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On Asymptotic Series of Symbols and of General Pseudo-Differential Operators.

S. ZAIDMAN (*)

Introduction.

In their study of an algebra of pseudo-differential operators [6], Kohn and Nirenberg presented concepts of canonical pseudo-differential operators, asymptotic series of such operators-related to the notion of order and true-order, —, homogeneous symbols of real degree, asymptotic sums of such symbols and the mutual connection between them. A very interesting existential result, apparently belonging to Hörmander is given in Theorem 2 (iv) of [6] (see also [4] and [1]).

Similar facts are known in various other classes of symbols and associated pseudo-differential operators (see [5], [8], [10], [11], [12]).

In the present work we introduce, following [6] and [13], a class of symbols which are only measurable with respect to $\xi \in R^n - \{0\}$ and C^∞ for $x \in R^n$ (precise definition will follow) and we indicate how the above mentionned concepts and results can be extended to this new situation. We will refer to [14] for a preliminary version of a part of this paper (general pseudo-differential operators were not considered there).

1. – We consider a class of symbols which is denoted by $\mathfrak{S}$; it consists of measurable complex—valued functions $\sigma(x, \xi)$ which are defined on $\mathbf{R}^n \times \mathbf{R}^n - \{0\}$.

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We assume the following properties:

a) $\lim_{x\to\infty} \sigma(x,\xi) = \sigma(\infty,\xi)$ exists for all $\xi \in \mathbf{R}^n - \{0\}$ and is a measurable bounded function there;

b) if $\sigma'(x,\xi)$ means the difference $\sigma(x,\xi) - \sigma(\infty,\xi)$ then

$$\sigma'(x,\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} \exp\,[ix\cdot\eta]\tilde{\sigma}'(\eta,\xi)\,d\eta\,, \quad \forall(x,\xi)\in\mathbf{R}^n\times\mathbf{R}^n - \{0\} \tag{1.1}$$

where $x\cdot\eta = \sum_1^n x_l\eta_l$, $\eta\in\mathbf{R}^n$, while $\tilde{\sigma}'(\eta,\xi)$ is a complex-valued function defined on $\mathbf{R}^n\times\mathbf{R}^n - \{0\}$ which is measurable in η for all $\xi\in \mathbf{R}^n - \{0\}$, is measurable in ξ for all $\eta\in\mathbf{R}^n$ and satisfies an estimate

$$|\tilde{\sigma}'(\eta,\xi)|\leqslant k(\eta)\,, \qquad \eta\in\mathbf{R}^n \tag{1.2}$$

where $(1+|\eta|^2)^l k(\eta)\in L^1(\mathbf{R}^n)$, $\forall l = 0, 1, 2, \ldots,$ $\left(|\eta|^2 = \sum_1^n \eta_l^2\right)$.

Remark. The function $\tilde{\sigma}'(\eta,\xi)$ is the Fourier transform in the sense of temperate distributions of the bounded function $\sigma'(x,\xi)$. Furthermore $\sigma'(x,\xi)$ and $\sigma(x,\xi)$ are $C^\infty(\mathbf{R}^n_x)$ and $\sup_{(x,\xi)} |D_x^\alpha \sigma(x,\xi)| < \infty$, $\forall\alpha = (\alpha_1 \ldots \alpha_n)$, a multi-index of non-negative integers.

Let us consider now an infinite sequence $\{a_j(x,\xi)\}_{j=0}^\infty$ of functions in $\mathfrak{S}$, and a strictly decreasing sequence of real numbers convergent to $-\infty$, $r_0 > r_1 > r_2 \ldots$; also, we consider a $C^\infty(\mathbf{R}^n)$ function $\zeta(\xi)$, which is non-negative, equals 0 for $|\xi|\leqslant\frac{1}{2}$, equals one for $|\xi|\geqslant 1$, such that $0\leqslant\zeta(\xi)\leqslant 1$, $\forall\xi\in\mathbf{R}^n$.

Our main interest in this section is about series of the form

$$\sum_{j=0}^\infty \zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j} a_j(x,\xi)\,, \qquad \text{where } |\xi|^2 = \sum_{l=1}^n \xi_l^2 \tag{1.3}$$

and $(t_j)_0^\infty$ is a (conveniently chosen) sequence of real numbers.

Note that for any fixed $\xi\in\mathbf{R}^n - \{0\}$ the sum is finite (it equals

$$\sum_{t_j<2|\xi|} \zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j} a_j(x,\xi)\,, \qquad x\in\mathbf{R}^n$$

and in any sphere $|\xi|\leqslant m$ it is written also as

$$\left.\sum_{1\leqslant t_j\leqslant 2m} \zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j} a_j(x,\xi)\right).$$

We may define the function

$$a(x, \xi) = \sum_{j=0}^{\infty} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} a_j(x, \xi);$$

Obviously that $a(x, \xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$, $x \in \mathbf{R}^n$ and that it is a measurable complex-valued function. The limit $a(\infty, \xi)$ exists for $\xi \in \mathbf{R}^n - \{0\}$ being given by

$$\sum_{1 \leqslant t_j \leqslant 2m} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} a_j(\infty, \xi) \qquad \text{when } |\xi| \leqslant m;$$

accordingly, the difference $a'(x, \xi) = a(x, \xi) - a(\infty, \xi)$ is expressed by

$$\sum_{j=0}^{\infty} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} a'_j(x, \xi) .$$

We see that $a(x, \xi)$ is infinitely differentiable with respect to x, for any $\xi \in \mathbf{R}^n - \{0\}$.

2. – In this section we shall give a global estimate (i.e. on $\mathbf{R}^n \times \times \mathbf{R}^n - \{0\}$)of the above defined function $a(x, \xi)$ and of the remainders of order N:

$$b_N(x, \xi) = a(x, \xi) - \sum_{j=0}^{N-1} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} a_j(x, \xi) = \sum_{j=N}^{\infty} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} a_j(x, \xi) .$$

Precisely we will prove the following

THEOREM 1. – *It is possible to choose a sequence of real numbers* $(t_j)_0^\infty$, $t_0 = 1$, *such that the inequalities*:

$$(2.1) \qquad |a(x, \xi)| \leqslant C|\xi|^{r_0}, \qquad |b_N(x, \xi)| \leqslant C_N |\xi|^{r_N},$$
$$(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n - \{0\}, \qquad N = 1, 2, \ldots$$

be satisfied. (In particular, the estimates

$$(2.2) \quad |a(\infty, \xi)| \leqslant C|\xi|^{r_0}, \qquad |b_N(\infty, \xi)| \leqslant C_N |\xi|^{r_N}, \qquad \xi \in \mathbf{R}^n - \{0\},$$

$N = 1, 2, \ldots$ are also true.)

PROOF. - Any function in the class $\mathfrak{S}$ is bounded on $R^n \times R^n - \{0\}$; let be $M_j = \sup_{(x,\xi)} |a_j(x, \xi)|$, $j = 0, 1, 2, \ldots$.

We indicate a choice of the sequence $(t_j)_0^\infty$ in such a way that the estimates

$$\left|\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j} a_j(x, \xi)\right| \leqslant \frac{1}{2^j}|\xi|^{r_{j-1}}, \qquad j = 1, 2, \ldots, (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n - \{0\} \tag{2.3}$$

be all verified.

Actually, for $|\xi| \leqslant t_j/2$, $\zeta(\xi/t_j) = 0$ and the estimates are obvious.

For $|\xi| > \frac{1}{2}t_j$ we remark firstly (using $0 \leqslant \zeta \leqslant 1$), the inequality

$$\left|\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j} a_j(x, \xi)\right| \leqslant M_j|\xi|^{r_j}.$$

Consequently, we can obtain (2.3) as a consequence of the stronger inequalities

$$M_j|\xi|^{r_j} \leqslant \frac{1}{2^j}|\xi|^{r_{j-1}}, \qquad j = 1, 2, \ldots, \ \xi \in \mathbf{R}^n - \{0\} \text{ and } |\xi| > \frac{1}{2}t_j$$

or

$$2^j M_j \leqslant |\xi|^{r_{j-1}-r_j}, \qquad j = 1, 2, \ldots, |\xi| > \frac{1}{2}t_j.$$

We know that $r_{j-1} - r_j > 0$, $\forall j = 1, 2, \ldots$ hence $(t_j/2)^{r_{j-1}-r_j} < |\xi|^{r_{j-1}-r_j}$.

We choose now t_j in such a way as to have

$$2^j M_j \leqslant \left(\frac{t_j}{2}\right)^{r_{j-1}-r_j}$$

that is

$$t_j \geqslant 2 \cdot 2^{j(r_{j-1}-r_j)^{-1}} \cdot M_j^{(r_{j-1}-r_j)^{-1}}, \qquad \forall j = 1, 2, \ldots.$$

This special choice of the sequence $(t_j)_1^\infty$ has (2.3) as a corollary. We have seen that $a(x, \xi) = 0$ for $x \in R^n$, $|\xi| \leqslant \frac{1}{2}$; consider now $\frac{1}{2} \leqslant |\xi| \leqslant 1$. In this case $|\xi/t_j| \leqslant 1/t_j$.

Assume $t_1 = 2$ or bigger. Then $|\xi/t_j| \leqslant \frac{1}{2}$, $j = 1, 2, \ldots$ and $\zeta(\xi/t_j) = 0$ for $j = 1, 2, \ldots$. Hence $a(x, \xi)$ reduces for these ξ to the single term $a(x, \xi) = \zeta(\xi)|\xi|^{r_0} a_0(x, \xi)$ and we get $|a(x, \xi)| \leqslant M_0|\xi|^{r_0}$, $0 < |\xi| \leqslant 1$, $x \in R^n$.

Consider now $|\xi|\geqslant 1$. In this case $|\xi|^{r_j}\leqslant|\xi|^{r_0}$, $\forall j=1,2,\ldots$. We can deduce that

$$|a(x,\xi)|\leqslant M_0|\xi|^{r_0}+|\xi|^{r_0}\left(\frac{1}{2}+\frac{1}{2^2}+\ldots\right)=(M_0+1)|\xi|^{r_0},$$

$$\forall x\in\mathbf{R}^n,\ |\xi|\geqslant 1.$$

Let us estimate now $b_N(x,\xi)$ for $N=1,2,\ldots$:

If $|\xi|\leqslant 1$ and because $t_1\geqslant 2$, $\zeta(\xi/t_j)=0$ for $j=N, N+1,\ldots$; thus $b_N(x,\xi)=0$ for $|\xi|\leqslant 1$, $x\in\mathbf{R}^n$. For $|\xi|\geqslant 1$ we have $|\xi|^{r_j}\leqslant|\xi|^{r_N}$ when $j>N$. Therefore we get

$$|b_N(x,\xi)|\leqslant M_N|\xi|^{r_N}+\sum_{N+1}^{\infty}\left|\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j}a_j(x,\xi)\right|\leqslant M_N|\xi|^{r_N}+\sum_{N+1}^{\infty}\frac{1}{2^j}|\xi|^{r_{j-1}}.$$

Consequently

$$|b_N(x,\xi)|\leqslant M_N|\xi|^{r_N}+|\xi|^{r_N}\sum_{N+1}^{\infty}\frac{1}{2^j}=\left(M_N+\frac{1}{2^N}\right)|\xi|^{r_N},\qquad N=1,2,\ldots.$$

This ends the proof.

3. – We shall use here formula:

$$a'(x,\xi)=a(x,\xi)-a(\infty,\xi)=\sum_{1\leqslant t_j\leqslant 2m}\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j}a'_j(x,\xi),$$

$$\text{for } |\xi|\leqslant m,\quad x\in\mathbf{R}^n.$$

It follows that $a'(x,\xi)$ admits a partial Fourier transform, with respect to $x\in\mathbf{R}^n$, taken in $\mathcal{S}'(\mathbf{R}^n)$-sense, which equals

$$\sum_{1\leqslant t_j\leqslant 2m}\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j}\tilde{a}'_j(\lambda,\xi),\qquad \text{for } |\xi|\leqslant m,\ \lambda\in\mathbf{R}^n\ (\xi\neq 0 \text{ as usual here})$$

We can write also:

$$\tilde{a}'(\lambda,\xi)=\sum_{j=0}^{\infty}\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j}\tilde{a}'_j(\lambda,\xi)$$

which is a finite sum for any $(\lambda,\xi)\in\mathbf{R}^n\times\mathbf{R}^n-\{0\}$.

Let us consider also, as in previous section, the remainders of order N:

$$b'_N(x,\xi)=\sum_{j=N}^{\infty}\zeta\left(\frac{\xi}{t_j}\right)|\xi|^{r_j}a'_j(x,\xi)$$

and their partial Fourier transform:

$$\tilde{b}'_N(\lambda, \xi) = \sum_{j=N}^{\infty} \zeta\left(\frac{\xi}{t_j}\right) |\xi|^{r_j} \tilde{a}'_j(\lambda, \xi) , \qquad N = 1, 2, \ldots .$$

Therefore we are now ready to state the following

THEOREM 2. *A convenient selection of the real sequence* $(t_j)_0^\infty$ *where* $t_0 = 1$, $t_1 \geq 2$, *allows inequalities*

$$(3.1) \qquad |\tilde{a}'(\lambda, \xi)| \leq K(\lambda) |\xi|^{r_0} , \qquad \lambda \in \mathbb{R}^n , \ \xi \in \mathbb{R}^n - \{0\}$$

$$(3.2) \qquad |\tilde{b}'_N(\lambda, \xi| \leq K_N(\lambda) |\xi|^{r_N} , \qquad \lambda \in \mathbb{R}^n , \ \xi \in \mathbb{R}^n - \{0\}$$

where

$$(1 + |\lambda|^2)^p K(\lambda) \in L^1(\mathbb{R}^n) , \qquad (1 + |\lambda|^2)^p K_N(\lambda) \in L^1(\mathbb{R}^n) ,$$
$$\forall p = 0, 1, 2, \ldots, \ N = 1, 2, \ldots .$$

PROOF. We know, because each $a_j(x, \xi)$ belongs to $\mathfrak{S}$, that $|\tilde{a}'_j(\lambda, \xi)| \leq k_j(\lambda)$ where

$$(1 + |\lambda|^2)^l k_j(\lambda) \in L^1(\mathbb{R}^n) , \qquad \forall l = 0, 1, 2, \ldots, \ j = 0, 1, 2, \ldots .$$

We shall define a real sequence $(t_j)_0^\infty$ in such a way that the inequalities

$$(3.3) \qquad \left|\zeta\left(\frac{\xi}{t_j}\right)\right| |\xi|^{r_j} |\tilde{a}'_j(\lambda, \xi)| \leq \frac{1}{2^j} |\xi|^{r_{j-1}} \frac{k_j(\lambda)}{\|k_j^*(\lambda)\|_{L^1}} , \qquad j = 1, 2, \ldots$$

be all verified $\left(k_j^*(\lambda) = \left(1 + |\lambda|^2\right)^j k_j(\lambda)\right)$.

We shall look for $(t_j)_0^\infty$ such that, if $|\xi| \geq \frac{1}{2} t_j$, then

$$(3.4) \qquad |\xi|^{r_j} \leq 2^{-j} \|k_j^*\|_{L^1}^{-1} |\xi|^{r_{j-1}} , \qquad j = 1, 2, \ldots .$$

This is equivalent with

$$(3.5) \qquad 2^j \|k_j^*\|_{L^1} \leq |\xi|^{r_{j-1} - r_j}$$

and is obviously implied by

$$(3.6) \qquad 2^j \|k_j^*\|_{L^1} \leq (\tfrac{1}{2} t_j)_{r_{j-1} - r_j} .$$

Thus, a good sequence $(t_j)_0^\infty$ should satisfy the estimates

$$t_j \geqslant 2^{j(r_{j-1}-r_j)^{-1}+1} \cdot \|k_j^*\|_{L^1}^{(r_{j-1}-r_j)^{-1}} . \tag{3.7}$$

We see now that (3.3) is indeed true.

This is obvious when $|\xi| \leqslant \frac{1}{2} t_j$, all terms at left are null.

For $|\xi| > \frac{1}{2} t_j$ we have obviously

$$\left|\zeta\left(\frac{\xi}{t_j}\right)\right| |\xi|^{r_j} |\tilde{a}_j'(\lambda, \xi)| \leqslant |\xi|^{r_j} k_j(\lambda)$$

and we apply (3.4) and get (3.3). We are now ready to establish the estimate (3.1). Again we see that $\tilde{a}'(\lambda, \xi) \equiv 0$ for $|\xi| \leqslant \frac{1}{2}$; next, for $\frac{1}{2} \leqslant |\xi| \leqslant 1$, if $t_0 = 1$ and $t_1 \geqslant 2$, we see that $\tilde{a}'(\lambda, \xi) = \zeta(\xi)|\xi|^{r_0} \tilde{a}_0'(\lambda, \xi)$ and accordingly will be:

$$|\tilde{a}'(\lambda, \xi)| \leqslant k_0(\lambda) |\xi|^{r_0}, \qquad (\lambda, \xi) \in \mathbf{R}^n \times \mathbf{R}^n - \{0\} .$$

Now, for $|\xi| \geqslant 1$ we see that $|\xi|^{r_{j-1}} \leqslant |\xi|^{r_0}$, $j = 1, 2, \ldots$; then

$$|\tilde{a}'(\lambda, \xi)| \leqslant |\xi|^{r_0} k_0(\lambda) + |\xi|^{r_0} \sum_{j=1}^{\infty} 2^{-j} k_j(\lambda) \|k_j^*(\cdot)\|^{-1} \quad \text{(after using (3.3))}$$

If $K(\lambda) = \sum_{j=1}^{\infty} 2^{-j} k_j(\lambda) \|k_j^*(\cdot)\|_{L^1}^{-1}$ (a measurable function which has obviously a finite integral, —using Th. of B. Levi—(see [3], [7]), because

$$\left.\sum_{j=1}^{\infty} 2^{-j} \left(\int_{\mathbf{R}^n} k_j(\lambda)\, d\lambda\right) \|k_j^*\|_{L^1}^{-1} \leqslant \sum_{j=1}^{\infty} 2^{-j} < \infty \right)$$

we obtained:

$$|\tilde{a}'(\lambda, \xi)| \leqslant |\xi|^{r_0} \big(k_0(\lambda) + K(\lambda)\big) , \qquad (\lambda, \xi) \in \mathbf{R}^n \times \mathbf{R}^n - \{0\}$$

and we have only to prove that

$$(1 + |\lambda|^2)^l K(\lambda) \in L^1(\mathbf{R}^n) , \qquad \forall l = 1, 2, \ldots .$$

Returning to the definition of $K(\lambda)$ we see that:

$$(1+|\lambda|^2)^l K(\lambda)=\sum_{j=1}^{\infty}2^{-j}(1+|\lambda|^2)^l k_j(\lambda)\|k_j^*(\cdot)\|_{L^1}^{-1}=$$
$$=\sum_{1\leqslant j\leqslant l}2^{-j}(1+|\lambda|^2)^l k_j(\lambda)\|k_j^*(\cdot)\|_{L^1}^{-1}+\sum_{l<j<\infty}2^{-j}(1+|\lambda|^2)^l k_j(\lambda)\|k_j^*(\cdot)\|_{L^1}^{-1}.$$

The first sum is finite, therefore $\in L^1$ as requested.

In the second sum l is smaller than j, therefore $(1+|\lambda|^2)^l k_j(\lambda)\leqslant\leqslant(1+|\lambda|^2)^j k_j(\lambda)=k_j^*(\lambda)$ and it follows that

$$\sum_{l<j<\infty}2^{-j}\|k_j^*(\cdot)\|_{L^1}^{-1}\cdot\int_{\mathbf{R}^n}(1+|\lambda|^2)^l k_j(\lambda)\,d\lambda\leqslant\sum_{l<j<\infty}2^{-j}<\infty.$$

Another application of B. Levi's theorem implies the result.

Similar reasonings apply in order to prove (3.2). For $|\xi|\leqslant 1$, $\tilde{b}_N'(\lambda,\xi)=0\ \forall\lambda\in\mathbf{R}^n$, $N=1,2,\ldots$. For $|\xi|\geqslant 1$ we shall use the same sequence $(t_j)_0^\infty$ as above. We get:

$$|\tilde{b}_N'(\lambda,\xi)|\leqslant|\xi|^{r_N}k_N(\lambda)+\sum_{j=N+1}^{\infty}2^{-j}|\xi|^{r_{j-1}}k_j(\lambda)\|k_j^*(\cdot)\|_{L^1}^{-1}\leqslant$$
$$\leqslant|\xi|^{r_N}\left\{k_N(\lambda)+\sum_{N+1}^{\infty}2^{-j}k_j(\lambda)\|k_j^*(\cdot)\|_{L^1}^{-1}\right\}.$$

The *integrable* function $k_N(\lambda)+\sum_{N+1}^{\infty}2^{-j}k_j(\lambda)\|k_j^*(\cdot)\|_{L^1(\mathbf{R}^n)}^{-1}$ (it is so again by monotone sequences theorem) will be denoted with $K_N(\lambda)$.

We still have to show that $(1+|\lambda|^2)^p K_N(\lambda)\in L^1(\mathbf{R}^n)$, $\forall p=1,2,\ldots,$ which is a consequence of:

$$(1+|\lambda|^2)^p\sum_{N+1}^{\infty}2^{-j}k_j\|k_j^*(\cdot)\|_{L^1}^{-1}\in L^1(\mathbf{R}^n)\,,\qquad\forall p=1,2,\ldots.$$

In fact, for $p\leqslant N+1$, $j\geqslant N+1$, we have $(1+|\lambda|^2)^p k_j(\lambda)\leqslant k_j^*(\lambda)$ while for $p>N+1$ we write:

$$(1+|\lambda|^2)^p\sum_{N+1}^{\infty}2^{-j}k_j(\lambda)\|k_j^*\|_{L^1}^{-1}=\sum_{j=N+1}^{p}2^{-j}(1+|\lambda|^2)^p k_j(\lambda)\|k_j^*\|_{L^1}^{-1}+$$
$$+\sum_{j>p}2^{-j}(1+|\lambda|^2)^p k_j(\lambda)\|k_j^*\|_{L^1}^{-1}.$$

The first sum at right is finite, thus $\in L^1$; in the second sum, $(1+|\lambda|^2)^p k_j(\lambda) \leqslant k_j^*(\lambda)$ so that again B. Levi's theorem will give the requested result.

4. – Here we start considerations on pseudo-differential operators associated to symbols in $\mathfrak{S}$; canonical pseudo-differential operators associated to symbols of degree r, (i.e. of the form $a(x,\xi)\zeta(\xi/t)|\xi|^r$, $r\in\mathbf{R}$, $t\in\mathbf{R}^+$) asymptotic series of such canonical operators and pseudo-differential operators which are sums of these series, associated to symbols verifying (2.1), (2.2), and (3.1). Our main interest, as indicated in the Introduction, is to state and prove an analogous of the existential result Theorem 2 (iv) of [6] for our-somewhat different-classes of symbols.

To start let us remark that the class $\mathfrak{S}$ is more general than that defined in our paper [13]; we kept i) and iv) but not ii) and iii). This is enough however in order to define the operator $a(x,D)=A$, $\mathcal{S}(\mathbf{R}^n)\to\mathcal{S}'(\mathbf{R}^n)$ by formula:

$$\widetilde{a(x,D)u}\,(\xi)=a(\infty,\xi)\tilde{u}(\xi)+(2\pi)^{-n/2}\int_{\mathbf{R}^n}\tilde{a}'(\xi-\eta,\xi)\tilde{u}(\eta)\,d\eta\,,\qquad \forall u\in\mathcal{S}(\mathbf{R}^n)$$

and prove that $\|a(x,D)u\|_{H^s}\leqslant C_s\|u\|_{H^s}$, $\forall s\in\mathbf{R}$, $\forall u\in\mathcal{S}(\mathbf{R}^n)$, where H^s are the Sobolev spaces of real order s (on $\mathbf{R}^n$).

Therefore $a(x,D)$ will extend by continuity to a linear continuous operator from $H^s(\mathbf{R}^n)$ into itself, $\forall s\in\mathbf{R}$, therefore from $H^\infty(\mathbf{R}^n)=\bigcap_{s\in\mathbf{R}}H^s(\mathbf{R}^n)$ into itself. Consider now a function $\zeta(\xi/t)$, $t\in\mathbf{R}^+$ as defined in section 1, and then for a real r, a function $\zeta_r(\xi/t)=0$ for $|\xi|\leqslant\frac{1}{2}t$, $=\zeta(\xi/t)|\xi|^r$, for $|\xi|\geqslant\frac{1}{2}t$.

Remark now that $|\xi|\leqslant(1+|\xi|^2)^{\frac{1}{2}}$ and $|\xi|^r\leqslant(1+|\xi|^2)^{r/2}$ for $r\geqslant 0$ while for $r<0$ and $|\xi|\geqslant\frac{1}{2}t$ it is $(1+|\xi|^2)|\xi|^{-2}=1+|\xi|^{-2}\leqslant 1+4t^{-2}$ and;

$$(1+|\xi|^2)^r\geqslant|\xi|^{2r}(1+4t^{-2})^r\,,\qquad |\xi|^r\leqslant C_{t,r}(1+|\xi|^2)^{r/2}\,.$$

Therefore we can derive

$$\left|\zeta_r\left(\frac{\xi}{t}\right)\right|\leqslant C(1+|\xi|^2)^{r/2}\,,\qquad\forall\xi\in\mathbf{R}^n\,,\quad\forall r\in\mathbf{R}\,.$$

We can define an operator $\zeta_r\big((1/t)D\big)$ given on $\mathcal{S}(\mathbf{R}^n)$ by the formula:

$$\zeta_r\left(\frac{1}{t}D\right)u=\mathcal{F}^{-1}\left(\zeta_r\left(\frac{\xi}{t}\right)\tilde{u}(\xi)\right),$$

$\mathcal{F}^{-1}$ being the inverse Fourier transform. Estimating the H_s-norm we get

$$\left\|\zeta_r\left(\frac{1}{t}D\right)u\right\|_{H^s} = \left(\int_{\mathbf{R}^n}(1+|\xi|^2)^s\left|\zeta_r\left(\frac{\xi}{t}\right)\right|^2|\tilde{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} \leqslant$$

$$\leqslant C\left(\int_{\mathbf{R}^n}(1+|\xi|^2)^{s+r}|\tilde{u}(\xi)|^2\,d\xi\right)^{\frac{1}{2}} = C\|u\|_{H^{s+r}}\,, \qquad \forall s\in\mathbf{R}\,,\ \text{and}\ \forall u\in\mathcal{S}(\mathbf{R}^n)\,.$$

Therefore, $\zeta_r\big((1/t)D\big)$ extends by continuity to an operator in $\mathcal{L}(H^s; H^{s-r})$ $\forall$ real s, and consequently it will map H^∞ into itself. We call, following [6], any operator $\zeta_r(D/t)a(x, D)$, $H^\infty\to H^\infty$ a canonical operator of degree r. We see that

$$\left\|\zeta_r\left(\frac{1}{t}D\right)a(x, D)u\right\|_{H^s}\leqslant C\|u\|_{H^{s+r}}\,, \qquad \forall s\in\mathbf{R}\,,\ \forall u\in H^\infty(\mathbf{R}^n)\,.$$

Now we shall define the concept of an asymptotic representation of a linear operator $H^\infty\to H^\infty$ through a sequence of canonical operators of degree r_j, $(r_j)_0^\infty$ being a strictly decreasing sequence of reals convergent to $-\infty$ (or a finite sequence, $r_0 > r_1 \ldots > r_N$). We need the definition of order and true order of a linear operator in H^∞ which is given below in a form slightly more detailed than in [6] (see [1] for a different, « non-linear » definition of the order). Denote by Lin (H^∞) the vector space of all linear operators, $H^\infty\to H^\infty$. Given an operator $L\in\mathrm{Lin}\,(H^\infty)$ we associate to it a subset of the real line, $\mathcal{O}(L)$ (order of L), which can be the empty set too, by means of the following:

DEFINITION 4.1. $\mathcal{O}(L) = \{r\in\mathbf{R}$ *such that,* $\forall s\in\mathbf{R}$, $\exists C_s\in\mathbf{R}^+$ *with property that* $\|Lu\|_{H^s}\leqslant C_s\|u\|_{H^{s+r}}$, $\forall u\in H^\infty\}$.

Let us remark that if $r_0\in\mathcal{O}(L)$ and $r' > r_0$, then $r'\in\mathcal{O}(L)$ too. Next, the true order of L is by definition:

$$\text{t.o.}\,(L) = \inf\mathcal{O}(L)\,, \qquad \text{the greatest lower bound of } \mathcal{O}(L)\,.$$

If $\mathcal{O}(L)$ is the whole real line, t.o. $(L) = -\infty$.

The true order can belong to the order set or not; a thorough discussion of these concepts in a slightly more general frame-work will be presented in [15].

We are thus able to give

DEFINITION 4.2. *Let $(r_j)_0^\infty \to -\infty$ be a strictly decreasing sequence of real numbers, and $\{\zeta_{r_j}((1/t_j)D)a_j(x, D)\}_0^\infty$ be a sequence of canonical operators of degree r_j, corresponding to a sequence of positive reals $(t_j)_0^\infty$ and to a sequence of symbols $\{a_j(x, \xi)\}_0^\infty \subset \mathfrak{S}$. A linear operator M, $H^\infty \to H^\infty$ is asymptotically expanded into the series $\sum_{j=0}^{\infty} \zeta_{r_j}((1/t_j)\, D)a_j(x, D)$ if the following holds:*

$$t\cdot o\left[M-\sum_{j=0}^{N}\zeta_{r_j}\left(\frac{1}{t_j}D\right)a_j(x, D)\right]< r_N \qquad \text{(strict inequality)}$$

We say that

$$M\sim\sum_{j=0}^{\infty}\zeta_{r_j}\left(\frac{1}{t_j}D\right)a_j(x, D)\,.$$

Our main goal in the remaining of this paper is to prove the following

THEOREM 3. *Let be given a sequence $\{a_j(x, \xi)\}_{j=0}^\infty$ of symbols in $\mathfrak{S}$ and a strictly decreasing to $-\infty$ infinite sequence of real numbers, $\{r_j\}_{j=0}^\infty$. Then, there exists a sequence of canonical operators of degree r_j, K_j and a linear operator P in H^∞, such that:*

i) $\text{t.o.}\,(P)\leqslant r_0$

ii) $P\sim\sum_{j=0}^{\infty}K_j$

PROOF. We shall construct canonical operators K_j in the following way: First consider a fixed function $\zeta(\xi)$, $0\leqslant\zeta(\cdot)\leqslant 1$, $\zeta(\xi)=0$ for $|\xi|\leqslant\frac{1}{2}$, $\zeta(\xi)=1$ for $|\xi|\geqslant 1$, $\zeta(\xi)\in C^\infty(\mathbf{R}^n)$.

Next define a sequence of real numbers $(t_j)_0^\infty$ as follows:

$$t_0=1\,,\ \ t_1\geqslant 2\,,\ \ t_j>\max\left\{2^{j(r_{j-1}-r_j)^{-1}+1}M_j^{(r_{j-1}-r_j)^{-1}},\ 2^{j(r_{j-1}-r_j)^{-1}+1}\|k_j^*\|^{(r_{j-1}-r_j)^{-1}}\right\}$$

where, we remember,

$$M_j=\sup_{x,\xi}|a_j(x, \xi)|\,,\qquad k_j^*(\lambda)=(1+|\lambda|^2)^j k_j(\lambda)$$

$k_j(\lambda)$ being associated to $a_j(x, \xi)$ as in (1.2).

This choice of the sequence $(t_j)_0^\infty$ will permit, according to Theorems 1 and 2, construction of a function $a(x, \xi)$ verifying (2.1), (2.2), (3.1), (3.2) simultaneously.

Now, K_j is by definition the operator $\zeta_{r_j}\big((1/t_j)D\big)a_j(x, D)$ corresponding to the symbol $a_j(x, \xi)$ and to the real r_j given and t_j chosen in the way indicated above. In terms of Fourier transform we have the following representation formula for the action of a canonical operator on $\mathcal{S}(\mathbb{R}^n)$:

$$K_j u = \mathcal{F}^{-1}\left[\zeta_{r_j}\left(\frac{\xi}{t_j}\right)\left\{a_j(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int_{\mathbf{R}^n}\tilde{a}_j'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta\right\}\right]$$

Let us define now the operator P; remark that

$$a(x, \xi) = \sum_{j=0}^{\infty}\zeta_{r_j}\left(\frac{\xi}{t_j}\right)a_j(x, \xi)$$

satisfies estimates which are somewhat different from those defining the class $\mathfrak{S}$; still we can associate to it a linear operator in H^∞ by means of similar formulas as those defining $a(x, D)$ when $a(x, \xi) \in \mathfrak{S}$. Therefore, let us put, for $u \in \mathcal{S}(\mathbb{R}^n)$

$$(Pu)(x) = \mathcal{F}^{-1}\left[a(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int_{\mathbf{R}^n}\tilde{a}'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta\right].$$

If we consider separately the operator $P(\infty, D) = \mathcal{F}^{-1}a(\infty, \cdot)\mathcal{F}$ we have immediately

$$\|P(\infty, D)u\|_{H^s} = \left(\int_{\mathbf{R}^n}(1+|\xi|^2)^s|a(\infty, \xi)\tilde{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \leqslant$$
$$\leqslant C\left(\int_{\mathbf{R}^n}(1+|\xi|^2)^s(1+|\xi|^2)^{r_0}|\tilde{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} = C\|u\|_{H^{s+r_0}}$$

(when we use (2.2) and the fact that $a(\infty, \xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$ because then $|\xi|^{r_0} \leqslant C(1+|\xi|^2)^{r_0/2}$ for $|\xi| \geqslant \frac{1}{2}$, for any real r_0). If we look now at the integral: $\int_{\mathbf{R}^n}\tilde{a}'(\xi-\eta, \xi)\tilde{u}(\eta)\,d\eta$, $u \in \mathcal{S}(\mathbb{R}^n)$ we see first its absolute convergence, for any $\xi \in \mathbb{R}^n$.

Remember that $\tilde{a}'(\lambda, \xi) = 0$ for $|\xi| \leqslant \frac{1}{2}$, therefore

$$\int_{\mathbf{R}^n} \tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta = 0$$

too for ξ in the same sphere.

For bigger $|\xi|$, we can use estimate (3.1); accordingly

$$|\tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)| \leqslant cK(\xi - \eta)(1 + |\xi|^2)^{r_0/2}|\tilde{u}(\eta)| \leqslant C_1(1 + |\xi|^2)^{r_0/2}K(\xi - \eta)$$

—because $\tilde{u}(\eta)$ is bounded on $\mathbf{R}^n$; now $\int_{\mathbf{R}^n} K(\xi - \eta)\, d\eta < \infty$.

Therefore, the function $G(\xi)$, $= 0$ for $|\xi| \leqslant \frac{1}{2}$, $= \int_{\mathbf{R}^n} \tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta$, for $\xi \in \mathbf{R}^n$ is well defined for $\xi \in \mathbf{R}^n$, verifying the inequality $|G(\xi)| \leqslant C_2(1 + |\xi|^2)^{r_0/2}$; $G(\xi)$ is also a measurable function as easily seen (the $2n$-integral $\int_{\Omega \times \Omega} |\tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)|\, d\eta\, d\xi$ is convergent when Ω is bounded).

Accordingly we can take the inverse Fourier-transform of $(2\pi)^{-n/2}G(\xi)$ in $\mathcal{S}'(\mathbf{R}^n)$ sense and obtain $P'(x, D)u \in \mathcal{S}'(\mathbf{R}^n)$.

We establish now an estimate in Sobolev spaces for P' too. We shall see that

$$\|P'(x, D)u\|_{H^s} \leqslant C_s \|u\|_{H^{s+r_0}}, \qquad \forall u \in \mathcal{S}(\mathbf{R}^n) .$$

We must therefore give an upper bound for the $L^2(\mathbf{R}^n)$ norm of the function

$$U_s(\xi) = (1 + |\xi|^2)^{s/2}G(\xi) = (2\pi)^{-n/2}\int_{\mathbf{R}^n} (1 + |\xi|^2)^{s/2}\, \tilde{a}'(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta =$$

$$(2\pi)^{-n/2}\int_{|\xi| \geqslant \frac{1}{2}} (1 + |\xi|^2)^{s/2}(1 + |\eta|^2)^{-s/2}\tilde{a}'(\xi - \eta, \xi)(1 + |\eta|^2)^{s/2}\tilde{u}(\eta)\, d\eta ;$$

it results

$$|U_s(\xi)| \leqslant C_s \int_{|\xi| \geqslant \frac{1}{2}} (1 + |\xi - \eta|^2)^{|s|/2}K(\xi - \eta)(1 + |\xi|^2)^{r_0/2}(1 + |\eta|^2)^{s/2}|\tilde{u}(\eta)|\, d\eta =$$

$$C_s \int_{|\xi| \geqslant \frac{1}{2}} (1 + |\xi - \eta|^2)^{|s|/2}K(\xi - \eta)(1 + |\xi|^2)^{r_0/2}(1 + |\eta|^2)^{-r_0/2}(1 + |\eta|^2)^{(r_0 + s)/2} \cdot$$

$$\cdot |\tilde{u}(\eta)|\, d\eta \leqslant C\int_{\mathbf{R}^n} (1 + |\xi - \eta|^2)^{(|s| + |r_0|)/2}K(\xi - \eta)(1 + |\eta|^2)^{(r_0 + s)/2}|\tilde{u}(\eta)|\, d\eta .$$

Substituting η' for $\xi - \eta$ and applying Minkowski's inequality for integrals we obtain the required result.

Accordingly, $P(x, D) = P(\infty, D) + P'(x, D)$ will verify the same inequality, for any real number s. Thus, $r_0 \in \mathcal{O}(P)$ and t.o. $(P) \leqslant r_0$.

The final part of our proof deals with upper bounds for the true orders of all operators $P - \sum_{j=0}^{N} K_j$, $N = 1, 1, 2, \ldots$.

Let us give first the representation formula for $\mathcal{F}\left[\sum_{j=0}^{N} K_j u\right]$, $u \in \mathcal{S}$. We know that K_j maps $\mathcal{S}$ in H^∞, we take $\mathcal{F}$ in $\mathcal{S}'(\mathbf{R}^n)$-sense. We have:

$$\mathcal{F}\left[\sum_0^N K_j u\right] = \sum_0^N \mathcal{F}[K_j u] = \sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) a_j(\infty, \xi)\tilde{u}(\xi) + \\ + \sum_0^N (2\pi)^{-n/2} \zeta_{r_j}\left(\frac{\xi}{t_j}\right) \int_{\mathbf{R}^n} \tilde{a}'_j(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta = \left[\sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) a_j(\infty, \xi)\right]\tilde{u}(\xi) + \\ + (2\pi)^{-n/2} \int_{\mathbf{R}^n} \left[\sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) \tilde{a}'_j(\xi - \eta, \xi)\right] \tilde{u}(\eta)\, d\eta$$

Therefore it is:

$$\mathcal{F}\left[Pu - \sum_0^N K_j u\right] = \left[a(\infty, \xi) - \sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) a_j(\infty, \xi)\right] \tilde{u}(\xi) + \\ + (2\pi)^{-n/2} \int_{\mathbf{R}^n} \left[\tilde{a}'(\xi - \eta, \xi) - \sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) \tilde{a}'_j(\xi - \eta, \xi)\right] \tilde{u}(\eta)\, d\eta\,.$$

Using notations in § 2 and § 3 we have:

$$a(\infty, \xi) - \sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) a_j(\infty, \xi) = b_{N+1}(\infty, \xi)$$

and because of (2.2) and of vanishing of b_N near the origin we get

$$|b_{N+1}(\infty, \xi)| \leqslant c(1 + |\xi|^2)^{r_{N+1}/2}\,;$$

also

$$\tilde{a}'(\xi - \eta, \xi) - \sum_0^N \zeta_{r_j}\left(\frac{\xi}{t_j}\right) \tilde{a}'_j(\xi - \eta, \xi) = \tilde{b}'_{N+1}(\xi - \eta, \xi)$$

and the estimate

$$|\tilde{b}'_{N+1}(\lambda, \xi)| \leqslant C K_{N+1}(\lambda)(1 + |\xi|^2)^{r_{N+1}/2}\,, \qquad (\lambda, \xi) \in \mathbf{R}^n \times \mathbf{R}^n\,.$$

Therefore

$$\mathcal{F}\left[Pu - \sum_0^N K_j u\right] = b_{N+1}(\infty, \xi)\tilde{u}(\xi) + (2\pi)^{-n/2}\int_{\mathbf{R}^n} \tilde{b}'_{N+1}(\xi - \eta, \xi)\tilde{u}(\eta)\, d\eta \, .$$

In the same way as above we derive:

$$\left\| Pu - \sum_0^N K_j u \right\|_{H^s} \leqslant C \|u\|_{H^{s+r_{N+1}}}, \qquad u \in \mathcal{S}(\mathbf{R}^n) \, , \; s \in \mathbf{R} \, .$$

Therefore t.o. $\left[P - \sum_0^N K_j\right] \leqslant r_{N+1}$ which is strictly $< r_N$ as requested. This ends our proof.

REFERENCES

[1] A. ANDREOTTI, *Operatori pseudodifferenziali* (Note di S. Spagnolo), Università degli Studi di Pisa (1966-67), pp. 1-171.

[2] J. BARROS-NETO, *An Introduction to the Theory of Distributions*, Marcel Dekker Inc., New York (1973), pp. 1-215.

[3] W. H. FLEMING, *Functions of several variables*, Addison-Wesley Publ. Comp. Inc. (1965).

[4] L. HÖRMANDER, *Pseudo-differential operators*, Comm. Pure Appl. Math., **18**, n. 3 (1965), pp. 501-517.

[5] L. HÖRMANDER, *Pseudo-differential operators and hypoelliptic equations* (Singular Integrals, Proc. Symp. Pure Math. X, Amer. Math. Soc. 1967), pp. 138-183.

[6] J. J. KOHN - L. NIRENBERG, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math., **18** (1965), pp. 269-305.

[7] I. P. NATANSON, *Theory of functions of a real variable* (russian), Moscow-Leningrad, 1950.

[8] L. NIRENBERG, *Pseudo-differential operators*, (Notes by W. Sweeney), Lectures at Global Analysis Symposium, Berkeley, 1968.

[9] R. SEELEY, *Topics in pseudo-differential operators*, Lectures at C.I.M.E., Stresa, 1968; Edizioni Cremonese, Roma, 1969.

[10] M. TAYLOR, *Pseudo-differential operators*, Springer-Verlag, vol. 416, Lecture Notes in Mathematics, 1974.

[11] F. TREVES, *An introduction to pseudo-differential operators and Fourier integrals operators*, Universidade Federal de Pernambuco, Instituto de Matematica, Recife, 1973.

[12] F. TREVES, *Boundary value problems for elliptic equations*, Lectures at C.I.M.E., Bressanone, 1967, Liguori Editore.
[13] S. ZAIDMAN, *Some non-homogeneous symbols and associated pseudo-differential operators*, Ann. Sc. Norm. Sup. Pisa, **31** (1967), pp. 547-574.
[14] S. ZAIDMAN, Notices Amer. Math. Soc., October 1978.
[15] S. ZAIDMAN, *Order and true order ...* (in preparation).

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