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ωμ-additive topological spaces

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0. Introduction.

In [8] R. Sikorski has introduced \( \omega_\mu \)-additive spaces, strengthening the axioms of Kuratowski for a topological space: namely for every cardinal number \( \omega_\mu \) he requires that the intersection of less than \( \omega_\mu \) many open sets is still an open set. This leads us to consider many properties which generalize the usual ones such as \( \omega_\mu \)-compactness, \( \omega_\mu \)-metrizability, locally \( \omega_\mu \) covers, and to study the related problems.

In this theory some results have had formulations quite similar to the classical ones, some results cannot have an analogous statement (e.g. the Tychonoff product theorem and the matter discussed in the remark of the second section); finally some other questions have been studied but have unsatisfactory solutions, partly because the considered context was not the right (and natural) one.

In the first section we point out that the « \( \omega_\mu \)-product topology » is the right topology in the product of \( \omega_\mu \)-additive spaces, and this remark enables us to improve a theorem of Yasui and give to it the most desirable formulation (theorem 1.1).

Although we have claimed that \( \omega_\mu \)-compactness is not productive, we show (theorem 2.3) that it is finitely productive: with this tool we get that \( \omega_\mu \)-compact spaces admit a unique \( \omega_\mu \)-additive uniformity (theorem 2.4), obtaining a significant improvement of a result of Reichel. By the way, notice that « \( \omega_1 \)-compact » means « Lindelöf »
and consider the statement: «the product of two Lindelöf spaces is Lindelöf»; everybody knows that it is false in the general case, but the theorem 2.3 says that it is true in the category of \(\omega_1\)-additive spaces (that is, \(P\)-spaces).

As regards the results which do not have analogues in the \(\omega_\mu\)-additive spaces, we observe that the reason for this pathological behaviour lies chiefly in the existence of singular cardinals (see again the remark of the second section): and this obstacle clearly cannot be overcome.

Going on, we state and prove a theorem (3.4) whose main result is that what we could define as a kind of «\(\omega_\mu\)-paracompactness» is really nothing but paracompactness.

In the last paragraph we remark that \(\omega_\mu\)-compact topologies are minimal among the \(\omega_\mu\)-additive (Hausdorff) ones and formulate some results concerning with the minimality of \(\omega_\mu\)-additive topologies.

1. **Topological and uniform \(\omega_\mu\)-products.**

Every topological space is assumed to be Hausdorff.

Throughout the paper \(\omega_\mu\) will denote an initial ordinal, \(X\) a Hausdorff topological space, \(U\) a uniformity given as a family of entourages of the diagonal.

We say that \(X\) is \(\omega_\mu\)-additive or \(\omega_\mu\)-topological if the intersection of less than \(\omega_\mu\) many open sets is an open set; \(U\) is an \(\omega_\mu\)-uniformity and \((X, U)\) is an \(\omega_\mu\)-uniform space if the intersection of less than \(\omega_\mu\) many entourages is an entourage.

Clearly if \(\omega_\mu > \omega_\nu\), «\(\omega_\mu\)-topological (uniform)» implies «\(\omega_\nu\)-topological (uniform)»; and it is easy to show that if \(\omega_\mu\) is a singular ordinal, then «\(\omega_\mu\)-topological (uniform)» implies «\(\omega_{\mu+1}\)-topological (uniform)»: hence dealing with these spaces we may assume that \(\omega_\mu\) is a regular (initial) ordinal: we shall make this assumption from now on.

Denote by \(\mathcal{X}\), \(\mathcal{U}\) respectively the categories of topological and uniform spaces and by \(\mathcal{X}_\mu\) (\(\mathcal{U}_\mu\)) the full subcategory of \(\mathcal{X}\) (\(\mathcal{U}\)) consisting of \(\omega_\mu\)-topological (-uniform) spaces; clearly \(\mathcal{X}_0 = \mathcal{X}\), \(\mathcal{U}_0 = \mathcal{U}\). \(\mathcal{X}_\mu\) and \(\mathcal{U}_\mu\) are closed under arbitrary products, coproducts and subspaces: however, while coproducts and subspaces coincide in \(\mathcal{X}_\mu\) and \(\mathcal{X}\) (\(\mathcal{U}_\mu\) and \(\mathcal{U}\)), we have a different behaviour of products: that is quite natural since the usual product fails to be \(\omega_\mu\)-additive for \(\mu \neq 0\). The \(\omega_\mu\)-topological product of spaces \(X_s\), \(s \in S\), is denoted by \(\mu \prod_{s \in S} X_s\) and defined as the Cartesian product of the \(X_s\) equipped with the topology
generated by the subsets of the form: \{ (x_s)_s \in S: x_s \in V_s, V_s \text{ open in } X_s, V_s \neq X_s \text{ for less than } \omega_\mu \text{ many indexes} \}. Similarly if \( U_s \) is an \( \omega_\mu \)-uniformity on \( X_s \), define the \( \omega_\mu \)-uniform product as the Cartesian product equipped with the uniformity generated by the entourages \{ ((x_s), (y_s)) \in U_s \times U_s: U_s \neq X_s \times X_s \text{ for less than } \omega_\mu \text{ many indexes} \}.

It turns out easily that the topology (uniformity) introduced above is the weakest \( \omega_\mu \)-topology (uniformity) such that all the projections are continuous (uniformly continuous); moreover if \( X_s \) is an \( \omega_\mu \)-topological space and its topology is induced by the uniformity \( U_s \), then the topology of \( \mu \prod_{s \in S} X_s \) is induced by the uniformity of the \( \omega_\mu \)-uniform product. These considerations show that the concept of \( \omega_\mu \)-topological (uniform) product (introduced in [MS]) arises quite naturally in the category \( \mathcal{X}_\mu \). Furthermore we recall that, for \( \mu \neq 0 \), \( \omega_\mu \)-additive regular spaces are 0-dimensional [E]: therefore such a space of weight \( m \) can be embedded in \( \{0, 1\}^m \): observe then that the usual diagonal embedding is still an embedding equipping \( \{0, 1\}^m \) with the \( \omega_\mu \)-product topology. Finally in this section we consider \( \omega_\mu \)-metrizable spaces: it is well known that these spaces are exactly the (\( \omega_\mu \)-additive) ones which admit a uniformity with a totally ordered (for the reversed inclusion) basis whose cofinality is \( \omega_\mu \); Yasui in [Y] obtained a partial result on the \( \omega_\mu \)-metrizability of a product of \( \omega_\mu \)-metrizable spaces: we extend his result and show that the analogy with the usual metric case is complete.

1.1. THEOREM. Let \( \{ X_\alpha: \alpha < \omega_\gamma \} \) be a family of \( \omega_\mu \)-metrizable spaces each of them containing at least two points; then \( \mu \prod_{\alpha < \omega_\gamma} X_\alpha \) is \( \omega_\mu \)-metrizable if and only if \( \omega_\gamma < \omega_\mu \).

PROOF. Necessity: suppose \( \omega_\gamma > \omega_\mu \) and choose a point \( x \in \mu \prod_{\alpha < \omega_\gamma} X_\alpha \); since in the \( \omega_\mu \)-metrizable spaces the weight at any point is less than or equal to \( \omega_\mu \), it is enough to show that the weight at \( x \) is greater than \( \omega_\mu \); in fact if we take a family of power less than or equal to \( \omega_\mu \) of basic open neighbourhoods of \( x \), then the collection of the indexes \( \alpha \) for which the projection \( p_\alpha \) of some neighbourhood is a proper subset of \( X_\alpha \) has cardinality less than or equal to \( \omega_\mu \).

Sufficiency: clearly it is enough to prove the result for \( \omega_\mu = \omega_\gamma \). Every \( X_\alpha \) admits a compatible uniformity with a totally ordered basis of entourages, say \( \{ V_{\alpha, \lambda}: \lambda < \omega_\mu, \lambda_1 < \lambda_2 \implies V_{\alpha, \lambda_1} \supseteq V_{\alpha, \lambda_2} \} \). Consider the sets:

\[ W_\xi = \{ ((x_\alpha)_\alpha < \omega_\mu, (y_\alpha)_\alpha < \omega_\mu): (x_\alpha, y_\alpha) \in V_{\alpha, \xi}, \forall \alpha < \xi \}; \]
using the fact that $\omega_\mu$ is regular observe that the family $\{W_\xi: \xi < \omega_\mu\}$ is a basis for the $\omega_\mu$-uniform product which, as we have remarked above, is admissible for $\mu \prod_{\alpha < \omega_\mu} X_\alpha$. ■

2. $\omega_\mu$-compact spaces.

**Definition.** We say that a space is $\omega_\mu$-compact if it is $\omega_\mu$-additive and every open cover has a subcover of power less than $\omega_\mu$.

Trivially, taking the complements, one sees that an $\omega_\mu$-additive space is $\omega_\mu$-compact if and only if every family of closed sets such that each intersection of less than $\omega_\mu$ many elements is non-empty, has non-empty intersection.

These spaces have been studied by several authors (see e.g. [MS], [R], [S], [ST], [Y]) and were introduced by R. Sikorski who used for them the term « $\omega_\mu$-bicompact ».

In the following proposition we collect some folkloristic properties of $\omega_\mu$-compact spaces.

2.1. **Proposition:**

i) closed subspaces of $\omega_\mu$-compact spaces are $\omega_\mu$-compact;

ii) $\omega_\mu$-compact subspaces of $\omega_\mu$-additive spaces are closed;

iii) $\omega_\mu$-additive continuous images of $\omega_\mu$-compact spaces are $\omega_\mu$-compact;

iv) $\omega_\mu$-compact spaces are normal.

The proof is trivial, using $\omega_\mu$-additivity. ■

Clearly a discrete space of power less than $\omega_\mu$ is $\omega_\mu$-compact; more generally the coproduct of less than $\omega_\mu$ many $\omega_\mu$-compact spaces is $\omega_\mu$-compact too; moreover we get an example of a non discrete $\omega_\mu$-compact space as follows: denote by $X \cup \{\infty\}$ a space where $X$ is a discrete space of power greater than or equal to $\omega_\mu$ (always assumed regular) and $\infty$ is a point whose neighbourhoods are the sets which contain $\infty$ and have complements of power less than $\omega_\mu$.

**Remark.** Given an $\omega_\mu$-additive space $X$, each of the following conditions implies the next one:

i) every subset of $X$ of power greater than or equal to $\omega_\mu$ has a complete accumulation point;
ii) $X$ is $\omega_\mu$-compact;

iii) every subset of $X$ whose power is a regular cardinal number
greater than or equal to $\omega_\mu$ has a complete accumulation point.

One can show i) $\Rightarrow$ ii) using the argument of [K], Problem 5 I,
with the suitable modifications; the proof of ii) $\Rightarrow$ iii) is standard.

It is clear that here and in other cases the difference of the behaviour between compact and $\omega_\mu$-compact spaces depends on the existence of singular initial ordinal numbers. We can give an example for which ii) $\Rightarrow$ i) fails to be true.

For $n \in \mathbb{N}$, let $X_n$ be the discrete space of power $\omega_n$, $X_n^+ = X_n \cup \sigma \{p_n\}$ where the neighbourhoods of the point $p_n$ are the sets: $\{p_n\} \cup V$ where $V \subseteq X_n$, $X_n \setminus V$ is countable; $X = \bigoplus X_n^+$. $X$ is $\omega_1$-compact since it is the coproduct of a countable family of $\omega_1$-compact spaces, and clearly its power is $\omega_\omega$; nevertheless the set of all the points of $X$ has no complete accumulation point since, as $X_n^+$ is an open set in $X$ for every $n \in \mathbb{N}$, every point has a neighbourhood of power less than $\omega_\omega$.

We say that a filter $\mathcal{F}$ is an $\omega_\mu$-filter if the intersection of less than $\omega_\mu$ many members of $\mathcal{F}$ belongs to $\mathcal{F}$. The $\omega_\mu$-compact spaces can be characterized in terms of $\omega_\mu$-filters.

2.2. PROPOSITION. Let $X$ be an $\omega_\mu$-additive space. The following are equivalent:

i) $X$ is $\omega_\mu$-compact;

ii) every $\omega_\mu$-filter has a cluster point;

iii) every $\omega_\mu$-filter is contained in a convergent filter.

PROOF. i) $\Rightarrow$ ii): by definitions.

ii) $\Rightarrow$ i): using the regularity of $\omega_\mu$, one can show that a family of closed sets such that each intersection of less than $\omega_\mu$ many sets is non-empty, can be enlarged to an $\omega_\mu$-filter, hence the conclusion follows.

ii) $\Rightarrow$ iii): standard using $\omega_\mu$-additivity.

iii) $\Rightarrow$ ii): trivial. ■

We say that an $\omega_\mu$-filter on $X$ is an $\omega_\mu$-ultrafilter if there is no $\omega_\mu$-filter which contains it properly; we show that an $\omega_\mu$-ultrafilter $\mathcal{F}$ is an ultrafilter: otherwise there would exist $A \subseteq X$, $A \notin \mathcal{F}$ such that
A \cap F \neq \emptyset, \forall F \in \mathcal{F}, \text{ and } A \cup \mathcal{F} \text{ could be embedded in an } \omega_\mu\text{-filter}
containing \mathcal{F} \text{ properly. Moreover observe that if } X \text{ has non-measurable power and } \mu \neq 0, \text{ there is no free } \omega_\mu\text{-ultrafilter, hence } \omega_\mu\text{-ultrafilters cannot give any information about } \omega_\mu\text{-compactness.}

It is known that } \omega_\mu\text{-compactness is not productive in the category } \mathcal{L}_\mu: \text{ indeed in this category the product of } \omega_\mu\text{ many compact spaces need not be even } \omega_\mu\text{-compact (see [MS]). However we can provide a result in the positive direction:}

\textbf{2.3. Theorem. If } X \text{ and } Y \text{ are } \omega_\mu\text{-compact spaces, then } X \times Y \text{ is } \omega_\mu\text{-compact.}

\textbf{Proof.} \text{ Let } \mathcal{W} \text{ be a cover of } X \times Y, \mathcal{U} \text{ a basic cover which refines } \mathcal{W} \text{ and denote by } p_1, p_2 \text{ the canonical projections onto } X \text{ and } Y \text{ respectively. For every } x \in X \text{ take } \{p_2(V): V \in \mathcal{U}, p_1(V) \ni x\}: \text{ this is an open cover of } Y \text{ and let } \mathcal{U}_{s(x),x} = \{V_{s,x}: s \in S(x)\} \text{ be a subcover of it, indexed in a set } S(x) \text{ of power less than } \omega_\mu; \text{ now for every } V_{s,x} \text{ choose an open set } U_{s,z} \text{ containing } x \text{ and such that } U_{s,z} \times V_{s,x} \text{ belongs to } \mathcal{U}_x = \bigcap_{s \in S(x)} U_{s,x} \text{ is still an open set containing } x \text{ since } X \text{ is } \omega_\mu\text{-additive, hence } \{U_x: x \in X\} \text{ is an open cover of } X \text{ and has a subcover } \{U_x: x \in X'\} \text{ indexed in a subset } X' \text{ of } X \text{ of power less than } \omega_\mu. \text{ Finally it is easy to show that } \{U_x \times V_{s,z}: x \in X', s \in S(x)\} \text{ is an open cover which refines } \mathcal{U} \text{ and its power is less than } \omega_\mu \text{ since } \omega_\mu \text{ is regular.} \]

It is well known (see [I], VII.30) that a completely regular space is } \omega_\mu\text{-additive if and only if it admits an } \omega_\mu\text{-uniformity (however notice that our definitions are a little more general than the ones used by Isbell). With the following theorem we provide a result which generalizes in the most natural direction a fundamental theorem in the theory of compact spaces; by the way we observe that it contains as a particular case the theorem 5.2 of [R].}

\textbf{2.4. Theorem. An } \omega_\mu\text{-compact space } X \text{ admits exactly one } \omega_\mu\text{-uniformity.}

\textbf{Proof.} \text{ By Proposition 2.1 iv), } X \text{ is completely regular, hence it admits an } \omega_\mu\text{-uniformity. Now let } U \text{ be a compatible } \omega_\mu\text{-uniformity, } V \text{ an open subset of } X \times X \text{ containing the diagonal } \Lambda; \text{ since } \Lambda = \cap \{U: U \in U\} \text{ and } X \times X \text{ is } \omega_\mu\text{-compact by the theorem 2.3, there exists a family } \{U_\alpha: \alpha < \omega_\gamma\}, \omega_\gamma < \omega_\mu, \text{ such that } \cap U_\alpha \subseteq V: \text{ therefore } V \text{ belongs to } U \text{ and } U \text{ is the fine uniformity.} \alpha < \omega_\gamma
3. Paracompactness.

Definition. A family $\mathcal{A}$ of subsets of a space $X$ is said to be locally $\omega_\mu$ if for every $x \in X$ there exists a neighbourhood $V$ of $x$ such that the family $\{ A \in \mathcal{A} : A \cap V \neq \emptyset \}$ has power less than $\omega_\mu$ (hence "locally $\omega_\mu$" means "locally finite").

3.1. Lemma. Let $X$ be an $\omega_\mu$-additive space, $\mathcal{A}$ a locally $\omega_\mu$ family of subsets of $X$; then, denoting by $\text{cl} \mathcal{A}$ the closure of the set $\mathcal{A}$, we have:

$$\bigcup \{ \text{cl} A : A \in \mathcal{A} \} = \text{cl} \bigcup \{ A : A \in \mathcal{A} \} .$$

The proof is plain using $\omega_\mu$-additivity.

3.2. Lemma. Let $X$ be an $\omega_\mu$-additive regular space, $\mu \neq 0$, such that every open cover is refined by a locally $\omega_\mu$ open cover; then $X$ is normal.

Proof. Let $A, B$ be disjoint closed subsets of $X$; for every $x \in B$ there exists a clopen set $V_x$ which contains $x$ and does not meet $A$. Let $\mathcal{U}$ be a locally $\omega_\mu$ open refinement of the cover $\{ X \setminus B \} \cup \{ V_x : x \in B \}$. Put $\mathcal{U}^* = \{ U \in \mathcal{U} : U \subseteq V_x \text{ for some } x \in B \}$; we have $A \cap \text{cl} U = \emptyset$ for every $U \in \mathcal{U}^*$ and $B \subseteq \bigcup \{ U \in \mathcal{U}^* \}$; finally $\text{cl} \bigcup \{ U : U \in \mathcal{U}^* \} = \bigcup \{ \text{cl} U : U \in \mathcal{U}^* \}$ does not meet $A$.

3.3. Lemma. Let $X$ be an $\omega_\mu$-additive normal space, $\mu \neq 0$. For every pair of disjoint closed sets $A, B$ there exists a clopen which contains $A$ and does not meet $B$.

Proof. Inductively choose a sequence of open sets $A_n$ such that:

$$A \subseteq A_1 \subseteq \text{cl} A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \text{cl} A_n \subseteq A_{n+1} \subseteq \ldots \subseteq X \setminus B$$

Then $\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \text{cl} A_n$ which is closed by $\omega_\mu$-additivity.

3.4. Theorem. Let $X$ be an $\omega_\mu$-additive regular space, $\mu \neq 0$. The following are equivalent:

i) every open cover is refined by a locally $\omega_\mu$ open cover;

ii) every open cover is refined by a locally $\omega_\mu$ arbitrary cover;
iii) every open cover is refined by a locally \( \omega_\mu \) closed cover;

iv) every open cover is refined by a locally \( \omega_\mu \) clopen cover;

v) \( X \) is ultraparacompact;

vi) \( X \) is paracompact.

**Proof.** Clearly part of the statement and of the proof of this theorem follows the theorem 5.1.4 of [E].

i) \( \Rightarrow \) ii): trivial.

ii) \( \Rightarrow \) iii): let \( \mathcal{U} \) be an open cover, \( \mathcal{V} \) an open cover such that 
\( \{ \text{cl } V : V \in \mathcal{V} \} \) is a refinement of \( \mathcal{U} \), \( \mathcal{W} \) a locally \( \omega_\mu \) cover which refines \( \mathcal{V} \): then the cover \( \{ \text{cl } W : W \in \mathcal{W} \} \) is a locally \( \omega_\mu \) refinement of \( \mathcal{U} \).

iii) \( \Rightarrow \) iv): let \( \mathcal{U} \) be an open cover, \( \mathcal{A} \) a locally \( \omega_\mu \) closed cover which refines \( \mathcal{U} \); for every \( x \in X \) let \( V(x) \) be a neighbourhood of \( x \) which meets less than \( \omega_\mu \) many elements of \( \mathcal{A} \), and take a locally \( \omega_\mu \) closed cover \( \mathcal{V} \) which refines \( \{ V(x) : x \in X \} \). For every \( A \in \mathcal{A} \) choose \( U_A \in \mathcal{U} \) such that \( U_A \supseteq A \) and put: \( W_A = U_A \setminus \bigcup \{ V : V \in \mathcal{V}, V \cap A = \emptyset \} \). Clearly \( W_A \) is open by lemma 3.1 hence \( \{ W_A : A \in \mathcal{A} \} \) is an open cover (since \( W_A \supseteq A \)); we show that this cover is locally \( \omega_\mu \): in fact if \( V \in \mathcal{V} \) does not meet \( A \), then \( V \) does not meet \( W_A \); moreover every \( V \in \mathcal{V} \) is contained in a certain \( V(x) \) and so it meets less than \( \omega_\mu \) many elements (of \( \mathcal{A} \) hence) of \( \{ W_A : A \in \mathcal{A} \} \). Now for every \( x \in X \) choose a neighbourhood \( I(x) \) of \( x \) such that \( I(x) \) meets less than \( \omega_\mu \) many elements \( V \in \mathcal{V} \); by the regularity of \( \omega_\mu \) and the previous observations we get: \( I(x) \cap W_A \neq \emptyset \) holds for less than \( \omega_\mu \) many \( W_A \). Finally by lemmas 3.2 and 3.3, for every \( A \in \mathcal{A} \) take a clopen set \( W_A' \) such that \( A \subseteq W_A' \subseteq W_A \); clearly \( \{ W_A' : A \in \mathcal{A} \} \) is a locally \( \omega_\mu \) clopen cover which refines \( \mathcal{U} \).

iv) \( \Rightarrow \) v): let \( \mathcal{U} \) be an open cover, \( \mathcal{V} \) a locally \( \omega_\mu \) clopen cover which refines \( \mathcal{U} \). For every \( x \in X \) put \( A_x = \cap \{ V : V \in \mathcal{U}, V \ni x \} \), \( B_x = \cup \{ V : V \in \mathcal{U}, V \not\ni x \} \), \( W_x = A_x \setminus B_x \); since \( x \in V \) holds for less than \( \omega_\mu \) many \( V \in \mathcal{U} \), \( A_x \) is open; furthermore \( B_x \) is closed by lemma 3.1 so that \( W_x \) is open (trivially it is closed and contains \( x \)). Now observe that if a point \( y \) belongs to \( W_x \), then \( V \in \mathcal{U} \) contains \( x \) if and only if it contains \( y \); hence \( W_x = W_y \) and conclude that the sets \( W_x \) form a partition of clopen sets which refines \( \mathcal{U} \).

v) \( \Rightarrow \) vi) \( \Rightarrow \) i): trivial. □
A standard argument (see again theorem 5.1.4 [E]) shows that the six conditions of theorem 3.4 are equivalent to:

vii) every open cover of \( X \) is refined by a cover which is union of \( \omega_\mu \) many locally \( \omega_\mu \) families of open sets.

Using this condition we can prove the following corollary which contains the third statement of VII.31 [I] and as a particular case that every \( \omega_\mu \)-compact space is ultraparacompact, for \( \mu \neq 0 \).

3.5. Corollary. For \( \mu \neq 0 \), every regular \( \omega_\mu \)-additive space \( X \) such that every open cover has a subcover of power at most \( \omega_\mu \) is ultraparacompact.  ■

Moreover using an \( \omega_\mu \)-metrization theorem due to Wang Shu Tang ([W], theorem 6) we can obtain as a corollary that, for \( \mu \neq 0 \), every \( \omega_\mu \)-metrizable space is ultraparacompact ([H]).

4. Minimal \( \omega_\mu \)-additive topologies.

By means of proposition 2.1 we get the following theorem:

4.1. Theorem. Let \( f : X \to Y \) be a continuous injective function, \( X \) \( \omega_\mu \)-compact, \( Y \) \( \omega_\mu \)-additive; then \( f \) is a homeomorphism onto its image. Hence the \( \omega_\mu \)-compact topologies are minimal among the \( \omega_\mu \)-additive ones.  ■

In view of this theorem, we shall conclude with some considerations about the minimal objects in the category \( \mathcal{X}_\mu \); we shall omit the details since the proofs in this case and in the usual topological spaces proceed at the same rate, naturally without losing sight of \( \omega_\mu \)-additivity. For references and definitions related to this matter, see [E], [PT].

4.2. Lemma. Given an \( \omega_\mu \)-additive space, the topology consisting of all open domains ([E]) is still \( \omega_\mu \)-additive.  ■

Definitions. We say that an \( \omega_\mu \)-additive space \( X \) is \( \omega_\mu \)-absolutely closed if it is a closed subset of every \( \omega_\mu \)-additive (Hausdorff) space containing it. A proximate cover is a family of subsets of \( X \) whose union is dense in \( X \).
4.3. THEOREM. An $\omega_\mu$-additive space $X$ is $\omega_\mu$-absolutely closed if and only if every open cover has a proximate subcover of power less than $\omega_\mu$. ■

4.4. PROPOSITION:
   i) closed domains of $\omega_\mu$-absolutely closed spaces are $\omega_\mu$-absolutely closed;
   ii) continuous $\omega_\mu$-additive images of $\omega_\mu$-absolutely closed spaces are $\omega_\mu$-absolutely closed.

PROOF. Use theorem 4.3. ■

4.5. THEOREM:
   i) let $f: X \rightarrow Y$ be a continuous injective function, $X$ $\omega_\mu$-absolutely closed and semiregular, $Y$ $\omega_\mu$-additive; then $f$ is a homeomorphism onto its image.
   ii) a topology on $X$ is minimal among the $\omega_\mu$-additive ones if and only if it is $\omega_\mu$-absolutely closed and semiregular.

PROOF. Use proposition 4.4 to prove i); use i) and lemma 4.2 to prove ii). ■

Finally, for $\omega_\mu$-additive regular spaces, we can state the following theorem which summarizes the previous results:

4.6. THEOREM. Let $X$ be an $\omega_\mu$-additive regular space. The following are equivalent:
   i) $X$ is $\omega_\mu$-compact;
   ii) the topology of $X$ is minimal with respect to the $\omega_\mu$-additive ones;
   iii) $X$ is $\omega_\mu$-absolutely closed;
   iv) every open cover of $X$ has a proximate subcover of power less than $\omega_\mu$. ■

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\[ \omega_\mu \text{-Additive topological spaces} \]


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