m_2$ with support in $K \Subset A$ and with order less than n. By c) above:

$$\operatorname{supp}(m_1 - Q_3 h + {}^tD_t p,\ m_2 + {}^tPh + {}^tD_t k,\ -Q_3 k - {}^tPp) \subseteq K(n)^3 ;$$

in $E'(A)$ we can solve the system:

1) $Q_3 h + {}^tD_t p = Q_3 h_1$, ${}^tPh + {}^tD_t k = {}^tPh_1 + {}^tD_t k_1$,

2) $Q_3 k - {}^tPp = -Q_3 k_1 - {}^tPp_1$,

(it's simple exercise).

So we have:

$$\operatorname{supp}(m_1 - Q_3 h_1) \subseteq K(n) ; \qquad \operatorname{supp}(m_2 + {}^tPh_1 + {}^tD_t k_1) \subseteq K(n) .$$

Because we can suppose:

$$K \subseteq K(n) ,$$

we have:

$${}^tPm_1 + Q_3 m_2 + Q_3\, {}^tD_t k_1$$

has its support in $K(n)$; this implies, choosing $K'(n)$ a little bigger than $K(n)$, that:

$$\operatorname{supp}(m_1 - Q_3 h_1, m_2 + {}^tPh_1) \subsetneq K'(n)^2 .$$

e) *d*) above shows: if ${}^tPm_1 + Q_3 m_2$ is continuous in relation with a semi-norma p over $C^\infty(A)$, there exists a semi-norm q over the compatible data $D_2 = \big((f, g): Pg = Q_3 f\big)$, in relation to which (m_1, m_2) are continuous over D_2.

Solvability of the system: $(Pu = f,\ Q_3 u = 0)$ over compact subsets of A, and the ellipticity of Q_3, from which comes out the fact that A is Q-convex, shows the thesis of the Theorem 1.

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