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Invariant Measures and a Linear Model of Turbulence.

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SUMMARY - A sufficient condition is shown for the existence of continuous measures, invariant and ergodic with respect to semidynamical systems on topological spaces. This condition is applied to a dynamical system generated by a first order linear partial differential equation.

1. Introduction.

Roughly speaking a motion of a flow is turbulent if its trajectory in the phase-space is complicated and irregular. There are several ways to make this description precise. The most straightforward one is to give a rigorous definition of turbulent trajectories and then to prove that they exist [1], [3], [7]. Another approach was proposed by G. Prodi in 1960. According to his theory, stationary turbulence occurs when the flow admits a nontrivial ergodic invariant measure [6] (see also [2], [4]). Both points of view are closely related. In fact the existence of turbulent trajectories implies via Kryloff-Bogoluboff theorem the existence of invariant measures, and vice-versa from the existence of an ergodic invariant measure it follows, by Birkhoff individual ergodic theorem, that almost all trajectories are complicated enough.

In the present paper we shall consider this interdependance in the framework of semidynamical systems and we shall prove new

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sufficient conditions for the existence of turbulent trajectories and invariant measures. Then we shall apply our general results to a linear first order partial differential equation. The equation depends upon a parameter λ having the role of Reynolds number. For λ sufficiently small $(\lambda < 1)$ all the solutions converge to the laminar solution $u \equiv 0$. For large values of λ $(\lambda \geqslant 2)$ the equation admits infinitely many turbulent solutions. This is rather a surprising result, since in general turbulence seems to be related with strongly nonlinear partial differential equations of higher order.

The paper has its origin in the joint research done by J. Yorke and the author [5].

2. Strictly turbulent trajectories.

Let X be a topological Hausdorff space and let $S_t\colon X \to X$, $t \geqslant 0$, be a semigroup of transformations, that is

$$\begin{aligned} S_0 &= I \qquad \text{(identity)}, \\ S_{t+s} &= S_t \circ S_s \qquad \text{for } s \geqslant 0,\ t \geqslant 0. \end{aligned} \tag{1}$$

We call the semigroup $\{S_t\}$ a semidynamical system if the mapping

$$R^+ \times X \ni (t, x) \to S_t x \in X, \qquad (R^+ = [0, \infty)),$$

is continuous in (t, x). We admit the usual notions

$$O(x) = \{S_t(x)\colon t \geqslant 0\}, \qquad L(x) = \bigcap_{t \geqslant 0} \operatorname{cl} O\big(S_t(x)\big), \qquad (\mathrm{cl} = \text{closure}),$$

for the trajectory (orbit) starting from the point x and the limit set, respectively. The set $L(x)$ is always closed and invariant under S_t, that is

$$S_t\big(L(x)\big) \subset L(x), \qquad x \in X,\ t \geqslant 0.$$

A point $x \in X$ is called periodic if there exists $t > 0$ such that $S_t(x) = x$. According to this definition any fixed point is periodic.

DEFINITION 1. A trajectory $O(x)$ is called strictly turbulent if the following two conditions hold

(i) $L(x)$ is a compact nonempty set,

(ii) $L(x)$ does not contain periodic points.

THEOREM 1. Let $\{S_t\}_{t\geqslant 0}$ be a semidynamical system acting on a topological Hausdorff space X. Assume that there exist a number $r > 0$ and two nonempty compact sets $A, B \subset X$ such that

$$S_r(A) \cap S_r(B) \supset A \cup B\,, \qquad A \cap B = \emptyset\,. \tag{2}$$

Then there exists a point $x_0 \in A_0$ such that the trajectory $O(x_0)$ is strictly turbulent.

PROOF. The proof consists of three parts. First, in order to define the point x_0, we shall follow the construction given in [5]. Then we shall prove an important property of x_0, namely the existence of the limit (3). At the end, using (3), we shall show that the trajectory $O(x_0)$ is strictly turbulent.

Write $A_0 = A$, $A_1 = B$ and $S = S_r$. Define a family $\{A_{k_1\ldots k_n}\}$ of subsets of X by formula

$$A_{k_1\ldots k_n} = A_{k_1\ldots k_{n-1}} \cap S^{-1}(A_{k_2\ldots k_n})$$

for $k_i = 0, 1$; $i = 1, \ldots, n$ and $n \simeq 2, 3, \ldots$. Using (2) it is easy to verify by induction argument that the sets $A_{k_1\ldots k_n}$ are nonempty, compact and that

$$S(A_{k_1\ldots k_n}) \subset A_{k_2\ldots k_n}\,.$$

Now choose an irrational number $\alpha \in (0, 1)$ and define a dyadic sequence α_n by setting

$$\alpha_{n+1} = \begin{cases} 0 & \text{if } n\alpha (\mathrm{mod}\ 1) \in [0, \alpha)\,, \\ 1 & \text{if } n\alpha (\mathrm{mod}\ 1) \in [\alpha, 1)\,. \end{cases}$$

$A_{\alpha_1 \ldots \alpha_n}$ is a decreasing sequence of nonempty compact sets. Thus the intersection

$$A_\infty = \bigcap_{n=1}^{\infty} A_{\alpha_1\ldots\alpha_n}$$

is also a nonempty set. Choose an $x_0 \in A_\infty$ and consider the « discrete orbit »

$$O^*(x_0) = \{S^n(x_0) \colon n = 1, 2, \ldots\}$$

and its « limit set »

$$L^*(x_0) = \bigcap_{n=1}^{\infty} \operatorname{cl}\{S^n(x_0), S^{n+1}(x_0), \ldots\}.$$

From the definition of A_∞ it follows that

$$S^n(x_0) \in \begin{cases} A_0 & \text{if } n\alpha(\mathrm{mod}\, 1) \in [0, \alpha), \\ A_1 & \text{if } n\alpha(\mathrm{mod}\, 1) \in [\alpha, 1), \end{cases}$$

and consequently $O^*(x_0) \subset A_0 \cup A_1$. Since A_0 and A_1 are compact this implies that $L^*(x_0) \subset A_0 \cup A_1$. We shall show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_0}(S^k(x)) = \alpha \qquad \text{for } x \in L^*(x_0). \tag{3}$$

In fact, from the Weyl equipartition theorem it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\alpha)}(s + k\alpha(\mathrm{mod}\, 1)) = \alpha$$

uniformly for all $s \in [0, 1)$. In particular

$$\lim \frac{1}{n} \sum_{k=0}^{n-1} 1_{[0,\alpha)}\big((m+k)\alpha(\mathrm{mod}\, 1)\big) = \alpha$$

uniformly for all m. Consider a point $x \in O^*(x_0)$, say $x = S^m(x_0)$. We have

$$1_{A_0}\big(S^k(x)\big) = 1_{A_0}\big(S^{m+k}(x_0)\big) = 1_{[0,\alpha)}\big((m+k)\alpha(\mathrm{mod}\, 1)\big)$$

which implies that the limit (3) exists uniformly for all $x \in O^*(x_0)$. Since A_0 and A_1 are compact disjoint and the trajectory $O^*(x_0)$ is contained in $A_0 \cup A_1$, the functions $1_{A_0}\big(S^k(x)\big)$ $(k = 0, 1, \ldots)$ restricted to $\operatorname{cl} O^*(x_0)$ are continuous. Thus the limit (3) exists also for $x \in \operatorname{cl} O^*(x_0)$ and, in particular, for $x \in L^*(x_0)$.

Now we are going to prove that the trajectory $O(x_0) = \{S_t(x_0)\colon t \geqslant 0\}$ is strictly turbulent. The proof of condition (i) (see Def. 1) for $O(x_0)$ is easy. We have namely

$$L^*(x_0) \subset L(x_0) \subset \bigcup \{S_t(A_0 \cup A_1)\colon 0 \leqslant t \leqslant r\} .$$

The first inclusion shows that $L(x_0)$ is nonempty and the second that $L(x_0)$ is contained in a compact set. Since any limit set is closed, this finishes the proof of (i). To prove (ii) suppose that a point $\bar{x} \in L(x_0)$ is periodic. We claim that $L^*(x_0) \cap O(\bar{x}) \neq \emptyset$. In fact, by the definition od $\bar{x}$ there exists a (generalized) sequence $t_\nu \to \infty$ such that $S_{t_\nu}(x_0) \to \bar{x}$. Let q_ν be such that $n_\nu = r^{-1}(t_\nu + q_\nu)$ is an integer and $0 \leqslant q_\nu \leqslant r$. Passing to subsequences, if necessary, we may assume that q_ν is convergent to a number $q \in [0, r]$. We have

$$S^{n_\nu}(x_0) = S_{n_\nu r}(x_0) = S_{t_\nu + q_\nu}(x_0) \to S_q(\bar{x}) .$$

This implies that the point $\tilde{x} = S_q(\bar{x})$ belongs to $L^*(x_0)$ and finishes the proof of the claim. Now we shall consider two cases:

(a) $\bar{x}$ is fixed point,

(b) $\bar{x}$ is periodic point with a positive period.

Assume (a). We have $\tilde{x} \in O(\bar{x}) = \{\bar{x}\}$ and consequently $S^n(\tilde{x}) = \bar{x}$ for all n. This in turn implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_0}(S^k(\tilde{x})) = \begin{cases} 1 & \text{if } \bar{x} \in A_0 , \\ 0 & \text{if } \bar{x} \in A_1 , \end{cases} \tag{4}$$

which contradicts (3). Assume (b) and denote by p the (smallest) period of $\bar{x}$. Consider the sequence $p_n = q + nr \pmod{p}$ of points from the interval $[0, p]$. Now the case (b) splits into two possibilities: (b_1) r/p is a rational number, (b_2) r/p is irrational. Assume (b_1) and write $r/p = l/m$ where l and m are relatively prime integers. The sequence $\{p_n\}$ is periodic with periiod m. Since $S^n(\tilde{x}) = S_{p_n}(\bar{x})$ this implies that $\{S^n(\tilde{x})\}$ is also periodic with period m. Consequently the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{A_0}(S^k(\tilde{x}))$$

exists and is a rational number. This also contradicts (3). Assume (b_2). Since r/p is irrational, the sequence $\{p_n\}$ is dense in $[0, p]$. This implies that the sequence $\{S^n(\tilde{x})\} = \{S_{p_n}(\bar{x})\}$ is dense in $O(\bar{x})$. On the other hand the sequence $\{S^n(\tilde{x})\}$ is contained in $L^*(x_0)$ ($\tilde{x} \in L^*(x_0)$ and $L^*(x_0)$ is S-invariant). Therefore the whole trajectory $O(\bar{x})$ is contained in $L^*(x_0)$. Since $L^*(x_0) \subset A_0 \cup A_1$ and $O(\bar{x})$ is a continuum, it must be either $O(\bar{x}) \subset A_0$ or $O(\bar{x}) \subset A_1$. In both cases $\{S^n(\tilde{x})\} \subset A_k$ with fixed subscript k. This implies (4) and, once more, contradicts (3). Thus in all cases the assumption that a point $\bar{x} \in L(x_0)$ is periodic leads to a contradiction. The proof of the property (ii) for $O(x_0)$ as well as the proof of Theorem 1 is completed.

3. Turbulence in dynamical systems.

From condition (2) it follows that the mapping S_r: $X \to X$ is not invertible. Thus the semigroup $\{S_t\}_{t \geqslant 0}$ cannot be extended to a group of transformations. Now we state a version of Theorem 1 which can be also applied to dynamical systems. An example of such application will be given in Section 5.

Let $\{S_t\}_{t \geqslant 0}$ and $\{T_t\}_{t \geqslant 0}$ be semidynamical systems defined on compact (Hausdorff) topological spaces X and Y respectively. Let, moreover, F be a continuous mapping from X onto Y.

THEOREM 2. Assume that for each $t \geqslant 0$ the diagram

(5)
$$\begin{tikzcd} X \arrow[r, "S_t"] \arrow[d, "F"'] & X \arrow[d, "F"] \\ Y \arrow[r, "T_t"] & Y \end{tikzcd}$$

commutes. Assume, moreover, that there exist a number $r > 0$ and two nonempty closed sets, $A, B \subset Y$ satisfying

(6)
$$T_r(Z) \cap T_r(B) \supset A \cup B\,, \qquad A \cap B = \emptyset\,.$$

Then there exists a point $x_0 \in X$ such that the trajectory $O_S(x_0) = \{S_t(x_0): t \geqslant 0\}$ is strictly turbulent.

PROOF. According to Theorem 1 there exists a strictly turbulent trajectory $O_T(y_0) = \{T_t(t_0): t \geq 0\}$ starting from a point $y_0 \in Y$. Choose $x_0 \in X$ such that $F(x_0) = y_0$. Since X is a compact space, condition (i) for the trajectory $O_S(x_0)$ is automaticly satisfied. From the commutativness of (5) it follows that for any periodic point $x \in X$ the corresponding point $y = F(x)$ is also periodic. Moreover, if $x \in L_S(x_0)$ (the limit set of the trajectory $O_S(x_0)$), then $F(x) \in L_T(y_0)$ (the limit set of $O_T(y_0)$). Thus, the lack of periodic points in $L_T(y_0)$ implies that $L_S(x_0)$ has the same property. This completes the proof.

4. Existence of invariant measures.

Since the basic notions of the ergodic theory are seldom formulated for semidynamical systems, we start from recalling some necessary definitions. Let $\{S_t\}_{t \geq 0}$ be a semidynamical system acting on a Hausdorff topological space X. By a measure on X we mean any regular probabilistic measure defined on the σ-algebra of Borel subsets of X. We say that μ is supported on a Borel set E if $\mu(E) = 1$. A measure μ is called invariant under $\{S_t\}$ if $\mu(E) = \mu(S_t^{-1}(E))$ for each t and each Borel subset E. A measure μ is called ergodic if for each given Borel subset E the condition

$$E = S_t^{-1}(E) \qquad \text{for } t \geq 0$$

implies $\mu(E)(1 - \mu(E)) = 0$. We admit also the following

DEFINITION 2. A measure μ is called non-trivial (with respect to $\{S_t\}$), if $\mu(P) = 0$ where P denotes the set of all periodic points.

It is easy to verify that if μ is a nontrivial invariant measure, then $\mu(O(x)) = 0$ for each $x \in X$. In particular any nontrivial invariant measure is continuous, that is $\mu(\{x\}) = 0$ for each singleton $\{x\} \subset X$. A relationship between strictly turbulent trajectories and nontrivial invariant measures is shown by the following

PROPOSITION 1. If $\{S_t\}_{t \geq 0}$ admits a strictly turbulent trajectory, then there exists for $\{S_t\}_{t \geq 0}$ a nontrivial ergodic invariant measure.

PROOF. Let $L(x_0)$ be the limit set of the strictly turbulent trajectory. Since $L(x_0)$ is compact and invariant, there exists an ergodic invariant measure supported on $L(x_0)$. The lack of periodic points in $L(x_0)$ implies that the measure is nontrivial.

5. The model of turbulence.

Consider the differential equation

$$\frac{\partial u}{\partial t} = \lambda u - x\frac{\partial u}{\partial x} \tag{7}$$

on the domain $t \geqslant 0$, $x \geqslant 0$. By solution of (7) we mean a continuously differentiable function $u(t, x)$ for which (7) is satisfied, for all $t \geqslant 0$, $x \geqslant 0$. We shall consider equation (7) with boundary value conditions

$$u(t, 0) = 0\ , \qquad u(0, x) = v(x)\ . \tag{8}$$

Denote by V the space of all continuously differentiable functions $v\colon R^{+} \to R$ such that $v(0) = 0$. The metric norm in V is defined by the formula

$$\|v\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|v\|_n}{1 + \|v\|_n}, \qquad \|v\|_n = \sup\{|v'(x)|\colon 0 \leqslant x \leqslant n\}\ .$$

For any $v \in V$ there exists exactly one solution of (7), (8), namely

$$u(t, x) = e^{\lambda t} v(xe^{-t})\ .$$

Thus the evolution in time of solutions of the boundary value problem (7), (8) is given by the group of transformations

$$(S_t v)(x) = e^{\lambda t} v(xe^{-t})\ , \qquad 0 \leqslant x < \infty\ . \tag{9}$$

We shall consider these transformations only for $t \geqslant 0$. The behavior of the trajectories of the semidynamical system $\{S_t\}_{t\geqslant 0}$ depends upon λ. We have, namely, the following

THEOREM 3. If $\lambda < 1$, then for each $v \in V$

$$\lim_{t\to\infty} \|S_t v\| = 0 \tag{10}$$

and the unique measure invariant under $\{S_t\}_{t\geqslant 0}$ is supported on the fixed point $v \equiv 0$. If $\lambda \geqslant 2$ the semidynamical system $\{S_t\}_{t\geqslant 0}$ admits a strictly turbulent trajectory and, consequently, there exists for $\{S_t\}_{t\geqslant 0}$ an ergodic nontrivial invariant measure.

PROOF. Assume first that $\lambda < 1$. From the definition of $\|\cdot\|_n$ it follows that

$$\|S_t v\|_n \leqslant e^{(\lambda-1)t}\|v\|_n .$$

Consequently $\|S_t v\|_n \to 0$ as $t \to \infty$ uniformly on each ball

$$B_n(r) = \{v\colon \|v\|_n \leqslant r\} .$$

According to the definition of $\|\cdot\|$ this implies (10) and shows that each (finite) invariant under $\{S_t\}$ measure μ is supported on the se $B_n(0)$. Thus μ is supported on the intersection

$$\bigcap_{n=1}^{\infty} B_n(0)$$

which contains the unique element $v \equiv 0$.

Now assume $\lambda \geqslant 2$ and denote by V_λ the space of all differentiable functions $v\colon [0, \infty) \to R$ such that

$$\text{(11)} \qquad \begin{cases} v(0) = v'(0) = 0 , \\ v' \text{ is locally absolutely continuous on } [0, \infty) , \\ |v''(x)| \leqslant x^{\lambda-2} . \end{cases}$$

It is easy to see that V_λ is a compact subspace of V. For each function $v\colon [0, \infty] \to R$ denote by $w = F(v)$ its restriction to the interval $[0, 1]$. Thus $W_\lambda = F(V_\lambda)$ contains all differentiable functions $w\colon [0, 1] \to R$ satisfying conditions analogous to (11). The set W_λ with the topology defined by the norm

$$\|w'\|_1 = \sup\{|w(x)|\colon 0 \leqslant x \leqslant 1\}$$

is a compact topological space and F is, obviously, a continuous mapping from V_λ onto W_λ. For each $w \in W_\lambda$ write

$$(T_t w)(x) = e^{\lambda t} w(xe^{-t}) , \qquad 0 \leqslant x \leqslant 1 , \ t \geqslant 0 .$$

It is easy to see that $S_t(V_\lambda) \subset V_\lambda$ and $T_t(W_\lambda) \subset W_\lambda$ for $t \geqslant 0$ and that the diagram

$$\begin{tikzcd} V_\lambda \arrow[r, "S_t"] \arrow[d, "F"'] & V_\lambda \arrow[d, "F"] \\ W_\lambda \arrow[r, "T_t"] & W_\lambda \end{tikzcd}$$

commutes. Now we define a number $r > 0$ and two sets A, $B \subset W_\lambda$ by setting $r = \ln 2$, $\alpha = \lambda - 1$ and

$$A = \{w \in W_\lambda \colon w'(x) = w'(\tfrac{1}{2}) \text{ for } \tfrac{1}{2} \leqslant x \leqslant 1\},$$

$$B = \{w \in W_\lambda \colon w'(x) = w'(\tfrac{1}{2}) + \alpha^{-1}(x - \tfrac{1}{2})^\alpha \text{ for } \tfrac{1}{2} \leqslant x \leqslant 1\}.$$

The sets A, B are evidently closed, disjoint, nonempty and an easy computation shows that

$$T_r(A) = T_r(B) = W_\lambda \supset A \cup B .$$

Thus according to the Theorem 2 there exists a point $v_0 \in V_\lambda$ such that the trajectory $\{S_t v_0 \colon t \geqslant 0\}$ is strictly turbulent with respect to the dynamical system $\{S_t\}_{t \geqslant 0}$ restricted to the space V_λ. Since V_λ is closed, the same trajectory is strictly turbulent on the whole space V. This completes the proof.

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