Franco Cardin

On pocket and empirical temperatures. An alternative choice for the heat flux vector in Eckart’s relativistic thermodynamics

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On Pocket and Empirical Temperatures.

FRANCO CARDIN (*)

SUMMARY - Some results related with pocket temperature are presented within a certain general version $\mathcal{C}_E$ of Eckart’s relativistic thermodynamics. They allow us to propose an (equally acceptable) alternative choice for the heat flux vector $\tilde{q}^a$ that, unlike Eckart’s $\tilde{q}^a$, does not involve intrinsic acceleration. It is shown that $\tilde{q}^a = [1 + O(c^{-4})]q^a$ for non-viscous fluids. By the above results in $\mathcal{C}_E$, the general solution of a fundamental differential relation, stated within Alts and Müller’s theory $\mathcal{C}_{AM}$, among empirical temperature $\vartheta$, absolute temperature $T$, and mass density is found. An agreement between $\mathcal{C}_{AM}$ and the Chernikov’s kinetic relativistic theory $\mathcal{C}_C$ is shown to hold up to $O(c^{-4})$. It is shown that $\mathcal{C}_E$ and $\mathcal{C}_{AM}$ are supported by $\mathcal{C}_C$ equally well.

1. Introduction.

In [1] Alts and Müller consider a relativistic theory of thermodynamics, say $\mathcal{C}_{AM}$, in which the usual absolute temperature $T$ is replaced by the empirical temperature $\vartheta$; this temperature is given

(*) Indirizzo dell’A.: Seminario Matematico, Università - Via Belzoni 7 - 35100 Padova.

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an operational meaning only in certain equilibrium processes called *E-equilibria*. In these processes $d\theta$ is shown in $\mathcal{C}_{AM}$ to equal a certain differential form $\alpha dT + \beta dk$ where $k$ is the conventional mass density—cf. [3], § 21; and by a natural physical assumption this relation, $d\theta = \alpha dT + \beta dk$, can be regarded as a completely determined differential link among $\theta, T,$ and $k$.

Within the general version $\mathcal{C}_E$ of Eckart's thermodynamics, that is presented in [3], I define $e$-equilibrium (N. 3)—a notion stronger then the analogue of $E$-equilibrium (used in $\mathcal{C}_{AM}$)—, by requiring the absence of heat flux ($q^x = 0$) and Born rigidity ($u_{(\alpha/\beta)} = 0$); furthermore I show that for a viscous fluid $\mathcal{F}$ that is capable of heat conduction and satisfies the 2nd principle—cf. (2.10)—the scalar field $\Theta = T(1 + c^{-2}\phi)^{-1}$, where $\phi$ is (Gibbs's) free enthalpy, is constant on regions of $e$-equilibrium ($\Theta_{\alpha/\beta} = 0$). Hence $\Theta$ appears as a generalization of the pocket temperature $T_{(\rho)} = T\sqrt{-g_{00}}$ studied by Tolman and Ehrenfest—cf. [3], § 45. This first results presented in this paper belongs to $\mathcal{C}_E$ and are in a tight agreement with some results obtained in [8] within a theory of relativistic thermostatics based on a variational version of the second principle $(1)$.

The relation between the metric tensor $g_{\alpha\beta}$ (precisely $g_{00}$) and $\phi$, or the Newtonian potential $U$ in the case of weak gravitation, is briefly shown in N. 4.

In N. 5 an alternative to Eckart's choice $q_{\alpha} = -\kappa(T_{/\alpha} + TA_{\alpha})$ for the heat flux vector $q_{\alpha}$ is presented: a vector $q_{\alpha} = -\kappa\Theta_{/\alpha}$, which I call pocket flux. It leads to a non equivalent but equally acceptable relativistic thermodynamics for heat conducting fluids. Indeed it can be shown (N. 5) that (i) $p_{\alpha} = [1 + O(c^{-r})]q_{\alpha}$ with $r = 2$ [$r = 4$] for viscous [non-viscous] fluids, where $c$ is the speed of light in vacuum and $O(c^{-r})$ means terms of the order of $c^{-r}$, and (ii) $q_{\alpha}$ $\equiv \tilde{q}_{\alpha} = 0$ in $e$-equilibria. By (i) the alternative $\mathcal{C}_P$ to $\mathcal{C}_E$ obtained from $\mathcal{C}_E$ by substituting $\tilde{q}_{\alpha}$ for $\tilde{q}_{\alpha}$ is in agreement with experiments and also complies with the theoretical considerations made to support Eckart's choice $q_{\alpha}$ for $q_{\alpha}$, which just required (ii) to hold—cf. [3], § 45.

$(1)$ The above result of mine in $\mathcal{C}_E$ concerning $\Theta$ and $T_{(\rho)}$, is taken from my thesis for the degree in physics in July 1978, which thesis was presented at the national competition for a Grant of the C.N.R. Bando no. 201.1.89 (term: 22th July 1978). After [8] appeared in 1979 the deduction of the aforementioned result in $\mathcal{C}_E$ is still interesting because of the difference between $\mathcal{C}_E$ and the thermostatic theory used in [8].
In N. 6 $\mathcal{C}_A$ and $\mathcal{C}_E$ are compared in connection with the fluids dealt within $\mathcal{C}_A$, the non-viscous ones. First this is done in the case of $E$-equilibrium; and then in connection with processes near those equilibrium processes. More in detail, for the above equilibrium differential relation $d\theta = \alpha dT + \beta dk$ obtained in $\mathcal{C}_A$, in [1] integrability conditions are written. Here (N. 6) the general solution of this relation is shown to be $\theta = f(\Theta)$, where $f$ is any mapping of $\mathbb{R}$ in $\mathbb{R}$ of class $C^{(1)}$.

Lastly in [1], N. 5, the authors assume (in $\mathcal{C}_A$) that along processes near $E$-equilibria the constitutive functions considered there have the same form as in equilibrium processes—i.e. do not involve $\theta$; This assumption is compatible with $\mathcal{C}_A$'s axioms up to $O(\varepsilon^{-4})$—cf. fnt (6) in N. 6. Under the above assumption the heat flux $q_\alpha$ in $\mathcal{C}_A$—where non-viscous fluids are treated—is shown to be a vector $\overset{\b}{q}_\alpha$ parallel with the analogous heat flux $\overset{c}{q}_\alpha$ obtained within Chernikov’s relativistic kinetic theory $\mathcal{C}_C$—see [5] to [7], and $\overset{\b}{q}_\alpha$ can be identified with $\overset{c}{q}_\alpha$. On the other hand, by (i) above, $\overset{\b}{q}_\alpha$ can be identified with $\overset{\b}{q}_\alpha$ and $\overset{c}{q}_\alpha$. Thus Alts and Müller’s assertion on $\mathcal{C}_A$—see [1], N. 7—that the theories $\mathcal{C}_A$ and $\mathcal{C}_C$ support each other, also holds for $\mathcal{C}_E$; and in either case the agreement occurs up to $O(\varepsilon^{-4})$.

2. Some basic notions and theorems of Eckart’s relativistic thermodynamics in its general version $\mathcal{C}_E$ presented in [3].

A) Preliminaries on space-time.

The notions and notations introduced in [3] are presupposed in this paper. Let $\mathcal{E}$ be an event point of the space-time $\mathcal{S}_4$ of general relativity, and let $x^\alpha (\alpha = 0, \ldots, 3)$ be its co-ordinates (2) in a given (admissible) reference frame—cf. [3], p. 37. For the metric at $\mathcal{E}$ we have

$$ds^2 = -g_{\alpha\beta} dx^\alpha dx^\beta, \quad g_{00} < 0, \quad \text{sign}[g_{\alpha\beta}] = +2.$$

Assume that $\mathcal{C}$ is a continuous body, $P^*$ is any material point of it, and $x^\alpha = x^\alpha(s)$ describes the world line $\mathcal{W}_{P^*}$ of $P^*$. Then for

(2) Greek [Latin] letters run from 0 [1] to 3.
the 4-velocity \( u^a \) and intrinsic acceleration \( A^a \) of \( P^* \) we have

\[
(2.2) \quad u^a = \frac{Dx^a}{Ds} \left( = \frac{dx^a}{ds} \right), \quad A^a = \frac{Du^a}{Ds}, \quad u^a u_\alpha = -1, \ A^a u_\alpha = 0.
\]

For any tensor field \( T_{\alpha\beta} \), \( T_{\alpha\beta;\alpha} \) denotes its covariant derivatives based on the metric (2.1) while

\[
(2.3) \quad \frac{\partial}{\partial x^\alpha} = \frac{DT_{\alpha\beta}}{Ds} = T_{\alpha\beta;\alpha} u^\alpha
\]

is its material derivative (in Römer units). Let us set

\[
(2.4) \quad g_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \quad T_{\alpha\beta;\alpha} = T_{\alpha\beta\gamma} g_{\alpha\gamma} (\text{whence } T_{\alpha\beta;\alpha} u^\alpha = 0).
\]

The index \( \alpha \) of \( T_{\alpha\beta} \) is said to be spatial if \( T_{\alpha\beta;\alpha} u^\alpha = 0 \).

B) Einstein gravitation and conservation equations for materials capable of heat conduction.

Let \( \mathcal{E}_E \) be C. Eckart’s theory of relativistic thermodynamics—cf. [9]— in the general version presented in [3], but in absence of electromagnetic phenomena and couple stress. Assume that \( c \) is the speed of light in vacuum, \( k_1 \) is conventional mass [gravitational mass (in energy units)] per unit proper volume—cf. [3], p. 54—, \( q_\alpha \) is the (spatial) heat flux (vector), \( Q_{\alpha\beta} \) is Eckart’s thermodynamic tensor, and \( X^{\alpha\beta} \) is the (completely spatial) Eulerian stress tensor. Then (3)

\[
(2.5) \quad Q_{\alpha\beta} = 2u_{(\alpha} q_{\beta)}, \quad q^\alpha u_\alpha = 0, \quad u_\alpha X^{\alpha\beta} = 0 = X^{\alpha\beta} u_\beta;
\]

furthermore the continuity equation and definition of the (actual) internal energy \( w \) per unit reference mass read—cf. [3]:

\[
(2.6) \quad \left( ku^\alpha \right)_{;\alpha} = 0, \quad \varrho = k (e^2 + w).
\]

Denoting by \( A_{\alpha\beta} (= A_{\beta\alpha}) \) and \( h \) Levi Civita’s tensor and Caven-dish’s constant respectively, in the framework of \( \mathcal{E}_E \) Einstein gravita-

\[
(3) \quad 2T_{(\alpha\beta)} = T_{\alpha\beta} + T_{\beta\alpha}, \quad 2T_{(\alpha\beta)} = T_{\alpha\beta} - T_{\beta\alpha}.
\]
tion equations read

\[ (2.7) \quad A_{\alpha\beta} + \frac{8\pi\hbar}{c^4} Q_{\alpha\beta} = 0 \]

with \( Q_{\alpha\beta} = q u_{\alpha} u_{\beta} + X_{\alpha\beta} + Q_{\alpha\beta} \), hence \( X_{(\alpha\beta)} = 0 \).

Of course the consequence \((2.7)_3\) of \((2.7)_{1,2}, (2.5)_1\), and the symmetry of \( A_{\alpha\beta} \), constitute the relativistic version of the 2nd Cauchy equation for non-polar continuous media. The spatial and temporal part of conservation equations—which constitute the consequence \( Q_{\alpha\beta} = 0 \) of \((2.7)_1\)—can be put into the respective forms—cf. [3], p. 62:

\[
\begin{align*}
q A_x &= -g_{\alpha\gamma}(X^\gamma + Q^\gamma)_{\gamma}\beta \\
\frac{1}{k} \frac{Dw}{D8} + \frac{\delta l^{(t)}}{D8} &= c^{-1} k q_{\text{ass}} \\
\end{align*}
\]

where \( g_{\alpha\gamma} Q^\gamma_{\gamma\beta} = k \left( g_{\alpha\gamma} \frac{D q^\gamma}{D8} k + u_{\alpha\beta} q^\gamma k \right) \),

\( k q_{\text{ass}} = c u_{\alpha} Q^\alpha_{\alpha\beta} \) and \( \frac{\delta l^{(t)}}{D8} = X^\alpha_{\alpha\beta} u_{\alpha\beta} \).

Assume that \( \mathcal{E} \in \mathcal{W}_C \), the world tube of \( C \), and that the frame \( (x) \) is natural and proper at \( \mathcal{E} \), i.e.

\( (2.9) \quad g_{\alpha\beta,\gamma} = 0, \quad g_{\alpha\gamma} = \delta_{\alpha\gamma}, \quad g_{\alpha\beta} = -1, \quad u^\alpha = \delta^\alpha_0 \quad (f,\gamma = \partial f/\partial x^\gamma) \)

hold there. Then, up to terms of order \( c^{-2} \), equations \((2.8)_{1,2}\) equal the 1st Cauchy dynamic equation for continuous media and the first principle, written within classical physics in a Euclidean frame that is locally freely gravitating and non-rotating with respect to Galileian frames. Hence they constitute acceptable relativistic versions of those laws.

C) Second principle of thermodynamics.

Let \( T (>0) \) be the absolute temperature of \( C \) at the typical event point \( \mathcal{E} \in \mathcal{W}_C \), measurable by observer locally joined to matter. The 2nd principle of thermodynamics reads in relativity theory substantially in the same way as in classical physics:

For every material point \( P^* \) of \( C \) there is a function \( \eta \) of the local intrinsic physical state of \( C \) at \( P^* \)—called the entropy function—such
that along every possible physical process we have—cf. [3 (25.1)]—

\begin{equation}
\frac{k}{D} \frac{D\eta}{D\tau} \geq \frac{q_{\text{ass}}}{T} \quad (s = c\tau).
\end{equation}

On the other hand, in classical physics, Clausius-Duhem inequality

\begin{equation}
\int_{\mathcal{V}} k \dot{\gamma} \, dv \geq \int_{\partial \mathcal{V}} \frac{\mathbf{q}^i}{T} \, da_i \quad (\mathbf{q}^i = cq^i),
\end{equation}

constitutes a much used version of the 2nd principle. It yields

\begin{equation}
k \dot{\gamma} \geq -\left(\frac{\mathbf{q}^i}{T}\right)_{i}.\end{equation}

This local form is relativized into

\begin{equation}
k \frac{D\eta}{D\tau} \geq -\left(\frac{q^s}{T}\right)_{s}.\end{equation}

Thus the classical divergence \((T^{-1}\mathbf{q})_{,i}\) is relativized into a space-time divergence (and not e.g. into \((T^{-1}q^s)_{,s}\)) in harmony with Cattaneo’s point of view—cf. [4]—justified by Bressan in [2] by means of kinematic considerations.

If \(C\) is capable of only reversible processes, i.e. processes that render (2.10) an equality in \(\mathcal{W}_C\), then by (2.8), and (2.5), (2.13) yields

\begin{equation}
q^s \theta_s < 0 \quad \text{where} \quad \theta_s = T_{i,s} + TA_{,s}.
\end{equation}

By considerations involving pocket temperature, \(\theta_s\) appears to be the relativistic analogue of the classical temperature gradient \(T_{,i}\) —cf. [3], §§ 25, 45. Then (2.14) is a natural relativization of the classical relation \(\bar{q}^i T_{,i} < 0\), assumed to hold for general processes and materials. Therefore is, besides (2.10), inequality (2.14) postulated (4).

\(^{(4)}\) This formulation constitutes substantially a relativistic version of the classical version of the 2nd principle in [13]: \(\gamma_{\text{loc}} \geq 0\), \(\gamma_{\text{con}} \geq 0\) where \(\gamma_{\text{loc}} = \dot{\eta} + (kT)^{-1}\bar{q}^i_{,i}\), \(\gamma_{\text{con}} = -k^{-1}T^{-2}\bar{q}^i T_{,i}\).
D) Explicit form of the heat flux vector. On some viscous fluids capable of heat conduction.

The following theorem is proved in [3], Theor. 25.1, p. 66:
Let \( q^\alpha \) be a function of the position gradient \( \alpha^\alpha_k, T, T^\alpha_{i\alpha}, \) and \( A_\alpha, \)
that is linear in \( T^\alpha_{i\alpha} \) and \( A_\alpha. \) Then inequality (2.14) implies, in relativity, the version

\[
(2.15) \quad q^\alpha = -\chi^{\alpha\beta}(T^\beta_{i\beta} + TA_\beta) \quad (\chi^{\alpha\beta}\xi^\alpha\xi_\beta > 0)
\]

of Fourier's law, with \( \chi^{\alpha\beta} \) spatial and depending at most on \( \alpha^\alpha_k \) and \( T. \)
Since for fluids Eckart proposed the special version

\[
(2.16) \quad q_\alpha = -\kappa(T^\alpha_{i\alpha} + TA_\alpha) \quad (\kappa > 0)
\]

of the relation (2.15), this is often called the Fourier-Eckart law of heat conduction.

A. Bressan proved—cf. [3], Theor. 25.3—that \( \chi^{\alpha\beta} = 0 \) iff \( q^{\alpha\beta} \) complies with the principle of material frame indifference—cf. [3], §§ 80-82—or more simply iff \( q^{\alpha\beta} \) is rotationally objective.

In [3] the viscous fluids are considered for which \( w, \eta, \) and \( X^{\alpha\beta} \)
are functions of \( k, T, u_{\alpha/\beta}, \) and also \( N \) unspecified physical parameters \( \xi_1 \) to \( \xi_N; \) after setting

\[
(2.17) \quad \psi = w - T\eta = \ddot{\psi}(k, T, u_{\alpha/\beta}, \xi_1, \ldots, \xi_N), \quad X^{\alpha\beta} = k^2 \frac{\partial \psi}{\partial k} \delta^{\alpha\beta} + X^{\alpha\beta}_{(irr)},
\]

so that \( \psi \) is the free energy, the constitutive relations

\[
(2.18) \quad \eta = -\frac{\partial \psi}{\partial T}, \quad \frac{\partial \psi}{\partial u_{\alpha/\beta}} = 0 = \frac{\partial \psi}{\partial \xi_i}, \quad i = 1, \ldots, N, \quad X^{\alpha\beta}_{(irr)} u_{\alpha/\beta} < 0
\]

are proved there with a procedure of the Coleman-Noll type. Set

\[
(2.19) \quad p = k^2 \frac{\partial \psi}{\partial T}, \quad \text{whence } \quad X^{\alpha\beta} = p g^{\alpha\beta} + X^{\alpha\beta}_{(irr)}.
\]

As is well known by Helmoltz's postulate and (2.18-19) the expressions \( T = \ddot{T}(k, \eta) \) and \( w = \ddot{w}(k, \eta) \) can be specified and the classical Gibbs's
relation

\begin{equation}
(2.20) \quad p = k^2 \frac{\partial \bar{w}(k, \eta)}{\partial k}, \quad T = \frac{\partial \bar{w}(k, \eta)}{\partial \eta}, \quad T \, d\eta = \bar{w} + \frac{1}{k} \nabla \cdot
\end{equation}

can be deduced.

In the sequel a (perhaps non-linearly) viscous fluid is considered. Let it be described by the constitutive equations (2.16) and (2.20) where—cf. (2.18)—

\begin{equation}
(2.21) \quad \bar{X}^{a\beta}(k, \eta, \omega_{\eta/\omega}) = \bar{X}^{a\beta}(k, \eta, 0) = 0.
\end{equation}


The conditions for classical thermodynamic equilibrium \((\bar{q} = 0, \bar{v} = 0)\) involve a rigid motion. Therefore it is natural to extend this notion of equilibrium to general relativity by means of a definition such as the following

**Definition 3.1.** The body \(C\) is said (in \(E\)) to be in (or to undergo a process of) \(e\)-equilibrium in the region \(R \subseteq W_C\), if in \(R\) we have

\begin{equation}
(3.1) \quad q^a = 0 \equiv \omega_{\omega/\omega}.
\end{equation}

Remark that if (a) \(C\) has a co-moving frame \((x)\) which is stationary \((g_{\alpha\beta,0} \equiv 0)\) or in particular static \((g_{\alpha\beta,0} = 0 \equiv g_{\alpha\sigma})\) in \(R\), and (b) no heat conduction takes place there, then (c) \(C\) is in \(e\)-equilibrium in \(R\). Indeed \(\omega_{\omega/\omega} = (\mathcal{V}_g)^{-1}(g_{\alpha\sigma} - g_{\alpha\sigma}g_{\sigma\omega}g_{\omega\omega})\), so that (a) yields (3.1)2. The converse is usually false in that (c) generally fails to imply (a).

**Theorem 3.1.** Let the viscous fluid \(F\)—cf. (2.16), (2.20), and (2.21)—be in \(e\)-equilibrium in \(R \subseteq W_F\). Then we have there

\begin{equation}
(3.2) \quad \frac{Dk}{Ds} = 0, \quad \frac{D\eta}{Ds} = 0, \quad A_\alpha = -\frac{p_k^{\lambda}}{\mathcal{V} + p}.\n\end{equation}

**Proof.** By (2.6)1 and (3.1)2, (3.2)1 holds. By (2.8)3, (2.20), and (2.21)2, (3.1) yield (3.2)2; and (3.2)3 follows from (3.1) by (2.8)1, (2.19)2, and (2.21)2. q.e.d.
By (3.2) and (2.20) in a region of e-equilibrium

$$\frac{Dg}{Ds} = 0 \quad \text{for } g = \hat{g}(k, \eta, T, p).$$

In particular this holds for Gibbs's function (or free enthalpy)

$$\phi = w + \frac{p}{k} - T\eta = \hat{\phi}(p, T).$$

As is well known

$$\frac{1}{k} = \frac{\partial \hat{\phi}}{\partial p}, \quad \eta = -\frac{\partial \hat{\phi}}{\partial T}.$$  

***

Let us now consider any process $\mathcal{P}$ for $\mathcal{C}$ in $\mathcal{E}_G$, for which the (field of the) intrinsic acceleration $A_\alpha$ is lamellar—cf. [10], p. 824—in the space time region $\Lambda$, i.e. for some scalar field $\varphi$

$$A_\alpha(x) = \varphi(x) \Gamma^\alpha_\beta, \quad \forall x \in \Lambda, \quad \text{where } \Lambda \subseteq \mathcal{W}_C.$$  

In this case can $q^\alpha$ be given a useful expression.

**Theorem 3.2.** In the process $\mathcal{P}$ for $\mathcal{C}$ let (3.6), (2.15), and the definitions

$$\Theta = T e^\varphi, \quad \chi'^\alpha{}_{\beta} = \chi^\alpha{}_{\beta} e^{-\varphi}$$

hold. Then

$$q^\alpha = -\chi'^\alpha{}_{\beta} \Theta_{/\beta}.$$  

The proof is obvious. Remark that $\Theta$ is a natural extension to lamellar fields of Tolman and Ehrenfest’s pocket temperature $T_{(p)}$—cf. [11], [12]—which notion was introduced for the first time in 1930, in connection with the equilibrium of a black body with respect to a static frame:

$$T_{(p)} = T \sqrt{-g_{00}}.$$
Indeed if the co-moving frame (for C) is stationary (or static),

\[ A_\alpha = u_{\alpha \beta} w^\beta = \frac{g_{00,\alpha}}{2g_{00}} = (\ln \sqrt{-g_{00}})_{,\alpha}, \]

so that, for \( \varphi = \ln \sqrt{-g_{00}} \) we have

\[ \Theta = T e^\varphi = T e^{\ln \sqrt{-g_{00}}} = T \sqrt{-g_{00}} = T(\varphi). \]

Therefore \( \Theta \) will be called \textit{pocket temperature} in the sequel.

Obviously \( A_\alpha \) generally fails to be lamellar for a body \( C \) (in \( \mathcal{C}_E \)).

In spite of this we can prove the following

**Theorem 3.3.** Let \( \mathcal{F} \) is a viscous fluids capable of heat conduction, defined by (2.16), (2.20), and (2.21); furthermore let \( \mathcal{F} \) be in c-equilibrium in \( \mathcal{R} (\subset \mathcal{W}_\mathcal{F}) \). Then, under definition (3.4), in \( \mathcal{R} \) we have

\[
\begin{align*}
A_\alpha &= -\left[ \ln \left( 1 + \frac{\phi}{\alpha} \right) \right]_{,\alpha}, \\
\Theta &= \frac{T}{1 + \phi/\alpha}, \quad \Theta_{,\alpha} \equiv 0.
\end{align*}
\]

**Proof.** By Theor. 3.1 (3.2) holds; furthermore by (3.4) and (2.6)

\[ \varrho + p = k(\phi + T\eta + \alpha^2). \]

Hence the first of the relations

\[ (\phi + T\eta + \alpha^2)A_\alpha = -\frac{\partial \phi}{\partial \eta} p_{,\alpha} = -\phi_{,\alpha} + \frac{\partial \phi}{\partial T} T_{,\alpha} =
\]

holds by (3.5), while (3.13)\( \alpha,\beta \) follow from (3.4) and (3.5)\( \alpha \): By (2.16), for \( \alpha \neq 0, q^\alpha \equiv 0 \) implies \( \eta TA_\alpha = -\eta T_{,\alpha} \), so that (3.13) yields

\[ A_\alpha = -\left[ \ln (\phi + \alpha^2) \right]_{,\alpha} = -\left[ \ln \left( 1 + \frac{\phi}{\alpha^2} \right) \right]_{,\alpha}. \]

In addition by (3.3) \( D\phi/Ds \equiv 0 \) in \( \mathcal{R} \). Then (3.12)\( 1 \) holds. This is
(3.6) for \( \varphi = -\ln (1 + e^{-2\phi}) \). Hence by Theor. 3.2 we have (3.7), i.e. (3.12)_2, and (3.8). Since \( \kappa' \), as well as \( \kappa \), is strictly positive definite, (3.8) and (3.1), yield \( \Theta_{j\delta} = 0 \) in \( \mathfrak{R} \); and the above definition of \( \varphi \) together with (3.3) yields \( D\Theta/Ds = 0 \) there. Then (3.12)_3 holds. q.e.d.

4. Expression of \( g_{00} \) in the stationary case of \( \epsilon \)-equilibrium. Comparison with the case of a weak (classical) gravitational field.

Let \( \mathfrak{F} \) be in \( \epsilon \)-equilibrium in \( \mathfrak{R} (\subset \mathfrak{W}_\mathfrak{F}) \) and let any co-moving frame of it be stationary (in \( \mathfrak{R} \)). Then, by (3.11), for some constant \( K \),

\[
(4.1) \quad \frac{\ln K}{K} + \ln \sqrt{-g_{00}} = \varphi = -\ln \left( 1 + \frac{\phi}{c^2} \right)
\]

hence \( g_{00} = -\frac{K^2}{(1 + \phi/c^2)^2} \).

The thermodynamic expression (4.1)_3 of \( g_{00} \) holds also in a strong gravitational field provided \( q^\alpha \) has a linear expression in \( T_{j\delta}^\alpha \) and \( A_\alpha \). Let us now show that if the gravitational field is weak (4.1)_3 becomes the well known relation

\[
(4.2) \quad g_{00} \simeq -\left( 1 - \frac{2U}{c^2} \right)
\]

up to an additive constant, where \( U \) is the newtonian potential of the gravitational field. Indeed the classical equations for the thermodynamic equilibrium of a viscous fluid in the above gravitational field read \( kU_{,i} = p_{,i} \) and \( T_{,i} = 0 \). By (3.5) they yield

\[
(4.3) \quad U_{,i} = \frac{\partial \phi}{\partial p} p_{,i} = \phi_{,i}, \quad \text{hence } U = \phi + K_1 \text{ (} K_1 \text{ = const).}
\]

Since \( e^{-2\phi} \ll 1, (1 + e^{-2\phi})^{-2} \simeq 1 - 2e^{-2\phi} = 1 - 2e^{-2U} \pm \text{const.} \) Hence (4.1)_3 becomes (4.2), up to the constant \( 2e^{-2K_1} \).

Now remark that since (4.1)_3 must be equivalent with (4.2), (4.1)_3 holds for \( K = 1 \) and the determination of \( \phi \) that fulfils condition (4.3)_3 with \( K_1 = 0 \).
5. An alternative to Eckart's choice of the heat flux vector that leads to a nonequivalent but equally acceptable relativistic thermodynamics for heat conducting fluids.

I want to show that in every process for a possibly viscous fluid $\mathcal{F}$—defined by (2.16), (2.20) and (2.21)—

\begin{equation}
\tag{5.1}
P \sigma = [1 + O(\sigma^{-2})] \hat{q}^\alpha,
\end{equation}

where $\hat{q}^\alpha = -\hat{\Theta}_i \hat{q}_i$ and

\[
\hat{\Theta} = \frac{(\sigma^2 + \phi)^2 k}{c^2 (\rho + p)}
\]

(\hat{q}^\alpha \text{ is the Fourier-Eckart heat flux vector } q^\alpha \text{ defined in (2.16)}), \text{ and that}

\begin{itemize}
  \item[(a)] $\hat{q}^\alpha$ vanishes in any $e$-equilibrium process.
\end{itemize}

Since the magnitude $\Theta$ is the pocket temperature—cf. N. 3—, I shall call $\hat{q}^\alpha$ the pocket heat flux.

Let $\mathcal{U}_{\alpha\beta}$ be Eckart's energy-momentum tensor for viscous fluids and let $\hat{\mathcal{U}}_{\alpha\beta}$ result from it by replacing $\hat{q}^\alpha$ with $\hat{q}_\alpha$. By (5.1),

\begin{equation}
\tag{5.2}
P \mathcal{U}_{\alpha\beta} = [1 + O(\sigma^{-2})] \hat{\mathcal{U}}_{\alpha\beta},
\end{equation}

where $\hat{\mathcal{U}}_{\alpha\beta} = \hat{q}_\alpha u_\beta + \hat{X}_{\alpha\beta} + 2 \hat{q}_\alpha u_\beta$.

(b) For non-viscous fluids one can strengthen (5.1) and (5.2) into

\begin{equation}
\tag{5.3}
P \mathcal{q}^\alpha = [1 + O(\sigma^{-4})] \hat{q}^\alpha, \quad P \mathcal{U}_{\alpha\beta} = [1 + O(\sigma^{-4})] \hat{\mathcal{U}}_{\alpha\beta}.
\end{equation}

The use of $\hat{\mathcal{U}}_{\alpha\beta}$ as the energy-momentum tensor in special or general relativity is in agreement with experiments by (5.1). By (a) this use also complies with the considerations made to support Eckart's proposal $q^\alpha$. Indeed these considerations substantially say that the relation $T_{\alpha\beta} = -TA_{\alpha\beta}$ must hold rigorously in thermodynamic equilibrium—cf. [3], § 45.

In order to prove (5.1) we first write an explicit expression of $d\Theta$ for any viscous fluid. Now I consider $\phi = \hat{\phi}(k, T) = \hat{\phi}(k, T)$
where—cf. N. 2—\( \hat{\rho}(k, T) = k^2 \partial \hat{\psi}(k, T)/\partial k \) and \( \hat{\omega}(k, T) \) and \( \hat{\eta}(k, T) \) are defined by (2.17) and (2.18). By (3.12) one easily obtains

\[
(5.4) \quad d\hat{\Theta}(k, T) = \frac{e^2}{(e^2 + \hat{\phi})^2 k} \left( \hat{\xi} + \hat{\rho} - T \frac{\partial \hat{\rho}}{\partial T} \right) \cdot \left( dT - T \frac{\partial \hat{\rho}/\partial k}{\hat{\xi} + \hat{\rho} - T(\partial \hat{\rho}/\partial T) dk} \right).
\]

On the other hand

\[
(5.5) \quad \frac{\partial \hat{\Theta}(k, T)}{\partial T} = \frac{1}{1 + \hat{\phi}/e^2} \frac{T (\partial \hat{\phi}/\partial p)(\partial \hat{\rho}/\partial T) + \partial \hat{\phi}/\partial T}{(1 + \hat{\phi}/e^2)^2} = \frac{e^2}{(e^2 + \hat{\phi})^2 k} \left( \hat{\xi} + \hat{\rho} - T \frac{\partial \hat{\rho}}{\partial T} \right),
\]

whence

\[
(5.6) \quad d\hat{\Theta}(k, T) = \frac{\partial \hat{\Theta}(k, T)}{\partial T} \left( dT - T \frac{\partial \hat{\rho}/\partial k}{\hat{\xi} + \hat{\rho} - T(\partial \hat{\rho}/\partial T) dk} \right).
\]

Now let us eliminate \( A_\alpha \) from the expression (2.16) of \( \bar{q}_\alpha \) in connection with the above typical viscous fluid. By (2.16) and (2.8), we obtain

\[
(5.7) \quad \bar{q}_\alpha = - \frac{\xi + \hat{\rho} - T(\partial \hat{\rho}/\partial T)}{\hat{\xi} + \hat{\rho}} \cdot \left[ T_{\alpha} - \frac{\partial \hat{\rho}/\partial k}{\hat{\xi} + \hat{\rho} - T(\partial \hat{\rho}/\partial T) k_{\alpha}} - T \frac{g_{\alpha e}(X_{(ir)}^{\sigma}) + Q_{(ir)}^{\sigma})}{\phi + \hat{p} - T(\partial \hat{\rho}/\partial T)} \right].
\]

By (5.4), under definitions (5.1), (5.7) becomes

\[
(5.8) \quad \bar{q}_\alpha = \bar{q}_\alpha + \frac{\kappa T}{\phi + \hat{p}} g_{\alpha e}(X_{(ir)}^{\sigma}) + Q_{(ir)}^{\sigma})/\sigma.
\]
Since, in units of ordinary sizes, \( \bar{q}^2 = cq^2 \) and \( u^\alpha = c^\alpha \beta^\alpha c^{-1}. \)
\( \cdot Dv^\alpha / D\tau, \) the members of (2.8) are \( O(c^{-2}) \) and \( \pi \) is \( O(c^{-1}) \) (with respect to ordinary size magnitudes). Hence

\[
(5.9) \quad \frac{\pi T}{\bar{q} + p} \frac{1}{g_{\alpha\beta}} X_{(irr)\alpha} \approx O(c^{-2}), \quad \frac{\pi T}{\bar{q} + p} g_{\alpha\beta} Q_{(irr)\alpha} \approx O(c^{-5}).
\]

Then by (5.8) and (5.9) we have (5.1). By (2.5) and (2.7) this yields (5.2) and (5.3) when \( X_{(irr)\alpha} \equiv 0. \)

Lastly in any \( e \)-equilibrium process \( \Theta_{(e)\alpha} \equiv 0 \)--cf. N. 3. Hence (5.1) yields (a). q.e.d.

6. Comparison of \( \mathcal{C}_E \) with Alts and Müller's theory \( \mathcal{C}_{AM} \).

A) Comparison of \( \mathcal{C}_E \) and \( \mathcal{C}_{AM} \) in equilibrium processes.

In [1] a relativistic thermodynamic theory \( \mathcal{C}_{AM} \) is presented by Alts and Müller. In this theory a magnitude \( \bar{\theta} \), called empirical temperature (or heat potential) is introduced. This temperature cannot be identified with the absolute one—as the deductions in [1] show—and in the general case it lacks any operative physical interpretation; furthermore \( E \)-equilibrium is defined in \( \mathcal{C}_{AM} \) by means of the condition \( \bar{\theta}_{(e)\alpha} \equiv 0. \)

Along \( E \)-equilibrium processes for (simple) non viscous fluids capable of heat conduction the validity of Gibbs's differential relation (2.20) is proved (4), so that the corresponding well known twoparameter thermodynamics holds. In this case the deductions made in [1] to differentiate \( \bar{\theta} \) thought of as a function of \( k \) and \( T \), lead to a result which, with the present notations, reads

\[
(6.1) \quad d\bar{\theta} = \frac{\partial \bar{\theta}}{\partial T} \left( dT - T \frac{\partial p / \partial k}{\bar{q} + p - T(\partial p / \partial T)} dk \right)
\]

—cf. [1 (5.19)]—where \( p \) and \( \bar{q} \) are suitable functions that can express the pressure and density of gravitational mass (in energy units) in terms of \( k \) and \( T \).

(4) The analogue for viscous fluids is not done in \( \mathcal{C}_{AM} \).
By comparing the relation (5.6), deduced in $\mathcal{E}_E$, with the result (6.1) of $\mathcal{E}_{AM}$, we see that the condition

\begin{equation}
T_{\alpha\beta} = T \frac{\partial p}{\partial k_{\alpha\beta}} \frac{\partial}{\partial T} k_{\alpha\beta}
\end{equation}

which characterizes $E$-equilibria in $\mathcal{E}_{AM}$, also holds in $e$-equilibria (in $\mathcal{E}_E$); and in them it is equivalent to the only condition on thermodynamic fields present in the definition of $e$-equilibria.

B) Determination of all choice for the equilibrium empirical temperature $\hat{\Theta}(k, T)$ in terms of Gibbs's function.

Remark that, while in [1] the usual integrability conditions of (6.1)—cf. [1 (7.8)]—are made explicit, here the analysis of $e$-equilibrium and in particular (5.6), where the definition (3.12) is presupposed, allows us to solve the differential condition (6.1) in the unknown function $\hat{\Theta}(k, T)$. It suffices to set

\begin{equation}
\hat{\Theta}(k, T) = f(\Theta) \quad (\Theta = T/(1 + e^{-\phi}))
\end{equation}

or in particular $\Theta = \Theta$. Hence the empirical temperature $\hat{\Theta}$ can be identified with the pocket temperature $\Theta$ as far as equilibrium is concerned.

Conversely (5.6) and (6.1) imply $\partial(\Theta, \hat{\Theta})/\partial(k, T) = 0$, which yields (6.3) for some differential function $f$. Thus (6.3) is the general solution of (6.1). So the empirical temperature $\hat{\Theta}$ is determined up to a change of the metric on the possible values of an arbitrarily prefixed choice of empirical temperature.

C) Comparison of $\mathcal{E}_E$ and $\mathcal{E}_{AM}$ in the non equilibrium case.

In order to compare $\mathcal{E}_E$ with $\mathcal{E}_{AM}$ in non equilibrium cases remark that, on the one hand, a choice of constitutive equations in $\mathcal{E}_{AM}$, that express $w, p, \eta, \chi, \varphi$, and $Q$ in terms of the magnitudes $k$ and $\hat{\Theta}$ (among which $\partial_{\alpha\beta}$ does not appear) is compatible with the restrictions due to the entropy principle—cf. [1], N. 3—up to $O(c^{-4})$ (4).

(4) To realize this directly, consider the fluids in $\mathcal{E}_{AM}$ that are capable of heat conduction and are defined by a sixtuple of constitutive functions that express $w, p, \eta, \chi, \varphi$, and $Q$ in terms of $k$ and $\hat{\Theta}$ (but not of $\partial_{\alpha\beta}$ as in general cases);
Hence in this case the heat flux has a linear expression $\mathcal{L}_{q_a}$ in $\partial_{/a}$:

\[
\mathcal{L}_{q_a} = -\chi \frac{\partial \theta}{\partial T} \left( T \frac{\partial p}{\partial k} - T \frac{\partial p}{\partial T} \right)
\]

—cf. [1 (5.21)]— and here $\chi, \theta, \varphi,$ and $p$ are thought of as functions of $k$ and $T$.

On the other hand in $\mathcal{C}_E$, the heat flux for non-viscous fluids has an expression, $\mathcal{F}_{q_a}$— cf. (5.1) and (5.3) — which differs from $\mathcal{L}_{q_a}$ to $O(c^{-4})$.

Lastly Alts and Müller conclude in [1] that $\mathcal{C}_{AM}$, which deals only with non-viscous fluids, is in good agreement with Chernikov's relativistic kinetic theory— cf. [5] to [7] —, say $\mathcal{C}_C$, in that a certain expression $\mathcal{C}_{q_a}$ for the heat flux obtained in $\mathcal{C}_C$ is suitably identifiable with the expression (6.4) for $\mathcal{L}_{q_a}$ (hence with $\mathcal{F}_{q_a}$ too). Therefore I can conclude that, since for same fluids $\mathcal{F}_{q_a}$ and $\mathcal{L}_{q_a}$ are suitably identifiable with $\mathcal{C}_{AM}$'s $\mathcal{C}_{AM}$'s axioms up to $O(c^{-4})$, $\mathcal{C}_C$ agrees with $\mathcal{C}_E$ at the same approximation order as with $\mathcal{C}_{AM}$.

REFERENCES


remark that for them the restriction relations (4.12) and (4.13) in [1] become

$$\Delta RQ = \Delta EQ/2, \quad \Delta Q = -\Delta R\chi c^{-2}, \quad \Delta Q/(2k) = 0,$$

where $R = (\chi + Q\theta)(\varphi + p - \chi \theta c^{-2})^{-1}$; and $R \approx O(c^{-2})$ because $\chi$, the counterpart of $\varphi$ in $\mathcal{C}_E$, is an ordinary size magnitude. (The magnitudes $\varphi$ and $Q$ have no counterparts in $\mathcal{C}_E$.) Hence, for $\chi$ and $\Lambda$ not vanishing (in $E$-equilibria $\Lambda^{-1}$ is the absolute temperature $T$), a suitable choice of the above constitutive functions is compatible with the axioms of $\mathcal{C}_{AM}$ if $Q(k, \theta) = 0$ and $Rc^{-2}(\approx O(c^{-4}))$ is regarded as negligible.


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