FRANCO PARLAMENTO

Binumerability in a sequence of theories

Rendiconti del Seminario Matematico della Università di Padova,
tome 65 (1981), p. 9-12

<http://www.numdam.org/item?id=RSMUP_1981__65__9_0>

© Rendiconti del Seminario Matematico della Università di Padova, 1981, tous droits réservés.

L’accès aux archives de la revue « Rendiconti del Seminario Matematico della Università di Padova » (http://rendiconti.math.unipd.it/) implique l’accord avec les conditions générales d’utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d’une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.
Binumerability in a Sequence of Theories.

FRANCO PARLAMENTO (*)

SUMMARY - In this note we answer a question raised by A. Ursini in [2]. In that work he defines a denumerable sequence of arithmetic theories $Q_n$, whose union is complete, and asks a question concerning the binumerability of the $\Delta_{n+1}$ relations in $Q_n$. We show that a relation is in $\Delta_{n+1}$ if and only if it is binumerable in $Q_n$.

We are going to follow the notations and terminology of [1] and [2]. In particular $K_0$ is the language of first order arithmetic, and a $K$-system is a set of sentences in the language $K$. $Prf_T(x, y)$ is the relation « $y$ is a proof of $x$ from axioms in $T$ ». Let's recall that, given a $K$-system, where $K_0 \subseteq K$, a numerical relation $R \subseteq \omega^n$ is called numerable in $T$ if there is a formula $\varphi(x_1, \ldots, x_n)$ in $K$ such that

$$R(k_1, \ldots, k_n) \text{ holds if and only if } T \vdash \varphi(\overline{k}_1, \ldots, \overline{k}_n).$$

In that case we say that $\varphi$ numerates $R$ in $T$.

A relation $R$ is binumerable in $T$ if there exists a formula $\varphi$ in $K$ such that $\varphi$ numerates $R$ and $\neg \varphi$ numerates $\omega^n \setminus R$.

The result expressed in the following proposition is obtained as a straightforward application of the « Rosser trick ».

**Proposition 1.** Let $T$ be a consistent $K$-system, where $K_0 \subseteq K$, such that $T \vdash x < \overline{n} \leftrightarrow x = \emptyset \lor x = 1 \lor \ldots \lor x = \overline{n}$ and $T \vdash x < \overline{n} \setminus \overline{n} < x$.

Let the relation $Prf_T(x, y)$ be binumerable in $T$ and $R \subseteq \omega^n$.

(*) Indirizzo dell'A.: University of California, Berkeley, Cal. and Università di Torino.
If both $R$ and $\omega^* - R$ are numerable in $T$ then $R$ is binumerable in $T$.

**Proof.** For notational convenience let's suppose $R \subseteq \omega$. Let $\varphi(x)$ numerate $R$ and $\psi(x)$ numerate $\omega - R$, and let $\text{Prf}(x, y)$ binumerate $\text{Prf}_T(x, y)$ in $T$.

Consider then the following formula

$$\chi(x) = \exists y (\text{Prf}(\varphi(x), y) \land \forall z \leq y \rightarrow \text{Prf}(\psi(x), z)).$$

We claim that $\chi(x)$ binumerates $R$ in $T$.

Since $T$ is consistent it is clearly enough to show that

(i) if $k \in R$ then $T \vdash \chi(k)$,

and

(ii) if $k \notin R$ then $T \vdash \neg \chi(k)$.

(i) If $k \in K$ then $T \vdash \varphi(k)$, since $\varphi$ numerates $R$ in $T$, therefore for some $n \in \omega$, $\text{Prf}_T(\varphi(k), n)$ holds and therefore

$$T \vdash \text{Prf}(\varphi(k), n).$$

On the other hand, since $\psi$ numerates $\omega - R$, we have

$$\forall i \leq n, \quad T \vdash \neg \text{Prf}(\psi(k), i)$$

and thus

$$T \vdash \forall z \leq n \rightarrow \neg \text{Prf}(\psi(k), z).$$

From (1) and (2) we have

$$T \vdash \chi(k).$$

(ii) Let's assume that $k \notin R$.

Since $\psi$ numerates $\omega - R$ in $T$, we have $T \vdash \psi(k)$ and therefore, as above, for some $r \in \omega$

$$T \vdash \text{Prf}(\psi(k), r).$$

On the other hand for $i \leq r \text{ Prf}_T(\varphi(k), i)$ doesn't hold, hence, as
above

\[ T \vdash \forall z \leq \bar{r} \rightarrow \text{Prf}(\varphi(k), z). \]

Therefore

(4) \[ T \vdash \text{Prf}(\varphi(k), y) \rightarrow y \leq \bar{r}. \]

From (3) and (4) we have

\[ T \vdash \text{Prf}(\varphi(k), y) \rightarrow \exists z \leq y \text{Prf}(\varphi(k), z) \]

namely

\[ T \vdash \lnot \chi(k). \]

In [2] two sequences \( \{Q_n : n \in \omega\} \) and \( \{R_n : n \in \omega\} \) with the following properties are defined.

1) \( Q_n = \text{Prf}_{R_n} \).

2) \( R_n \) is a \( \Sigma_{n+1} \) valid \( K_0 \)-system containing Robinson’s Arithmetic \( Q \).

3) \( R \in \Sigma_{n+1} \) if and only if it is numerable in \( R_n \).

4) \( R_n \) is binumerable in \( Q_n \) via a formula \( \alpha_n \) (Proposition 8 in [2]).

Applying Proposition 1 we have the following result,

PROPOSITION 2. \( R \in \Lambda_{n+1} \) if and only if \( R \) is binumerable in \( Q_n \).

PROOF. Let's first notice that by 1) numerability (and binumerability) of a relation in \( R_n \) or in \( Q_n \) are equivalent. Hence, by 4) we have that \( R_n \) is binumerable in \( R_n \) via \( \alpha_n \). Thus the relation \( \text{Prf}_{a_n}(x, y) \) is binumerable in \( R_n \), via, say \( \text{Prf}_{a_n}(x, y) \), since it is just a primitive recursive combination of \( R_n \) and \( PR \) relations, which are binumerable in \( R_n \) by 2). Also from 2) we have that \( R_n \) is consistent and,

\[ R_n \vdash x \leq \bar{n} \leftrightarrow x = \bar{0} \lor x = \bar{1} \lor \ldots \lor x = \bar{n} \quad \text{and} \quad R_n \vdash x \leq \bar{n} \land \bar{n} \leq x. \]

If \( R \in \Lambda_{n+1} \) we have that both \( R \) and \( \omega - R \) are numerable in \( R_n \).

We can thus apply Proposition 1 to get that \( R \) is binumerable in \( R_n \). Conversely if \( R \) is binumerable in \( R_n \) it follows immediately from 3) that both \( R \) and its complement are in \( \Sigma_{n+1} \), namely \( R \in \Lambda_{n+1} \).
REFERENCES


Manoscritto pervenuto in redazione il 28 gennaio 1980.