JINDŘICH BEČVÁŘ

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Abelian Groups in which Every Pure Subgroup is an Isotype Subgroup.

JINDŘICH BEČVÁŘ (*)

All groups in this paper are assumed to be abelian groups. Concerning the terminology and notation we refer to [2]. In addition, if $G$ is a group then $G_t$ and $G_p$ are the torsion part and the $p$-component of $G$, respectively. Let $G$ be a group and $p$ a prime. Following Rangaswamy [11], a subgroup $H$ of $G$ is said to be $p$-absorbing, resp. absorbing in $G$ if $(G/H)_p = 0$, resp. $(G/H)_t = 0$. Obviously, every $p$-absorbing subgroup of $G$ is $p$-pure in $G$. Recall that a subgroup $H$ of $G$ is isotype in $G$ if $H \cap p^\alpha G = p^\alpha H$ for all primes $p$ and all ordinals $\alpha$. For example, $G_t$ and every $G_p$ are isotype in $G$, every basic subgroup of $G_p$ is isotype in $G_p$. If $H$ is an isotype subgroup of $G$ and $A$ a subgroup of $G$ containing $H$ then $H$ is isotype in $A$. If $G$ is torsion then a subgroup $H$ of $G$ is isotype in $G$ iff every $H_p$ is isotype in $G_p$. Each absorbing subgroup of $G$ is isotype in $G$ (see lemma 103.1, [2]).

The notion of isotype subgroups has been introduced by Kulikov [7] and investigated by Irwin and Walker [4]. It is well-known that there are groups in which not every pure subgroup is isotype (see e.g. [4] or ex. 6, 7, § 80, [2]).

The purpose of this paper is to describe the classes of all groups in which every pure subgroup is an isotype subgroup, every isotype subgroup is a direct summand, every isotype subgroup is an absolute direct summand, every neat subgroup is an isotype subgroup and every isotype subgroup is an absorbing subgroup.

(*) Indirizzo dell’A.: Matematicko-Fyzikální Fakulta, Sokolovská 83, 18600 Praha 8 (Cecoslovacchia).
Note that the classes of all groups in which every subgroup is a neat, resp. a pure subgroup, resp. a direct summand, resp. an absolute direct summand have been described in [12] and [5], resp. [3], resp. [6], resp. [12]; these classes coincide with the class of all elementary groups. The classes of all groups in which every neat subgroup is a pure subgroup, resp. a direct summand, resp. an absolute direct summand have been described in [9], [12] and [14] (see theorem 4), resp. [12] and [8], resp. [12] (see theorem 3). The classes of all groups in which every pure subgroup is a direct summand, resp. an absolute direct summand have been described in [15] and [3] (see theorem 2), resp. [12] (see theorem 3). The class \( \mathcal{A} \) of all groups in which every direct summand is an absolute direct summand has been described in [12]; \( G \in \mathcal{A} \) iff either \( G \) is a torsion group in which each \( p \)-component is either divisible or a direct sum of cyclic groups of the same order or \( G \) is divisible or \( G = G_t \oplus R \), where \( G_t \) is divisible and \( R \) is indecomposable. The class \( \mathcal{B} \) of all groups in which every absorbing subgroup is a direct summand has been described in [11] and [12]; \( G \in \mathcal{B} \) iff \( G = T \oplus D \oplus N \), where \( T \) is torsion, \( D \) is divisible and \( N \) is a direct sum of a finite number mutually isomorphic rank one torsion free groups. Finally, in [12] have been described the classes of all groups in which every neat, resp. pure subgroup is an absorbing subgroup (see theorem 6).

**Definition.** Let \( \mathcal{C} \) be the class of all groups in which every pure subgroup is an isotype subgroup.

**Lemma 1.** The class \( \mathcal{C} \) is closed under pure subgroups.

**Proof.** Obvious.

**Lemma 2.** Let \( G \) be a group, \( p \) a prime and \( S \) a \( p \)-pure subgroup of \( G \). If \( G_p \) is a direct sum of a divisible and a bounded group then \( p^\alpha S = S \cap p^\alpha G \) for every ordinal \( \alpha \).

**Proof.** Let \( S \) be a \( p \)-pure subgroup of \( G \). Since \( S_p \) is pure in \( G_p \), \( S_p = D \oplus B \), where \( D \) is divisible and \( B \) is bounded (see e.g. lemma 4.2, [1]). Now, \( S = S_p \oplus H \), \( p^\alpha S = p^\alpha S_p \oplus p^\alpha H = D \oplus p^\alpha H \) and obviously \( p^\alpha S \) is \( p \)-divisible. Consequently, \( p^\alpha S = S \cap p^\alpha G \) for every ordinal \( \alpha \).

**Lemma 3.** Let \( G \) be a group, \( p \) a prime and \( k \) a natural number. If \( p^{\alpha+k}G_p \) is not essential in \( p^\alpha G_p \) and either \( p^{\alpha+k+1}G_p \) is nonzero or \( p^{\alpha+k+1}G \) is not torsion then \( G \notin \mathcal{C} \).
PROOF. There is a nonzero element $n \in p^{\omega}G[p]$ such that $\langle n \rangle \cap \cap p^{\omega+k}G = 0$. Let $g \in p^{\omega+k}G$ and either $0 \neq pg \in G$ or $o(g) = \infty$. Write $X = \langle p^{\omega+k}G[p], pg, n + g \rangle$. Now, $\langle n \rangle \cap X = 0$. For, if $n = x + apg + b(n + g)$, where $a, b$ are integers and $x \in p^{\omega+k}G[p]$, then

$$(1 - b)n = x + apg + bg \in \langle n \rangle \cap p^{\omega+k}G = 0.$$ 

Hence $p|1 - b$, $(ap + b)pg = 0$—a contradiction. Let $H$ be an $\langle n \rangle$-high subgroup of $G$ containing $X$. Since $\langle n \rangle \subset p^\omega G$, $H$ is pure in $G$ (see [10]). Now, $pg \in p^{\omega+k+1}G \cap H \setminus p^{\omega+k+1}H$. For, if $pg = ph$ for some $h \in p^{\omega+k}H$ then $g - h \in p^{\omega+k}G[p] \subset H$, $g \in H$ and $n \in H$—a contradiction. Consequently, $G \not\subseteq C$.

**Lemma 4.** If $G$ is a $p$-group then $G \in C$ iff either $G$ is a direct sum of a divisible and a bounded group or $G^1$ is elementary.

**Proof.** If $G$ is a direct sum of a divisible and a bounded group then $G \in C$ by lemma 2. If $G^1$ is elementary and $S$ is a pure subgroup of $G$ then $p^\omega S = S \cap p^\omega G$ and $p^{\omega+1}S = S \cap p^{\omega+1}G = 0$. Hence $G \in C$.

Conversely, let $G \in C$. Let $G^1 = D \oplus R$, where $D$ is divisible and $R$ is reduced. If both $D$ and $P$ are nonzero then write $R = \langle a \rangle \oplus R'$, where $o(a) = p^k$, $k > 0$. Now, $p^k G^1$ is not essential in $G^1$, $p^{k+1} G^1 \neq 0$ and lemma 3 implies a contradiction. If $G^1$ is reduced and not bounded then $G^1 = \langle a \rangle \oplus \langle b \rangle \oplus R'$, where $o(a) = p^k$, $o(b) = p^j$, $j - k > 2$. Now, $p^s G^1$ is not essential in $G^1$, $p^{s+1} G^1 \neq 0$ and lemma 3 implies a contradiction. Consequently, $G^1$ is either divisible or bounded. Let $G^1$ be nonzero divisible, write $G = G^1 \oplus H$. By [13], if $H$ is not bounded then for any nonzero element $a \in G^1[p]$ there is a pure subgroup $P$ of $G$ such that $P \cap G^1 = \langle a \rangle$. Obviously, $P$ is not isotype in $G$. Hence in this case, $G$ is a direct sum of a divisible and a bounded group. Let $G^1$ be bounded; suppose that $pG^1 \neq 0$. If $H$ is any high subgroup of $G$ then $H$ is not bounded. By [13], if $a$ is a nonzero element of $pG^1[p]$ then there is a pure subgroup $P$ of $G$ such that $P \cap G^1 = \langle a \rangle$. Obviously, $P$ is not isotype in $G$. Hence in this case, $G^1$ is elementary.

**Lemma 5.** Let $G$ be a torsion group. Then $G \in C$ iff $G_p \in C$ for every prime $p$.

**Proof.** Obvious.
LEMMA 6. Let $G$ be a group, $p$ a prime and $a$ an element of $G$ such that $o(a) = \infty$ or $p \mid o(a)$. If $H$ is a subgroup of $G$ maximal with respect to the conditions $pa \in H$, $a \notin H$, then $H$ is $q$-absorbing in $G$ for each prime $q \neq p$.

PROOF. Let $g \in G \setminus H$ and $gg \in H$, where $q$ is a prime, $q \neq p$. Evidently, $a \in \langle H, g \rangle$, i.e. $a = h + ng$, where $h \in H$ and $n$ is an integer, $(n, q) = 1$. Now, $pa = ph + png$ and hence $png \in H$. Therefore $q/pn$—a contradiction. Consequently, $H$ is $q$-absorbing in $G$.

THEOREM 1. Let $G$ be a group. The following are equivalent:

(i) Every pure subgroup of $G$ is isotype in $G$ (i.e. $G \in C$).

(ii) For every prime $p$ either $G_p$ is a direct sum of a divisible and a bounded group or $G_p$ is unbounded, $(G_p)^1$ is elementary and $p^\omega G$ is torsion.

PROOF. Suppose that (ii) holds. Let $S$ be a pure subgroup of $G$. By lemmas 1, 4 and 5, $S_t$ is isotype in $G_t$. Let $p$ be any prime. If $p^\omega G$ is torsion and $\alpha$ an ordinal, $\alpha > \omega$, then

$$p^\alpha S = p^\alpha S_t = S_t \cap p^\alpha G_t = S \cap p^\alpha G_t = S \cap p^\alpha G.$$ 

If $G_p$ is a direct sum of a divisible and a bounded group then by lemma 2, $p^\alpha S = S \cap p^\alpha G$ for every ordinal $\alpha$. Consequently, the subgroup $S$ is isotype in $G$.

Conversely suppose that $G \in C$. By lemmas 1 and 4, for every prime $p$ either $G_p$ is a direct sum of a divisible and a bounded group or $(G_p)^1$ is elementary. If for some prime $p$ $(G_p)^1$ is a nonzero elementary group and $p^\omega G$ is not torsion then $p(G_p)$ is not essential in $(G_p)^1$, $p^{\omega+2}G$ is not torsion and lemma 3 implies a contradiction. Consequently, if $(G_p)^1$ is a nonzero elementary group then $p^\omega G$ is torsion.

To finish the proof it is sufficient to show that if $G_p$ is unbounded, $(G_p)^1 = 0$ and $p^\omega G$ is not torsion then $G \notin C$. In this case, there is a linearly independent set $\{b_1, b_2, \ldots\}$ in $G$ such that $o(b_i) = p^i$. Let $g \in p^\omega G$ be an element of infinite order; there are elements $g_1, g_2, g_3, \ldots$ such that $p^{i-1}g_i = g$ for every $i = 1, 2, 3, \ldots$. Put $X = \langle pg, g_1 + b_1, g_2 + b_2, \ldots \rangle$. We show that $g \notin X$. Suppose $g \in X$, i.e.

$$g = z_0 pg + z_1 (g_1 + b_1) + \ldots + z_k (g_k + b_k),$$
where $z_0, \ldots, z_k$ are integers. Then

\[(*) \quad -(z_1b_1 + \ldots + z_kb_k) = z_0pg - g + z_1g_1 + \ldots + z_kg_k.\]

From (\(\ast\)) follows

\[ -p^{k-1}z_kb_k = p^{k-1}(z_0pg - g + z_1g_1 + \ldots + z_kg_k) \in G_p \cap p^\infty G = 0 \]

and hence $p \mid z_k$. From (\(\ast\)) follows

\[ -p^{k-2}z_{k-1}b_{k-1} - p^{k-2}z_kb_k = p^{k-2}(z_0pg - g + \ldots + z_kg_k) \in G_p \cap p^\infty G = 0 \]

and hence $p \mid z_{k-1}$ and $p^2 \mid z_k$. Finally we have $p^{k-1} \mid z_k$, $p^{k-2} \mid z_{k-1}, \ldots, p \mid z_2$. Now, from (\(\ast\)) follows

\[ z_1b_1 + \ldots + z_kb_k \in G_p \cap p^\infty G = 0 \]

and hence $p \mid z_1, \ldots, p \mid z_k$. Write $z_2 = p^2z'_2, \ldots, z_k = p^{k-1}z'_k$; from (\(\ast\)) follows

\[ (z_0p - 1 + z_1 + z'_2 + \ldots + z'_k)g = 0 \]

—a contradiction, since $p \mid z_1, \ldots, p \mid z_k$.

Let $H$ be a subgroup of $G$ maximal with respect to the properties $X \subset H$, $g \notin H$. By lemma 6, $H$ is $q$-pure in $G$ for every prime $q \neq p$. Moreover, $H$ is $p$-pure in $G$. For, the inclusion $p^iG \cap H \subset p^iH$ holds for $i = 0$, suppose that holds for $i$. Let $p^{i+1}a \in H$ for some $a \in G$, we may suppose that $p^ia \notin H$. Now, $g \in \langle p^i a, H \rangle$, i.e. $g = rp^ia + h$, where $h \in H$ and $r$ is an integer, and evidently, $(r, p) = 1$. Further, $rp^ia \in \langle g, H \rangle$, $tp^ia \in H$ and therefore $p^ia \in \langle g, H \rangle$, i.e. $p^ia = kg + h'$ for some $h' \in H$ and some integer $k$. Hence

\[ p^ia - kp^ig_{i+1} = h' \in p^iG \cap H. \]

By induction hypothesis, $h' = p^ih''$ for some $h'' \in H$. Now,

\[ p^{i+1}a = pkg + p^{i+1}h'' = p^{i+1}(kg_{i+1} + kb_{i+1} + h'') \]

and hence $p^{i+1}a \in p^{i+1}H$. Finally, the subgroup $H$ is not isotype in $G$. For, if $pg = ph$ for some $h \in p^\infty H$ then $g - h \in G_p \cap p^\infty G = 0$—a contradiction; hence $pg \in H \cap p^\infty G \setminus p^\infty H$. Consequently, $G \notin \mathcal{C}$. 
THEOREM 2. Let $G$ be a group. The following statements are equivalent:

(i) Every isotype subgroup of $G$ is a direct summand of $G$.

(ii) Every pure subgroup of $G$ is a direct summand of $G$.

(iii) $G = T \oplus D \oplus N$, where $T$ is a torsion group in which each $p$-component is bounded, $D$ is divisible and $N$ is a direct sum of a finite number mutually isomorphic torsion-free rank one groups.

PROOF. Obviously, (ii) implies (i). Assume (i). Since every absorbing subgroup of $G$ is isotype in $G$, every absorbing subgroup of $G$ is a direct summand of $G$. By [11], $G = T \oplus D \oplus N$, where $T$ is torsion reduced, $D$ divisible and $N$ is a direct sum of a finite number mutually isomorphic torsion free groups of rank one. Moreover, $T_p$ is bounded for every prime $p$. Otherwise, $T_p$ contains a proper basic subgroup $B$, $B$ is isotype in $T_p$ and hence in $G$. Consequently, $T_p = B \oplus C$, where $C$ is divisible—a contradiction. By theorem 1, every pure subgroup of $G$ is isotype in $G$ and hence a direct summand of $G$. Consequently, (ii) holds. The equivalence (ii) $\iff$ (iii) is proved in [15].

THEOREM 3. Let $G$ be a group. Then the following are equivalent:

(i) Every isotype subgroup of $G$ is an absolute direct summand of $G$.

(ii) Every pure (neat) subgroup of $G$ is an absolute direct summand of $G$.

(iii) Either $G$ is a torsion group each $p$-component in which is either divisible or a direct sum of cyclic groups of the same order or $G = G_t \oplus R$, where $G_t$ is divisible and $R$ is a group of rank one or $G$ is divisible.

PROOF. The equivalence (ii) $\iff$ (iii) is proved in [12]. Obviously, (ii) implies (i). If every isotype subgroup of $G$ is an absolute direct summand of $G$ then each isotype subgroup of $G$ is a direct summand of $G$ and every direct summand of $G$ is an absolute direct summand of $G$. Now, theorem 2 and [12] imply (iii).

THEOREM 4. Let $G$ be a group. The following are equivalent:
(i) Every neat subgroup of $G$ is isotype in $G$.

(ii) Every neat subgroup of $G$ is pure in $G$.

(iii) Either $G$ is a torsion group in which every $p$-component is either divisible or a direct sum of cyclic groups of orders $p^i$ and $p^{i+1}$ or $G_i$ is divisible.

**Proof.** The equivalence (ii) $\iff$ (iii) is proved in [9], the implication (i) $\iff$ (ii) is trivial. Suppose that every neat subgroup of $G$ is pure in $G$; hence (iii) holds. By theorem 1, every pure subgroup of $G$ is isotype in $G$. Consequently, (i) holds.

**Theorem 5.** Let $G$ be a group. The following are equivalent:

(i) Every subgroup of $G$ is isotype in $G$.

(ii) $G$ is elementary.

**Proof.** It follows from [3] and [6].

**Theorem 6.** Let $G$ be a group. The following statements are equivalent:

(i) Every isotype subgroup of $G$ is an absorbing subgroup of $G$.

(ii) Every pure (neat) subgroup of $G$ is an absorbing subgroup of $G$.

(iii) Either $G$ is torsion free or $G$ is cocyclic.

**Proof.** Since the equivalence (ii) $\iff$ (iii) is proved in [12], it is sufficient to show that (i) implies (iii). If $G$ is torsion then $G$ is indecomposable and hence cocyclic. If $G$ is mixed then $G_i$ is cocyclic, $G$ splits—a contradiction.

**References**


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