RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Distributional boundary values in $\mathfrak{D}'_{L^p}$ (IV)

Rendiconti del Seminario Matematico della Università di Padova, tome 63 (1980), p. 203-214

<http://www.numdam.org/item?id=RSMUP_1980__63__203_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 63 (1980)

Distributional Boundary Values in $\mathcal{D}'_{L^p}$ (IV).

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1. – Introduction.

In this paper we add information to [3, section IV] where we have obtained results concerning the Cauchy and Poisson integrals of distributions in $\mathcal{D}'_{L^p}$ corresponding to generalized half planes. Here we show that many of the results of [3, section IV] hold for further values of p than previously obtained and also prove additional results.

The n-dimensional notation to be used in this paper will be exactly as described in [2, section II] and in [3, section II]. We note especially the following notation. Throughout this paper $\sigma = (\sigma_1, \ldots, \sigma_n)$, n being the dimension, is an n-tuple where $\sigma_j = \pm 1$, $j = 1, \ldots, n$. For each of the 2^n n-tuples σ we put $C_\sigma = \{y \in \mathbf{R}^n : \sigma_j y_j > 0, j = 1, \ldots, n\}$. For each of these 2^n octants C_σ we correspondingly define the 2^n generalized half planes in $\mathbf{C}^n$ as $B_\sigma = \mathbf{R}^n + iC_\sigma = \{z \in \mathbf{C}^n : \sigma_j \operatorname{Im}(z_j) > 0, j = 1, \ldots, n\}$. The reader should review the definitions and properties of the function spaces $\mathcal{S}$, $\mathcal{D}_{L^p}$, $\mathcal{B} \equiv \mathcal{D}_{L^\infty}$, and $\dot{\mathcal{B}}$ and the generalized function spaces $\mathcal{S}'$ and $\mathcal{D}'_{L^p}$ contained in Schwartz [7, pp. 199-205 and pp. 233-248]. All other needed definitions, such as that of Fourier transform, are contained in [3, section II].

2. – The Cauchy and Poisson kernel functions.

For each of the $2^n \sigma$ put

$$(2.1) \qquad R_\sigma(z - t) = (2\pi i)^{-n} \prod_{j=1}^{n} \frac{\operatorname{sgn}(y_j)}{t_j - z_j}, \quad z = x + iy \in B_\sigma, \qquad t \in \mathbf{R}^n,$$

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where

$$\operatorname{sgn}(y_j) = \begin{cases} 1, & y_j > 0, \\ & \qquad j = 1, \dots, n. \\ -1, & y_j < 0, \end{cases}$$

$R_\sigma(z-t)$ is the Cauchy kernel corresponding to the generalized half plane B_σ. It is implicit by the analysis of Tillmann [8] that $R_\sigma(z-t) \in \mathcal{D}_{L^q}$, $(1/p) + (1/q) = 1$, $1 < p < \infty$, as a function of $t \in \mathbf{R}^n$, for arbitrary $z \in B_\sigma$. But $\mathcal{D}_{L^q} \subset \dot{\mathcal{B}} \subset \mathcal{B} \equiv \mathcal{D}_{L^\infty}$ for every q, $1 \leqslant q < \infty$ by [7, pp. 199-200]. We thus have proved the following fact.

LEMMA 2.1. *For each n-tuple σ, let $z \in B_\sigma$. As a function of $t \in \mathbf{R}^n$,*

$$(2.2) \qquad R_\sigma(z-t) \in \dot{\mathcal{B}} \cap \mathcal{D}_{L^q} \quad \textit{for all } q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \leqslant p < \infty.$$

We note two false statements in [3, p. 259, lines 5-7]. As we have shown above $R_\sigma(z-t)$ is an element of $\dot{\mathcal{B}}$ contrary to the false assertion in [3, p. 259, lines 5-6]. Further, as we shall see in section 3 of this paper, the Cauchy integral $C(U; z \in B_\sigma)$ is well defined for $U \in \mathcal{D}'_{L^1}$ and [3, Theorem 3] does hold for $p = 1$.

Now put

$$(2.3) \qquad \begin{aligned} K_\sigma(t; z) &= (4\pi)^n \left(\prod_{j=1}^{n} (\operatorname{sgn}(y_j))\, y_j\right) R_\sigma(z-t)\, \overline{R_\sigma(z-t)} \\ &= (\pi)^{-n} \prod_{j=1}^{n} \frac{(\operatorname{sgn}(y_j))\, y_j}{(t_j - x_j)^2 + y_j^2} \end{aligned}$$

for each σ where $z = x + iy \in B_\sigma$ and $t \in \mathbf{R}^n$. $K_\sigma(t; z)$ is the Poisson kernel corresponding to B_σ. Let α be any n-tuple of nonnegative integers and let $z \in B_\sigma$ be arbitrary but fixed. By the generalized Leibnitz rule we have

$$(2.4) \qquad D_t^\alpha(K_\sigma(t; z)) = \\ = (4\pi)^n \left(\prod_{j=1}^{n} (\operatorname{sgn}(y_j))\, y_j\right) \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} D_t^\beta(R_\sigma(z-t))\, D_t^\gamma\big(\overline{R_\sigma(z-t)}\big),$$

where the differential operator D_t^α is defined in [2, p. 37]. From (2.2), $D_t^\beta(R_\sigma(z-t)) \in L^2 \cap L^\infty$ and similarly $D_t^\gamma\big(\overline{R_\sigma(z-t)}\big) \in L^2 \cap L^\infty$ as functions of $t \in \mathbf{R}^n$. Thus by (2.4), $D_t^\alpha(K_\sigma(t; z)) \in L^1 \cap L^\infty$. But $L^1 \cap L^\infty \subseteq L^p$, $1 \leqslant p \leqslant \infty$. We conclude that $K_\sigma(t; z) \in \mathfrak{D}_{L^q}$ for all q, $1 \leqslant q \leqslant \infty$; and $K_\sigma(t; z) \in \dot{\mathfrak{B}}$ also since $\mathfrak{D}_{L^q} \subset \dot{\mathfrak{B}}$ for every q, $1 \leqslant q < \infty$, [7, pp. 199-200]. This proves the following result.

LEMMA 2.2. *For each n-tuple σ, let $z \in B_\sigma$. As a function of $t \in \mathbf{R}^n$,*

$$K_\sigma(t; z) \in \dot{\mathfrak{B}} \cap \mathfrak{D}_{L^q} \text{ for all } q, \quad 1 \leqslant q \leqslant \infty. \tag{2.5}$$

3. – The Cauchy integral.

$\mathfrak{D}'_{L^p}$, $1 < p \leqslant \infty$, is the dual space (space of continuous linear functionals) of $\mathfrak{D}_{L^q}$, $(1/p) + (1/q) = 1$; while $\mathfrak{D}'_{L^1}$ is the dual space of $\dot{\mathfrak{B}}$ [7, p. 200]. Thus let $U \in \mathfrak{D}'_{L^p}$ for any p, $1 \leqslant p < \infty$. For each n-tuple σ put

$$C(U; z \in B_\sigma) = \langle U_t, K_\sigma(z-t)\rangle, \quad z \in B_\sigma, \tag{3.1}$$

which is the Cauchy integral of U corresponding to B_σ. According to Lemma 2.1, $C(U; z \in B_\sigma)$ is a well defined function of $z \in B_\sigma$.

THEOREM 3.1. *Let $U \in \mathfrak{D}'_{L^p}$, $1 \leqslant p < \infty$. For each σ, $C(U; z \in B_\sigma)$ is an analytic function of $z \in B_\sigma$ such that*

$$|C(U; z \in B_\sigma)| \leqslant M \prod_{j=1}^{n} \left(|y_j|^{-(1/p)} + |y_j|^{-(1/p)-m_j}\right), \quad z = x + iy \in B_\sigma, \tag{3.2}$$

where M is a positive constant, which is independent of $z \in B_\sigma$, and each m_j, $j = 1, \ldots, n$, is a nonnegative integer.

PROOF. For $1 < p < \infty$ the desired results have been proved by Tillmann [8]. We now prove these facts for $p = 1$. By Schwartz [7, p. 201], $U \in \mathfrak{D}'_{L^1}$ implies

$$U = \sum_{|\alpha| \leqslant k} D_t^\alpha(f_\alpha(t)), \quad f_\alpha \in L^1, \tag{3.3}$$

where k is some nonnegative integer and the α are n-tuples of nonnegative integers. Recall our definition of the differential operator D_t^α

given in [2, p. 37]. Using (2.1) and (3.3) we have

$$(3.4)\qquad C(U; z\in B_\sigma) = \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}\langle f_\alpha(t), D_t^\alpha(R_\sigma(z-t))\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n}\prod_{j=1}^{n}(\operatorname{sgn}(y_j))\left\langle f_\alpha(t), D_t^\alpha\left(\prod_{j=1}^{n}\frac{1}{t_j-z_j}\right)\right\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n}\prod_{j=1}^{n}(\operatorname{sgn}(y_j))\cdot$$

$$\cdot\left\langle f_\alpha(t), (2\pi i)^{-|\alpha|}\prod_{j=1}^{n}(-1)^{\alpha_j}(\alpha_j)!(t_j-z_j)^{-\alpha_j-1}\right\rangle =$$

$$= \sum_{|\alpha|\leqslant k}(-1)^{|\alpha|}(2\pi i)^{-n-|\alpha|}\cdot$$

$$\cdot\left(\prod_{j=1}^{n}(-1)^{\alpha_j}(\operatorname{sgn}(y_j))(\alpha_j)!\right)\int_{\mathbf{R}^n} f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\,dt\,.$$

For each α in (3.4) put

$$(3.5)\qquad F_\alpha(z) = \int_{\mathbf{R}^n} f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\,dt\,,\qquad z\in B_\sigma\,.$$

Let S be an arbitrary compact subset of B_σ and let z vary over S for the moment; there exist numbers $\gamma_j > 0$, $j = 1, \ldots, n$, depending only on S such that $|y_j| \geqslant \gamma_j > 0$ for all $y = (y_1, \ldots, y_n)$ for which $z = x + iy \in S$. Thus for all $z = x + iy \in S$ and all $t\in\mathbf{R}^n$ we have

$$\left|f_\alpha(t)\prod_{j=1}^{n}(t_j-z_j)^{-\alpha_j-1}\right| = |f_\alpha(t)|\prod_{j=1}^{n}((t_j-x_j)^2+y_j^2)^{-(\alpha_j+1)/2}$$

$$(3.6)\qquad \leqslant |f_\alpha(t)|\prod_{j=1}^{n}|y_j|^{-\alpha_j-1}$$

$$\leqslant |f_\alpha(t)|\prod_{j=1}^{n}(\gamma_j)^{-\alpha_j-1}\,.$$

Recalling that each $f_\alpha(t)\in L^1$, we see that the right side of (3.6) is an L^1 function of $t\in\mathbf{R}^n$ that is independent of $z = x + iy \in S$. Thus by [1, p. 295, Theorem B.4], each $F_\alpha(z)$ defined in (3.5) is analytic in B_σ; hence so is $C(U; z\in B_\sigma)$ because of (3.4). By analysis as in (3.6) we have for $z \in B_\sigma$ that

$$
\begin{aligned}
|F_\alpha(z)| &\leqslant \int_{\mathbf{R}^n} |f_\alpha(t)| \left| \prod_{j=1}^n (t_j - z_j)^{-\alpha_j - 1} \right| dt \\
&\leqslant \prod_{j=1}^n |y_j|^{-\alpha_j - 1} \int_{\mathbf{R}^n} |f_\alpha(t)|\, dt\,.
\end{aligned} \tag{3.7}
$$

The growth (3.2) for $p = 1$ follows easily now by combining (3.4) and (3.7) where $F_\alpha(z)$ is defined in (3.5) for each α, $|\alpha| \leqslant k$. The proof is complete.

Of course $R_\sigma(z - t)$ does not belong to $\mathcal{D}_{L^1}$ as a function of $t \in \mathbf{R}^n$ for z arbitrary in B_σ. Thus we can not let $p = \infty$ in Theorem 3.1 because $C(U; z \in B_\sigma)$ does not exist for $U \in \mathcal{D}'_{L^\infty}$. Theorem 3.1 extends the corresponding information of Tillmann [8] to the case $p = 1$.

Now consider any of the 2^n n-tuples σ and the corresponding generalized half plane B_σ. Let $U \in \mathcal{D}'_{L^p}$, $1 \leqslant p \leqslant 2$, such that $U = \hat{V}$, where $V \in \mathcal{S}'$ and $\operatorname{supp}(V) \subseteq S^0_\sigma = \{t: -\infty < \sigma_j t_j \leqslant 0, j = 1, \ldots, n\}$. Let $H_\sigma(t)$ denote the characteristic function of S^0_σ and define the C^∞ function $\alpha(t)$ as in [3, p. 258] corresponding to S^0_σ. Notice that

$$
\mathcal{F}[H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle]; x] = (-2\pi i)^{-n} \prod_{j=1}^n \frac{\operatorname{sgn}(y_j)}{z_j}\,, \qquad z = x + iy \in B_\sigma\,, \tag{3.8}
$$

as in [3, p. 258, lines 19-20], where the Fourier transform in (3.8) is the L^1 transform and hence also the $\mathcal{S}'$ Fourier transform. Thus because of (3.8),

$$
\mathcal{F}[H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle]; x] \in \mathcal{D}_{L^2} \subset \mathcal{D}'_{L^2} \tag{3.9}
$$

as a function of $x \in \mathbf{R}^n$ for arbitrary $y \in C_\sigma$, and (3.8) implies

$$
H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle] = \mathcal{F}^{-1}\left[(-2\pi i)^{-n} \prod_{j=1}^n \frac{\operatorname{sgn}(y_j)}{z_j}\right], \qquad z = x + iy \in B_\sigma\,, \tag{3.10}
$$

with this inverse Fourier transform being in $\mathcal{S}'$ [7, p. 250]. Using the fact (3.9) and the proof of [7, p. 270, lines 3-17] we now have

$$(3.11)\qquad \mathcal{F}^{-1}\left[U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right)\right] = \\ = \mathcal{F}^{-1}[U]\,\mathcal{F}^{-1}\left[(-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right]$$

with this equality holding in $\mathcal{S}'$ and the convolution on the left being the distributional convolution [7, Chapter 6]. But $U=\hat{V}$ in $\mathcal{S}'$ implies $V=\mathcal{F}^{-1}[U]$ in $\mathcal{S}'$. Combining this with (3.10) and (3.11) we get

$$\mathcal{F}^{-1}\left[U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right)\right] = H_\sigma(t)\alpha(t)\exp\left[2\pi\langle y, t\rangle\right]V$$

in $\mathcal{S}'$ and hence

$$(3.12)\qquad \mathcal{F}[H_\sigma(t)\alpha(t)\exp[2\pi\langle y, t\rangle]V] = U*\left((-2\pi i)^{-n}\prod_{j=1}^{n}\frac{\operatorname{sgn}(y_j)}{z_j}\right), \\ z = x+iy\in B_\sigma\,,$$

in $\mathcal{S}'$. Our method of obtaining (3.12) gives an alternate method of obtaining the equality [3, p. 258, (12)], and note that we have this equality now under the assumption $U\in\mathcal{D}'_{L^p}$, $1\leqslant p\leqslant 2$. (Recall that [3, Theorem 3] did not include the case $p=1$.) With the equality (3.12) now obtained under the specified assumptions for $1\leqslant p\leqslant 2$ and with Theorem 3.1 above, we now state that [3, Theorem 3] holds for $p=1$ also in which case $q=\infty$ there. The techniques to prove the stated conclusions are the same for $p=1$ as for $1<p\leqslant 2$ with the exception that we now use our proof of (3.12) above to obtain [3, p. 258, (12)]. In addition we can now state a growth condition on $C(U; z\in B_\sigma)$ because of Theorem 3.1. For completeness we now state our extension of [3, Theorem 3].

THEOREM 3.2. *Let B_σ be any of the 2^n generalized half planes in $\mathbf{C}^n$. Let $U\in\mathcal{D}'_{L^p}$, $1\leqslant p\leqslant 2$, such that $U=\hat{V}$, where $V\in\mathcal{S}'$ and* $\operatorname{supp}(V)\subseteq$ $\subseteq S^0_\sigma=\{t\colon -\infty<\sigma_j t_j\leqslant 0, j=1,\ldots,n\}$. *Then* $V=\sum_{|\beta|\leqslant m}(-1)^{|\beta|}t^\beta h_\beta(t)$, $h_\beta(t)\in$ $\in L^q$, $(1/p)+(1/q)=1$; *$C(U; z\in B_\sigma)$ is analytic in B_σ and satisfies* (3.2);

$$(3.13)\qquad C(U; z\in B_\sigma) = \langle V, \exp[-2\pi i\langle z, t\rangle]\rangle\,,\qquad z\in B_\sigma\,,$$

as elements of $\mathcal{S}'$; *and* $C(U; z \in B_\sigma) \to U \in \mathcal{D}'_{L^p}$ *in the strong (and weak) topology of* $\mathcal{S}'$ *as* $\mathrm{Im}(z) \to 0$.

The above convergence of $C(U; z \in B_\sigma) \to U \in \mathcal{D}'_{L^p}$ in the strong topology of $\mathcal{S}'$ as $\mathrm{Im}(z) \to 0$ is proved as follows. After obtaining (3.13) we use the same proof as in [3, Theorem 3] to show that

$$C(U; z \in B_\sigma) = \langle V, \exp[-2\pi i \langle z, t\rangle]\rangle \to \hat{V} = U \tag{3.14}$$

in the weak topology of $\mathcal{S}'$ as $\mathrm{Im}(z) \to 0$. But $\mathcal{S}$ is a Montel space [7, p. 235]; hence by [4, p. 510, Corollary 8.4.9] the convergence in (3.14) is in the strong topology of $\mathcal{S}'$.

As a result of Theorem 3.2, the extension of [3, Theorem 3], and its proof, the results [3, Corollary 1 and Theorem 5] hold also for $1 \leqslant p \leqslant 2$ by using the same proofs as before. Note that [3, Theorem 6] has already been obtained for $1 \leqslant p \leqslant 2$.

4. – The Poisson integral.

Let $U \in \mathcal{D}'_{L^p}$ for any p, $1 \leqslant p \leqslant \infty$. For each σ put

$$P(U; z \in B_\sigma) = \langle U_t, K_\sigma(t; z)\rangle, \quad z \in B_\sigma, \tag{4.1}$$

which is the Poisson integral of U. By Lemma 2.2, $P(U; z \in B_\sigma)$ is a well defined function of $z \in B_\sigma$. Note that the Poisson integral $P(U; z \in B_\sigma)$ is well defined for $U \in \mathcal{D}'_{L^\infty}$ while $C(U; z \in B_\sigma)$ is not defined for $U \in \mathcal{D}'_{L^\infty}$; this is because $K_\sigma(t; z) \in \mathcal{D}_{L^1}$ while $R_\sigma(z - t)$ does not. In general $P(U; z \in B_\sigma)$ is not an analytic function which is in contrast to the Cauchy integral. However, the result [3, Theorem 7] holds for $1 \leqslant p \leqslant \infty$ by the same proof as before for the case $1 < p < \infty$; hence we have extended this result to include the case $p = 1$ and $p = \infty$ now; and $P(U; z \in B_\sigma)$ is an n-harmonic function for $U \in \mathcal{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$. We note a misprint in the proof of [3, Theorem 7]; [3, p. 262, line 19] should read

$$\prod_{j=1}^{n} \frac{y_j}{\pi |t_j - z_j|^2} = \prod_{j=1}^{n} \left(\frac{1}{2\pi i}\right)\left(\frac{1}{t_j - z_j} - \frac{1}{t_j - \bar{z}_j}\right).$$

[3, Theorems 8 and 10] related the Poisson integral with the Cauchy integral and the Fourier-Laplace transform. Because of the preceding information in this paper, these two theorems can now be seen to hold for $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$, by the same proofs as given in [3, Thorems 8 and 10] since we now know that the analysis on which these proofs are based holds for $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq 2$.

We now extend [3, Theorem 9] by obtaining this result for $U \in \mathfrak{D}'_{L^p}$ for any p, $1 \leq p \leq \infty$, and give a separate proof. Further, our extension is slightly more general than [3, Theorem 9]. Our result is as follows.

THEOREM 4.1. *Let $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq \infty$. For any of the 2^n n-tuples σ we have*

$$\lim_{\substack{y \to 0 \\ y \in C_\sigma}} \langle P(U; (x+iy) \in B_\sigma), \varphi(x) \rangle = \langle U, \varphi \rangle \tag{4.2}$$

for every $\varphi \in \mathfrak{D}_{L^1}$.

Theorem 4.1 is more general than [3, Theorem 9] since $\mathcal{S} \subset \mathfrak{D}_{L^1}$. Our present proof of Theorem 4.1 relies on the following two lemmas.

LEMMA 4.1. *Let $U \in \mathfrak{D}'_{L^p}$, $1 \leq p \leq \infty$. For any of the 2^n n-tuples σ we have*

$$\begin{aligned} \langle P(U; (x+iy) \in B_\sigma), \varphi(x) \rangle &= \\ &= \langle U, \langle K_\sigma(t; x+iy), \varphi(x) \rangle \rangle, \qquad y \in C_\sigma, \end{aligned} \tag{4.3}$$

for every $\varphi \in \mathfrak{D}_{L^1}$.

PROOF. Let $\varphi \in \mathfrak{D}_{L^1}$. A change of variable yields

$$\int_{\mathbf{R}^n} K_\sigma(t; x+iy)\varphi(x)\,dx = \int_{\mathbf{R}^n} K_\sigma(x; y)\varphi(x+t)\,dx \tag{4.4}$$

for all $y \in C_\sigma$ and $t \in \mathbf{R}^n$ where

$$K_\sigma(x; y) = (\pi)^{-n} \prod_{j=1}^{n} \frac{(\operatorname{sgn}(y_j))\, y_j}{x_j^2 + y_j^2}, \quad x \in \mathbf{R}^n, \quad y \in C_\sigma. \tag{4.5}$$

By the proof of Lemma 2.2, $K_\sigma(x; y) \in \dot{\mathfrak{B}} \cap \mathfrak{D}_{L^q}$ for all q, $1 \leq q \leq \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C_\sigma$ arbitrary. Thus by [7, p. 201, Théorème XXV] we have $K_\sigma(x; y) \in \mathfrak{D}'_{L^q}$ for every q, $1 \leq q \leq \infty$, as a function of $x \in \mathbf{R}^n$ for $y \in C_\sigma$ arbitrary. Hence for $U \in \mathfrak{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, we have that the distributional convolution

$$U * K_\sigma(x; y) \in \mathfrak{D}'_{L^\infty}, \quad y \in C_\sigma, \tag{4.6}$$

by [7, p. 203, Théorème XXVI]. Thus for any $\varphi \in \mathfrak{D}_{L^1}$, $\langle U * K_\sigma(x; y), \varphi\rangle$ exists because of (4.6); and

$$\langle U * K_\sigma(x; y), \varphi\rangle = \langle U, \langle K_\sigma(x; y), \varphi(x + t)\rangle\rangle, \quad y \in C_\sigma, \tag{4.7}$$

by the definition of distributional convolution ([7, Chapter 6] or [3, p. 251].) Combining (4.4) and (4.7) we obtain for $y \in C_\sigma$ that

$$\langle U, \langle K_\sigma(t; x + iy), \varphi(x)\rangle\rangle = \langle U, \langle K_\sigma(x; y), \varphi(x + t)\rangle\rangle = \\ = \langle U * K_\sigma(x; y), \varphi\rangle$$

which proves that the right side of (4.3) is well defined for any $y \in C_\sigma$. For $U \in \mathfrak{D}'_{L^p}$, $1 \leqslant p \leqslant \infty$, we have by the characterization theorem of Schwartz [7, p. 201, Théorème XXV] that

$$U = \sum_{|\alpha| \leqslant m} D_t^\alpha(f_\alpha(t)), \quad f_\alpha \in L^p, \tag{4.8}$$

for some nonnegative integer m. Using (4.8), a change of order of integration, which is valid here, and the fact that differentiation can be taken under the integral sign as needed below, we obtain for any $y \in C_\sigma$ that

$$\begin{aligned} \langle U, \langle K_\sigma(t; x + iy), \varphi(x)\rangle\rangle &= \\ &= \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \int_{\mathbf{R}^n} f_\alpha(t) \int_{\mathbf{R}^n} D_t^\alpha(K_\sigma(t; x + iy))\varphi(x)\, dx\, dt \\ &= \sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} \int_{\mathbf{R}^n} \varphi(x) \int_{\mathbf{R}^n} f_\alpha(t) D_t^\alpha(K_\sigma(t; x + iy))\, dt\, dx \\ &= \left\langle \left\langle \sum_{|\alpha| \leqslant m} D_t^\alpha(f_\alpha(t)), K_\sigma(t; x + iy)\right\rangle, \varphi(x)\right\rangle \\ &= \langle P(U; (x + iy) \in B_\sigma), \varphi(x)\rangle \end{aligned}$$

which proves (4.3). The proof is complete.

LEMMA 4.2. *For any of the 2^n n-tuples σ, let $z = x + iy \in B_\sigma$. Let $\varphi \in \mathfrak{D}_{L^1}$. We have*

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \int_{\mathbf{R}^n} K_\sigma(t; x+iy)\varphi(x)\,dx = \varphi(t) \tag{4.9}$$

in the topology of $\mathfrak{D}_{L^q}$ for all q, $1 \leqslant q \leqslant \infty$, and in the topology of $\dot{\mathfrak{B}}$.

PROOF. For $\varphi \in \mathfrak{D}_{L^1}$ and any n-tuple α of nonnegative integers, we have using (4.4) that

$$D_t^\alpha(\langle K_\sigma(t; x+iy), \varphi(x)\rangle = \int_{\mathbf{R}^n} D_t^\alpha(\varphi(x+t))\,K_\sigma(x; y)\,dx\,, \qquad y \in C_\sigma\,, \tag{4.10}$$

where $K_\sigma(x; y)$ is defined in (4.5) and the differentiation under the integral sign is valid. Now $\varphi \in \mathfrak{D}_{L^1}$ implies $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathfrak{D}_{L^1}$. By [7, p. 200], $\mathfrak{D}_{L^1} \subseteq \mathfrak{D}_{L^q} \subset L^q$ for all q, $1 \leqslant q \leqslant \infty$, and $\mathfrak{D}_{L^1} \subset \dot{\mathfrak{B}} \subset \mathfrak{D}_{L^\infty}$. Now $K_\sigma(t; z)$ defined in (2.3) is the Poisson kernel function for the tube B_σ in $\mathbf{C}^n$ corresponding to the cone C_σ in $\mathbf{R}^n$; hence $K_\sigma(t; z)$ is an approximate identity [6, Proposition 2]. ($K_\sigma(x; y)$ is also an approximate identity.) Using (4.10) and [6, Proposition 2] we have

$$D_t^\alpha\Big(\int_{\mathbf{R}^n} K_\sigma(t; x+iy)\varphi(x)\,dx\Big) - D_t^\alpha(\varphi(t)) = \tag{4.11}$$
$$= \int_{\mathbf{R}^n} (\Psi_\alpha(x+t) - \Psi_\alpha(t))\,K_\sigma(x; y)\,dx$$

where $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathfrak{D}_{L^1} \subseteq \mathfrak{D}_{L^q} \subset L^q$ for all q, $1 \leqslant q \leqslant \infty$, as noted above. Now using (4.11), [6, Proposition 2], and the same method of proof used in [5, Theorem on pp. 17-19, Theorem on p. 32] we have

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \Big\| D_t^\alpha\Big(\int_{\mathbf{R}^n} K_\sigma(t; x+iy)\varphi(x)\,dx\Big) - D_t^\alpha(\varphi(t))\Big\|_{L^q} = \tag{4.12}$$
$$= \lim_{\substack{y\to 0\\ y\in C_\sigma}} \Big\| \int_{\mathbf{R}^n} (\Psi_\alpha(x+t) - \Psi_\alpha(t))\,K_\sigma(x; y)\,dx \Big\|_{L^q} = 0$$

for any q, $1 \leqslant q < \infty$, any n-tuple α of nonnegative integers, and any $\varphi \in \mathfrak{D}_{L^1}$. (4.12) thus proves (4.9) in the topology of $\mathfrak{D}_{L^q}$ for all q,

$1 \leq q < \infty$. Further, $\varphi \in \mathcal{D}_{L^1} \subset \dot{\mathcal{B}} \subset \mathcal{D}_{L^\infty}$ implies $\Psi_\alpha(t) = D_t^\alpha(\varphi(t)) \in \mathcal{D}_{L^1} \subset \subset \dot{\mathcal{B}} \subset \mathcal{D}_{L^\infty}$ for any n-tuple α of non-negative integers; and by the definition of $\dot{\mathcal{B}}$, $\Psi_\alpha(t) \to 0$ as $|t| \to \infty$ with $\Psi_\alpha(t)$ being continuous and bounded on $\mathbf{R}^n$. This implies that $\Psi_\alpha(t) = D_t^\alpha(\varphi(t))$ is uniformly continuous and bounded for $t \in \mathbf{R}^n$. Thus by the proof of [6, Proposition 3, (*b*)] we have

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \int_{\mathbf{R}^n} \Psi_\alpha(x+t) K_\sigma(x; y)\, dx = \Psi_\alpha(t)$$

uniformly for $t \in \mathbf{R}^n$. From this and (4.10) it follows that

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \left\| D_t^\alpha\left(\int_{\mathbf{R}^n} K_\sigma(t; x+iy)\varphi(x)\, dx\right) - D_t^\alpha(\varphi(t))\right\|_{L^\infty} = 0$$

which proves (4.9) in the topology of $\dot{\mathcal{B}}$ and in the topology of $\mathcal{B} \equiv \mathcal{D}_{L^\infty}$. The proof is complete.

We now give the

PROOF OF THEOREM 4.1. For any $\varphi \in \mathcal{D}_{L^1}$ the proof of Lemma 4.1 yields that $\langle U, \langle K_\sigma(t; x+iy), \varphi(x)\rangle\rangle$ exists for $y \in C_\sigma$. The continuity of $U \in \mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, and Lemma 4.2 combine to prove

$$\lim_{\substack{y\to 0\\ y\in C_\sigma}} \langle U, \langle K_\sigma(t; x+iy), \varphi(x)\rangle\rangle = \langle U, \varphi\rangle \tag{4.13}$$

and $\langle U, \varphi\rangle$ is well defined for $\varphi \in \mathcal{D}_{L^1}$ since $\mathcal{D}_{L^1} \subseteq \mathcal{D}_{L^q}$ for all q, $1 \leq q \leq \infty$, and $\mathcal{D}_{L^1} \subset \dot{\mathcal{B}}$. The desired result (4.2) follows now by combining (4.13) and (4.3). The proof of Theorem 4.1 is complete.

If $p = \infty$ in Theorem 4.1, then (4.2) proves that $P(U; (x+iy) \in \in B_\sigma) \to U$ in exactly the weak topology of $\mathcal{D}'_{L^\infty}$ as $y \to 0$, $y \in C_\sigma$, since (4.2) holds for each $\varphi \in \mathcal{D}_{L^1}$ whose dual space is $\mathcal{D}'_{L^\infty}$.

5. – Acknowledgement.

The research on this paper began while the author was Visiting Associate Professor at Iowa State University. The author thanks the Department of Mathematics of Iowa State University for its support during 1978-1979.

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Manoscritto pervenuto l'8 novembre 1979.