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Approximation of Eigenvalues and a Koehler's Type Method.

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SUMMARY - A new deduction of a Koehler's type method is presented that emphasizes the general properties required of a problem for the technique to work. Suggestions on strategies to follow when applying the method are also given.

1. Introduction.

In mechanics the most important examples of eigenvalue problems arise in the study of the dynamics of bodies and in stability or bifurcation problems. Eigenvalue-eigensolution pairs describe, respectively, either natural frequencies and modes of the body or critical values of the data and the corresponding non-trivial equilibrium configurations. Examples from different areas are flutter phenomena in aeroelasticity [1], Sensenig and Ericksen's problems in finite elasticity [2] [3], and the Taylor cells [4] or the Benàrd problem [5] in fluid mechanics. At some stage of their development all of these problems are phrased as eigenvalue problems whose solution singles out transition points between behaviors with completely different features.

The dramatic physical nature of these transitions highlight the

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importance of obtaining solutions to such eigenvalue problems, primarily for practical purposes. Therefore, as the evaluation of the eigenvalues is in general obtained by approximate methods, it is extremely useful in the applications to have a direct estimate of the error from techniques providing two-side bounds for the eigenvalues.

The study of this quantitative aspect is largely lacking for non-linear problems; on the contrary, there is a rather large litterature concerning linear problems, covering wide classes of operators of interest in mechanics. Here we fix our attention on linear problems.

To obtain lower bounds for the eigenvalues is the hardest part of the matter, upper bounds being easily found by the Rayleigh-Ritz method. The common basis of all techniques is the comparison between the eigenvalues of the problem under consideration with those of some other problem whose spectral properties are known. Comparisons are often mere applications of the monotonicity theorems based upon the min-max principe [6]. For discussing a specific problem, the availability of a comparison problem is the basic information that is always needed in order to start any method.

The early studies on lower bounds are sporadic applications of the monotonicity theorems in their simpler form, as is the case in the works of Morrow (1905), Prescott (1920) and Southwell (1921), see [7]. Systematic studies start only after the paper of Temple was published (1928); the main contributions are due to Weinstein (1935-63), Aronszajn (1948-51), Weinberger (1959) and Fichera (1965), see [6]. All the methods above differ in the quality and quantity of the basic informations they require, but are close to each other under several respects, as shown in [6] for the methods of Weinstein and Aronszajn, and in [8] for Weinstein and Weinberger's.

In what follows we wish to present a method for obtaining lower bounds to eigenvalues that is the natural development of results obtained discussing the infinitesimal stability of homogeneously deformed plates and shells [9]. Although this method has been worked out independently, it comes close to a technique proposed by Koehler [10] dealing with estimates for the eigenvalues of infinite matrices. The approach is different, however, and is more advantageous in that it puts in evidence the general properties required of a problem in order to apply the technique, and facilitates comparison with some of the methods previously mentioned. Moreover, the proof the method is based upon does not require a condition, perhaps minor, that is present in Koehler's paper.

While the deduction of the method and the comparison with others are given in [11] in full detail, in the present article we elaborate upon it and illustrate through few examples some strategies that can be used in the applications.

2. Description of the method.

Our aim is to estimate the eigenvalues of the following problem

$$Au = \mu u \qquad u \in \mathfrak{D} \subset \mathfrak{H} ,$$

when A is a lower bounded self-adjoint operator on a dense linear manifold $\mathfrak{D}$ in a Hilbert space $\mathfrak{H}$, with a scalar product $(\cdot, \cdot)$. Assume that

$$A = A' + B , \tag{1}$$

with A' self-adjoint on $\mathfrak{D}$, B defined on $\mathfrak{D}_B$, $\mathfrak{D} \subset \mathfrak{D}_B \subset \mathfrak{H}$, and both forms $(B^* B \cdot, \cdot)$ and $(\cdot, \cdot)$ are completely continuous with respect to $(A' \cdot, \cdot)$.

The method is based on the knowledge of the following:

a) The spectral properties of A';

b) A pair of numbers $\varrho < 1$ and $k(\varrho)$ with the property

$$|(Bu, u)| \leqslant \varrho(A'u, u) + k(\varrho)(u, u) \qquad u \in \mathfrak{D} . \tag{2}$$

REMARK 1. The existence of such a pair is a consequence of the assumption on B^*B, as $(B \cdot, \cdot)$ also is completely continuous with respect to $(A' \cdot, \cdot)$.

Under the hypotheses above, A' has an unbounded and discrete spectrum $\{\mu_1', \mu_2', \mu_3', \ldots\}$, and any corresponding family of eigenfunctions $\{u_1, u_2, u_3, \ldots\}$ is a basis for $\mathfrak{H}$. Let $\mathfrak{U}_n$ be the subspace spanned by $\{u_1, u_2, \ldots, u_n\}$:

$$\mathfrak{U}_n = \operatorname{span}\{u_1, \ldots, u_n\} ,$$

and call P_n the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{U}_n$.

The method is founded on the lemma:

LEMMA. A is positive if and only if $\exists n$ such that

$$\min_{\mathfrak{U}_n \cap \mathfrak{B}(1)} (Au, u) > \frac{\beta_n^2}{\delta_n} \geqslant 0 , \qquad \mathfrak{B}(1) = \{u \in \mathfrak{H}, (u, u) = 1\} , \tag{3}$$

where

$$\delta_n \equiv \mu'_{n+1}(1-\varrho) - k(\varrho) \quad \text{and} \quad \beta_n^2 \equiv \sup_{\substack{u \in \mathfrak{U}_n \cap \mathfrak{B}(1) \\ v \in \mathfrak{U}_n^\perp \cap \mathfrak{D} \cap \mathfrak{B}(1)}} (Bu, v)^2 .$$

REMARK 2. As $\mathfrak{U}_n$ is finite-dimensional, (Bu, v) is weakly continuous in $\mathfrak{U}_n \times \mathfrak{U}_n^\perp \cap \mathfrak{D}$ with respect to the norm induced by $(\cdot, \cdot)$. If we extend it to $\mathfrak{U}_n \times \mathfrak{U}_n^\perp$ by continuity, the extended form $(Bu, v)^2$ achieves a maximum on $\mathfrak{U}_n \cap \mathfrak{B}(1) \times \mathfrak{U}_n^\perp \cap \mathfrak{B}(1)$. Then we can write

$$\sup_{\mathfrak{U}_n \cap \mathfrak{B}(1) \times \mathfrak{U}_n^\perp \cap \mathfrak{D} \cap \mathfrak{B}(1)} (Bu, v)^2 = \max_{\mathfrak{U}_n \cap \mathfrak{B}(1) \times \mathfrak{U}^\perp \cap \mathfrak{B}(1)} (Bu, v)^2 = \max_{\mathfrak{U}_n \cap \mathfrak{B}(1)} \left\{ \max_{\mathfrak{U}^\perp \cap \mathfrak{B}(1)} (Bu, v)^2 \right\},$$

and, by using the projection theorem:

$$\beta_n^2 = \max_{\mathfrak{U}_n \cap \mathfrak{B}(1)} \{(Bu, Bu) - (P_n Bu, P_n Bu)\} .$$

Hence computation of all quantities in (3) involves finite algebra only. In particular, the following inequality

$$\beta_n^2 \leqslant \max_{\mathfrak{U}_n \cap \mathfrak{B}(1)} (B^* Bu, u) \tag{5}$$

holds. This can provide a cruder, but still valuable, estimate of β_n^2 to be used in (3).

We will not dwell upon the proof, see [11], but rather emphasize that sufficiency of (3) hinges upon the inequality

$$(Au, u) \geqslant \delta_n \qquad u \in (1 - P_n)\mathfrak{D} \cap \mathfrak{B}(1) , \tag{6}$$

whereas necessity requires in addition that

$$\operatorname{Lim}_{n \to \infty} \frac{\beta_n^2}{\delta_n} = 0 . \tag{7}$$

The former follows from (2); the latter is a consequence of the complete continuity of B^*B:

$$(8) \qquad \forall \varepsilon > 0,\ \exists \boldsymbol{k}(\varepsilon):\ (B^*Bu, u) \leqslant \varepsilon(A'u, u) + \boldsymbol{k}(\varepsilon)(u, u) \qquad u \in \mathfrak{D}\,.$$

By restricting our attention to $\mathfrak{U}_n$, (5) and (8) imply in fact

$$\beta_n^2 \leqslant \varepsilon \mu_n' + \boldsymbol{k}(\varepsilon)\,,$$

and hence (7).

Now if we put $A_\gamma = A + \gamma 1$ with $\gamma \in \mathbb{R}$, it follows that

$$(9) \qquad (A_\gamma u, u) > 0 \qquad \forall u \in \mathfrak{D} \setminus \{0\} \leftrightarrow -\gamma < \min_{\mathfrak{D} \cap \mathfrak{B}(1)} (Au, u) \equiv \mu_1\,.$$

By the lemma, then, A is positive if and only if for some n the inequalities

$$(10) \qquad \min_{\mathfrak{U}_n \cap \mathfrak{B}(1)} (Au, u) + \gamma > \frac{\beta_n^2}{\delta_n + \gamma} \geqslant 0$$

hold true. For any fixed n, we may regard (10) as a condition on γ. Thus, from (9) and by simply discussing a second grade equation, we obtain

$$(11) \qquad -\gamma_1^{(n)} = \frac{(\mu_1^{(n)} + \delta_n) - \sqrt{(\mu_1^{(n)} - \delta_n)^2 + 4\beta_n^2}}{2} \leqslant \mu_1\,,$$

where $\mu_1^{(n)} = \min\limits_{\mathfrak{U}_n \cap \mathfrak{B}(1)} (Au, u)$ is the Rayleigh-Ritz approximation of μ_1. On the other hand, since for any γ such that $-\gamma$ bounds μ_1 from below (10) are satisfied for some n, it follows that

$$(12) \qquad \operatorname*{Lim}_{n \to \infty} -\gamma_1^{(n)} = \mu_1\,.$$

Consideration of the higher order eigenvalues is only slightly more difficult, see [11]. However, it is possible to show that formula (11) carries over to all the eigenvalues μ_k, $k \leqslant n$, by proper use of the min-max principle, obtaining the estimate

$$(13) \qquad -\gamma_k^{(n)} \equiv \frac{(\mu_k^{(u)} + \delta_n) - \sqrt{(\mu_k^{(n)} - \delta_n)^2 + 4\beta_n^2}}{2} \leqslant \mu_k\,.$$

Here it does not follow from the lemma that $-\gamma_k^{(n)} \to \mu_k$ when $n \to \infty$, as shown in [11]; however convergence follows from the property

$$\lim_{n\to\infty} (\mu_k^{(n)} + \gamma_k^{(n)}) = 0 \,, \tag{14}$$

$\mu_k^{(n)}$ being upper bounds of μ_k. Equality (14) still depends upon (7).

REMARK 3. Incidentally, (13) and (14) state the convergence of the Rayleigh-Ritz approximations.

3. Comments and conclusions.

Dealing with estimates for the eigenvalues of infinite matrices in [10], Koehler finds estimates that are similar to the ones given above. However, our proof seems to be more convenient. Rephrased in the present terms, in fact, Koehler's proof asks for A to be positive whereas ours does not require it, showing that estimates (11) and (13) provide lower bounds for the eigenvalues of A anyhow. Moreover, the approach here emphasizes the role of certain properties that are sufficient in order to apply the method.

In this paper we will not present any numerical application of formulae (11) and (13); simple example has been considered in [11]. It is perhaps more interesting to give a brief discussion of the range of validity of the method and, also, to outline some simple strategies which make it applicable.

Convergence of the estimates based on conditions (3) depends upon (7) either directly, as for the higher order eigenvalues, or indirectly through the lemma, as for μ_1. Complete continuity of B^*B is sufficient for (7) to hold.

The condition of B^*B occurs, for instance, in the intermediate problems both of Weinstein and Aronszajn's methods, where B is a finite-dimensional perturbation of A'. Therefore, the use of formulae (11) and (13) is an alternative to the study of Weinstein's determinants in order to obtain lower bounds to the eigenvalues of A.

Another important class of problems where complete continuity of B^*B is amenable to standard results in analysis is when A is an elliptic differential operator. In this case, a classical interpolation theorem implies the complete continuity of B^*B if the order of B is less than m; $2m$ being the order of A', see [11].

Finally, it is also worth mentioning that (7) holds true when the $\mathfrak{U}_n$ approximate the analogous spans of the eigenvectors of A, $\mathfrak{M}_n$, in the following sense:

« There exist two finite-dimensional subspaces $\mathfrak{C}$ and $\mathfrak{D}$ such that for any given n there is some m with the property

$$\mathfrak{U}_u \subset \mathfrak{M}_m \vee \mathfrak{C}\,, \qquad \mathfrak{U}_n^\perp \subset \mathfrak{M}_m^\perp \vee \mathfrak{D}\,».$$

Under this hypothesis β_n^2 are equibounded, no matter whether B^*B is completely continuous. Unfortunately, however, properties involving comparisons between eigenspaces are difficult to recognize and thus this is not an explicit characterization of the operator B.

If one gives up convergence of the method, the cases it applies to increase considerably. In fact, it is easily seen that validity of inequalities (11) and (13) hinges upon sufficiency of conditions (3) and then depends on (6) only. From this we infer that Koehler's type of lower bounds are deducible whenever a pair $(\mathfrak{B}, \delta_n)$, with $\delta_n > 0$, is known such that

$$(15) \qquad (Au, u) \geqslant \delta_n(u, u) \qquad u \in \mathfrak{B}^\perp \cap \mathfrak{D}\,,$$

where $\mathfrak{B}^\perp \subset \mathfrak{H}$ is some subspace of finite deficiency n. If we decompose $\mathfrak{H}$ according to $\mathfrak{H} = \mathfrak{B} \oplus \mathfrak{B}^\perp$ and proceed as in the lemma, in fact, we still find that A is positive if (3) are satisfied with β_n^2 replaced now by

$$\beta_n^2 = \sup_{\substack{u \in \mathfrak{B} \cap \mathfrak{D} \cap \mathfrak{B}(1) \\ v \in \mathfrak{B}^\perp \cap \mathfrak{D} \cap \mathfrak{B}(1)}} (Au, v)^2\,.$$

Hence formulae (11) and (13) keep on giving lower bounds for the first n eigenvalues. Here no assumption has to be made either on any previous decomposition of A or on the knowledge of any special basis of $\mathfrak{H}$, and the quantity of information that is required comes to a minimum.

It is interesting to observe that, in this case, the required information is the same as in the *truncated operator method* proposed by Weinberger, see [6].

REMARK 4. Connections with Weinberger's method are even deeper in that it is possible to obtain arbitrarily close lower bounds for the eigenvalues of the truncated operator by a suitable use of Koehler's method, see [11].

The information expressed in (15) is available when the known part of A is the dominant one, at least asymptotically. In the elliptic differential problems, for instance, this occurs if A' contains the principal part of A. In that case it is rather easy to obtain perhaps crude inequalities like (15) in some space $\mathfrak{B}^{\perp}$ with finite deficiency. A skillful use of mappings from $\mathfrak{H}$ to some suitable Hilbert space $\mathfrak{H}'$, or changes of variables in the domain where the functions u are defined may help in order to achieve this situation. In conclusion, we wish to elucidate this by means of two examples.

Consider first Hadamard stability of elastic parallelograms or cylindrical shells, described by the director model, that are homogeneously strained and inequilibrium under given displacements along a pair of opposite edges. As shown in [9], if we restrict ourselves to discuss stability in the class of cylindrical or axisymmetric deformations, we are led to discuss the positiveness of the second variation of the total potential energy

$$(16)\qquad J(\boldsymbol{V}) = \int_0^1 \{(\boldsymbol{V}_{,t}, \mathfrak{L}^{tt}\boldsymbol{V}_{,t}) + 2(\boldsymbol{V}, \mathfrak{L}^{t}\boldsymbol{V}) + (\boldsymbol{V}, \mathfrak{L}\boldsymbol{V}_{,t})\}\,dy \geqslant 0 \qquad \text{for } \boldsymbol{V} \in H_0^1(0, 1)\,.$$

Here the R^6-vector valued functions $\boldsymbol{V}$ describe the displacements of the mean surface and of the directors with respect to the gound state, and the coefficients $\mathfrak{L}^{tt}$, $\mathfrak{L}^{t}$ and $\mathfrak{L}$ are constant 6×6 matrices depending on stresses and loads in the ground state, with

$$(17)\qquad \mathfrak{L}^{tt} \in \mathrm{Sym}\,.$$

Strenghtening the Legendre-Hadamard condition, assume that

$$(18)\qquad \mathfrak{L}^{tt} > 0\,.$$

Although apparently the problem is not an eigenvalue problem, it is of the kind discussed in the present paper for lower bounds to the minimum of J in $H_1^0 \cap \mathfrak{B}(1)$ are still of importance.

If we consider the mapping of H_0^1 onto itself defined by the successive changes of variables

$$V \to \boldsymbol{W} = \mathfrak{L}^{tt^{\frac{1}{2}}} V \qquad \text{and} \qquad \boldsymbol{W} \to W = e^{\mathcal{M}t}\boldsymbol{W}$$

with

$$\mathcal{M} = +\tfrac{1}{2}\mathfrak{L}^{tt^{-\frac{1}{2}}}(\mathfrak{L}^t - \mathfrak{L}^{t^{\mathsf{T}}})\mathfrak{L}^{tt^{-\frac{1}{2}}} \in \mathrm{Skw}\,,$$

(16) becomes

$$J(\boldsymbol{W}) = \int_0^1 \{(\boldsymbol{W}_{,t}, \boldsymbol{W}_{,t}) + (\boldsymbol{W}, e^{-\mathcal{M}t}(\mathcal{M}^2 + \mathcal{N})e^{\mathcal{M}t}\boldsymbol{W})\}\,dt \geqslant 0 \tag{19}$$

$$\text{for } \boldsymbol{W} \in H_0^1(0,1)\,,$$

where $\mathcal{N} = \mathfrak{L}^{tt^{-\frac{1}{2}}}\mathfrak{L}\mathfrak{L}^{tt^{-\frac{1}{2}}}$. Put it in the terms of the present paper, we may regard J as the quadratic form associated with the operator $A = A' + B$ where the operators

$$A'\boldsymbol{W} \equiv -\boldsymbol{W}_{,tt} \qquad \text{and} \qquad B\boldsymbol{W} \equiv e^{-\mathcal{M}t}(\mathcal{M}^2 + \mathcal{N})e^{\mathcal{M}t}\boldsymbol{W}$$

satisfy the condition required in order to obtain arbitrarily close lower bounds to the minimum of J from the method described above.

REMARK 5. In particular, under regularity assumptions on the coefficients of the principal part, the eigenvalue problems for Sturm-Liouville operators can always be reduced to a suitable form and be given convergent estimates of the eigenvalues by Koehler's method.

As a second example, consider the eigenvalue problem

$$\begin{cases} (Ku'')'' = \mu u\,, \\ u(0) = u(1) = u''(0) = u''(1) = 0\,, \end{cases} \tag{20}$$

with $K = K(x)$ sufficiently smooth and positive. In mechanical terms problem (20) describes the study of the natural frequencies of an elastic beam with bending stiffness K, simply supported at the ends.

Lower bounds to the fundamental frequency are those values of μ for which

$$\int_0^1 (Ku''^2 - \mu u^2)\,dx \geqslant 0 \qquad \text{for } \forall u \in H^2 \cap H_0^1\,. \tag{21}$$

In order to use conditions (3), we put

$$a(x) = K(x)^{\frac{1}{3}}$$

and introduce a new variable $y = y(x)$, $y \in (0, b)$, defined by

$$\frac{dy}{dx} = \frac{1}{a(x)}\,. \tag{22}$$

Accordingly, if $U(y) = u(x(y))$ and dots denote derivation with respect to y, problem (21) becomes

$$-a'\dot{U}^2\big|_0^b + \int_0^b \ddot{U}^2\,dy + \int_0^b \{(a'' + a'^2)\,\dot{U}^2 - \mu a U^2\}\,dy \geqslant 0 \tag{23}$$

$$U \in H^2 \cap H_0^1(0, b)\,.$$

The expression $-a'\dot{U}^2|_0^b + \int_0^b \ddot{U}^2\,dy$ is the form associated with the eigenvalue problem

$$\left\{\begin{array}{ll} \ddddot{U} & = \nu U\,, \\ \ddot{U}(0) & = a'(0)\,\dot{U}(0)\,, \\ \ddot{U}(b) & = a'(1)\,\dot{U}(b)\,, \\ U(0) & = U(b) = 0\,, \end{array}\right. \tag{24}$$

whose solution can be found by the characteristic exponents method. As $-a'\dot{U}^2|_0^b + \int_0^b \ddot{U}^2\,dy$ contains the principal part of (23), inequalities like (15) are easily found if we use the eigenfunctions of (24) as a basis for the space $H^2 \cap H_0^1(0, b)$ in (23); then we can write conditions (3) explicitly. Any value of μ that satisfies (3) is a lower bound for the fundamental frequency.

REMARK 6. As the remainder $\int_0^b \{(a'' + a'^2)\,\dot{U}^2 - \mu a U^2\}\,dy$ contains the first derivative of U, it is not ensured that we can find arbitrarily close bounds. It is however interesting to observe that what has been done for the problem (20) applies to differential problems of any order provided they are one-dimensional.

REFERENCES

[1] V. V. Bolotin, *Nonconservative problems of the theory of elastic stability*, G. Hermann ed., Pergamon Press, Oxford (1963).

[2] C. Sensenig, *Some buckling problems in nonlinear elasticity*, in *Bifurcation theory and nonlinear eigenvalue problems*, J. B. Keller and S. Antman ed., W. A. Benjamin Inc., New York (1969).

[3] J. L. Ericksen, *Bending a prism to helical form*, Arch. Rational Mech. Anal., **66** (1977), pp. 1-18.

[4] J. E. Marsden - M. McCracken, *The Hopf bifurcation and its applications*, Appl. Math. Sc., vol. 19, Springer-Verlag, New York (1976).

[5] P. Rabinowitz, *Existence and nonuniqueness of rectangular solutions of the Benàrd problem*, Arch. Rational Mech. Anal., **29** (1968), pp. 32-57.

[6] H. F. Weinberger, *Variational methods for eigenvalue approximation*, Reg. Conf. Series in Appl. Math., S.I.A.M., Philadelphia (1974).

[7] G. Temple, *The computation of characteristic numbers and characteristic functions*, Proc. London Math. Soc. II, **29** (1928), pp. 257-280.

[8] D. W. Fox - J. T. Stadter, *An eigenvalue estimation method of Weinberger and Weinstein's intermediate problems*, SIAM J. Math. Anal., **8** (1977), pp. 491-503.

[9] C. Davini, *One-dimensional problems in the stability of thin shells*, J. Elasticity, **10** (1980), pp. 241-254.

[10] F. Koehler, *Estimates for the eigenvalues of infinite matrices*, Pac. J. Math., **7** (1957), pp. 1391-1404.

[11] C. Davini, *A Koehler's type method for the approximation of eigenvalues* (to appear).

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