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An algorithm for the one-phase Stefan problem

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An Algorithm for the One-Phase Stefan Problem

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1. Introduction.

Consider the following one-phase one-dimensional Stefan problem.

\[
\begin{align*}
Lu &= u_{xx} - u_t = 0 \quad \text{in} \quad D_T = \{0 < x < s(t)\} \times (0, T] \\
 u(x, 0) &= h(x), \quad 0 < x < s(0) = b, \quad (b > 0) \\
 u_s(0, t) &= g(u(0, t), t), \quad 0 < t < T \\
 u(s(t), t) &= 0, \quad 0 < t < T \\
 u_s(s(t), t) &= -\delta(t), \quad 0 < t < T
\end{align*}
\]

(\(S\))

where \(x \rightarrow h(x)\), \((\xi, t) \rightarrow g(\xi, t)\) are given functions on \((0, b]\) and \(R \times (0, T]\) respectively.

Under suitable assumptions on \(h(\cdot)\) and \(g\), (S) admits a unique classical solution. For such results we refer to the survey article [13] and other papers given in the extensive bibliography.

The aim of this paper is to propose an algorithm to construct the solution, which consists in solving the heat equation in progressively increasing rectangles, whose size is controlled by the Stefan condition \(u_s(s(t), t) = -\delta(t)\).

Such an algorithm arises as a natural modification of Hübner's

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method \([15, 10, 1]\) and can be described in a simple fashion as follows.

First the interval \([0, T]\) is divided in \(n\) intervals of length \(\theta = T/n\), then for \(t \in [0, \theta]\) we set \(s_\theta(t) = b\) and solve the problem

\[
\begin{align*}
\mathcal{F}_1: & \quad u^1_{xx} - u^1_t = 0 \quad \text{in} \quad R_1 \equiv \{0 < x < b\} \times (0, \theta) \\
& \quad u^1_x(0, t) = g(u^1(0, t), t), \quad 0 < t \leq \theta \\
& \quad u^1(x, 0) = h(x), \quad 0 < x \leq b \\
& \quad u^1(b, t) = 0, \quad 0 < t < \theta.
\end{align*}
\]

We compute the number \(u^1_x(s_\theta(b) = b, \theta)\) and determine the rectangle

\[R_2 \equiv \{0 < x < x_2 = b - u^1_x(b, \theta) \theta \} \times (\theta, 2\theta),\]

setting \(s_\theta(t) = x_2\) for \(t \in (\theta, 2\theta]\). In \(R_2\) we solve a problem similar to \((\mathcal{F}_1)\), and proceed in this fashion.

The convergence of schemes where at each time step the free boundary is approximated by a vertical segment was conjectured by Datzeff \([5, 6]\).

A proof of convergence has been given by Fasano-Primicerio-Fontanella \([11]\). Their scheme however is somewhat more complicated than the one we propose here, both in the construction of the sequence of rectangles and in the boundary conditions on \(x_j, (j-1)\theta < t < j\theta\) which are not homogeneous, being given as a relationship linking the distance \(t - (j-1)\theta\) with the values \(u^1_{x-1}(x_{j-1}, (j-1)\theta)\).

Thus the scheme we have described has a two-fold simplicity: the rectangular geometry and the homogeneity of the boundary data on the approximating free boundary.

We treat the problem for boundary data on \(x = 0\) of variational type since such a condition is the «natural» one, as pointed out and discussed in \([7]\). The method could handle the Dirichlet boundary data as well.

We give an estimate of the speed of convergence which turns out to be of the order of \(\sqrt{\theta}\).

As a related work we mention the methods of \([14]\) based on enthalpy considerations, which yield a rate of convergence of the order of \([\theta \ln \theta^{-1}]^4\).

The methods of proof are simple in that we exploit both the rectangular geometry and the homogeneity of the data to represent the approximating solutions by means of elementary heat potentials.
Section 2 contains the precise description of the algorithm, assumptions and statement of results. In Section 3 and 4 we produce basic estimates and prove the convergence of the approximating solution, to the solution of (S3').

The error estimate is given in Section 5. We conclude the paper by discussing some variants of the scheme.

2. Assumptions and statement of results.

Throughout the paper we will make the following assumptions on the data.

[A1] \( x \mapsto h(x) \) is a positive Lipschitz continuous function on \([0, b]\), with Lipschitz constant \( H \), \( h(0)^{+} > 0 \) and \( h(b) = 0 \).

[A2] \( (\xi, t) \mapsto g(\xi, t) \) is non-positive on \( \mathbb{R} \times (0, T] \), continuous with respect to \( t \in (0, T] \), Lipschitz continuous in \( \xi \), uniformly in \( t \), with Lipschitz constant \( G_1 \) and \( g(h(0), 0) = h'(0) \). Moreover there exists a non-negative constant \( G_2 \) such that

\[
|g(\xi, t)| < G_1 |\xi| + G_2, \quad (\xi, t) \in \mathbb{R} \times (0, T].
\]

For \( n = 1, 2, \ldots \), set \( \theta = T/n \) and consider the sequence of problems \( \mathcal{P}_j \), \( j = 1, 2, \ldots, n \), defined by

\[
\begin{align*}
&Lu^j = 0 \quad \text{in} \quad R_j = \{0 < x < x_j\} \times \{(j - 1) \theta < t < j \theta\} \\
&u^j(x, (j - 1) \theta) = \begin{cases} 
0, & 0 < x < x_{j-1} \\
u^{j-1}(x, (j - 1) \theta), & x_{j-1} < x < x_j 
\end{cases}, \\
u_j^0(0, t) = g(u^j(0, t), t), & (j - 1) \theta < t < j \theta \\
u^j(x_j, t) = 0, & (j - 1) \theta < t < j \theta
\end{align*}
\]

where the sequence \( \{x_j\}_{j=0}^n \) is recursively defined by

\[
x_0 = x_1 = b; \quad x_j = x_{j-1} - u_{x}^{j-1}(x_{j-1}, (j - 1) \theta) \cdot \theta, \quad j = 2, 3, \ldots, n.
\]

and

\[
u^0(x, 0) = u^1(x, 0) \equiv h(x) \quad \text{in} \ (0, b].
\]
By virtue of \([A_1]-[A_2]\) each \((f_j)\) admits recursively a unique classical solution \(u^j\) whose derivative \(u^j_\ast\) exists up to the lateral boundaries of \(R_j\). Consequently the sequence \(\{x_j\}\) is well defined.

Setting

\[ s_0(0) = b ; \quad s_0(t) = x_j \quad \text{for} \quad (j - 1) \theta < t < j \theta, \ j = 1, 2, \ldots, n, \]

we obtain a left-continuous, piecewise constant function defined in \([0, T]\). By elementary considerations and the maximum principle \([7, 8]\) \(u^j(x, t) > 0\) and the numbers \(u^j_\ast(x_j, j \theta)\) are non-positive so that \(x_{j+1} > x_j\) and \(s_0(\cdot)\) is non-decreasing.

On the domain

\[ \mathbb{D}_s = \bigcup_{j=1}^n R_j, \]

we define the function \((x, t) \to u_\theta(x, t), (x, t) \in \mathbb{D}_s\), by setting

\[ u_\theta(x, t) = u^j(x, t), \quad (x, t) \in R_j, \quad j = 1, 2, \ldots, n. \]

We will think of \(u(\cdot, \cdot)\) the solution of \((83')\) and \(u_\theta\) as defined in the whole half strip \(S = (0, \infty) \times (0, T]\), by setting them to be equal to zero outside \(D_\tau\) and \(\mathbb{D}_s\) respectively. We will use this device for the various functions appearing in what follows without specific mention.

For bounded functions \((x, t) \to w(x, t), t \to f(t)\) defined in \(S\) and \((0, t]\) respectively we set

\[ \|w\|_{\infty, S} = \sup_{(x, t) \in S} |w(x, t)| \]

\[ \|f\|_t = \sup_{0 < \tau \leq t} |f(\tau)|. \]

We can now state our main result.

**Theorem.** As \(\theta \to 0\), \(u_\theta(x, t) \to u(x, t)\) uniformly in \(S\) and \(s_\theta(t) \to s(t)\) uniformly in \([0, T]\). Moreover there exists a constant \(C\) depending upon \(H, b, G_1, G_2, T\) such that

\[ \|u_\theta - u\|_{\infty, S} < C \sqrt{\theta} \]

\[ \|s_\theta - s\|_2 < C \sqrt{\theta}. \]
REMARK. (i) In view of the stability of (85') (see [2, 3]) the Lipschitz condition in \([A_1]\) can be replaced by

\[
[A_1]' \quad x \to h(x) \text{ is essentially bounded in } [0, b].
\]

(ii) The signum condition on \(g\) in \([A_3]\) can be dropped, provided we assume \(g(0, t) = 0, \ t \in (0, T]\). In this case we set \(G_2 = 0\) in the growth condition for \(g\).

3. Some basic estimates.

Let

\[
\Gamma(x, t; \xi, \tau) = \frac{1}{2\sqrt{\pi(t - \tau)}} \exp \left[ -\frac{(x - \xi)^2}{4(t - \tau)} \right]
\]

be the fundamental solution of the heat equation and let \(G(x, t; \xi, \tau), N(x, t; \xi, \tau)\) be the Green and Neumann's functions respectively, defined by

\[
G(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) - \Gamma(-x, t; \xi, \tau),
\]

\[
N(x, t; \xi, \tau) = \Gamma(x, t; \xi, \tau) + \Gamma(-x, t; \xi, \tau).
\]

In the \(j\)-th rectangle \(R_j\) the solution \(u^j\) of \((3_j)\) can be implicitly represented as

\[
(3.1) \quad u^j(x, t) = \int_0^{x_j} N(x, t; \xi, (j - 1)\theta) u^j(\xi, (j - 1)\theta) d\xi -
\]

\[
- \int_{(j-1)\theta}^t N(x, t; 0, \tau) g(u^j(0, \tau), \tau) d\tau + \int_{(j-1)\theta}^t N(x, t; x_j, \tau) u^j_x(x_j, \tau) d\tau
\]

for \((x, t) \in R_j\). Taking the derivative with respect to \(x\) in (3.1) and letting \(x \to x_j\) we obtain

\[
(3.2) \quad \frac{1}{2} u^j_x(x_j, t) = \int_0^{x_j} G(x_j, t; \xi, (j - 1)\theta) u^j_x(\xi, (j - 1)\theta) d\xi -
\]
The calculations leading to (3.1)-(3.2) are routine and we refer to \cite{4, 9, 12} for details.

Let us fix $1 \leq j \leq n$ and $(x, t) \in R_j$ and integrate the Green identity

$$
\frac{\partial}{\partial \xi_k} (Nu_\xi - u N \xi) - \frac{\partial}{\partial \tau} (Nu) = 0
$$

over $R_k$, $1 \leq k < j$. Since in $R_k$ we are away from the singularity we obtain

$$
(3.3) \int_0^{x_k} \int_{\xi} N(x, t; \xi, k\theta) \, d\xi
$$

$$
- \int_0^{x_k} N(x, t; \xi, (k-1)\theta) \, d\xi =
$$

$$
= - \int_{(k-1)\theta}^{k\theta} \int N(x, t; 0, \tau) \, d\tau + \int_{(k-1)\theta}^{k\theta} \int N(x, t; \xi, \tau) \, d\tau
$$

for $k = 1, 2, \ldots, (j-1)$.

By virtue of our definition of $(\mathcal{I}_j)$, the second integral on the left hand side of (3.3) can be rewritten as

$$
- \int_0^{x_{k-1}} \int N(x, t; \xi, (k-1)\theta) \, d\xi.
$$

Consequently adding the identities (3.3) for $k = 1, 2, \ldots, (j-1)$
with (3.1) and recalling the definition of \( u_\theta(x, t) \) we obtain

\[
(3.4) \quad u_\theta(x, t) = \int_0^b N(x, t; \xi, 0) h(\xi) \, d\xi - \int_0^t N(x, t; 0, \tau) g(u_\theta(0, \tau), \tau) \, d\tau + \sum_{k=1}^{j-1} \int_{(k-1)\theta}^{k\theta} N(x, t; x_k, \tau) u_\theta^k(x_k, \tau) \, d\tau + \int_{(j-1)\theta}^t N(x, t; x_j, \tau) u_\theta^j(x_j, \tau) \, d\tau.
\]

**Lemma 3.1.** For each \( \theta \) the following estimates are valid

\[
\|
\|
\]

(a) \( \|u_\theta(0, \cdot)\|_T \leq \left[ Hb + \frac{2G_2 \sqrt{T}}{\sqrt{\pi}} \right] \exp \left[ 2G_1 \sqrt{T}/\sqrt{\pi} \right] \equiv C_0, \]

(b) \( \|u_\theta\|_{\infty, S} \leq C_0. \)

**Proof.** By virtue of the maximum principle \( u_\theta \geq 0 \) in \( D_s \) and \( u_\theta^j(x_j, t) < 0, \ t \in ((j-1)\theta, j\theta] \), therefore dropping the non-positive terms on the right hand side of (3.4) and letting \( x \to 0 \) we obtain

\[
0 < u_\theta(0, t) \leq \int_0^b 2\Gamma(0, t; \xi, 0) h(\xi) \, d\xi - \frac{1}{\sqrt{\pi}} \int_0^t g(u_\theta(0, \tau), \tau) \, d\tau \leq \]

\[
\leq \frac{Hb}{\sqrt{\pi}} \int_0^b \frac{1}{\sqrt{t}} \exp \left[ -\xi^2/4t \right] d\xi + \frac{G_1}{\sqrt{\pi}} \int_0^t u_\theta(0, \tau) \, d\tau + \frac{2G_2 \sqrt{T}}{\sqrt{\pi}} \sqrt{T}.
\]

Statement (a) is now a consequence of standard calculations and Gronwall's inequality.

Statement (b) follows from the maximum principle applied recursively to \((3.1)\).

**Lemma 3.2.** For each \( j = 1, 2, \ldots, n \)

(a) \( |u_\theta^j(x_j, t)| \leq \tilde{H} \exp [\tilde{T}] + 8G\tilde{T} \exp [\tilde{T}] \equiv C_1, \ \tilde{T} = T/\sqrt{\pi}b^2 \)

(b) \( \left| \frac{\partial}{\partial x} u_\theta(x, t) \right| \leq C_1, \ (x, t) \in D_s \),

where \( G = G_1 C_0 + G_2 \), and \( \tilde{H} = \max \{ H, G \} \).
Proof. We employ an induction argument by making use of formulae (3.2). First we prove that if for some \( j = 1, 2, \ldots, (n - 1) \) we have

\[
|u_z^j(x, j\theta)| < P, \quad 0 < x < x_j
\]

for some positive constant \( P \), then

\[
|u_z^{j+1}(x_{j+1}, t)| < P \exp \left[ \frac{\theta}{\sqrt{\pi b^2}} \right] + 8G \frac{\theta}{\sqrt{\pi b^2}} \exp \left[ \frac{\theta}{\sqrt{\pi b^2}} \right],
\]

\( t \in (j\theta, (j + 1)\theta) \]

where

\[
\max_{(0,T)} |g(u_0(0, t), t)| < G_1C_0 + G_2 = G.
\]

Consider (3.2) written for the integer \( j + 1 \)

\[
\frac{1}{2} u_z^{j+1}(x_{j+1}, t) = I_1 + I_2 + I_3
\]

and estimate the integrals \( I_i, \ i = 1, 2, 3 \) separately as follows.

\[
|I_1| = \left| \int_0^{x_{j+1}} G(x_{j+1}, t; \xi, j\theta) u_z^{j+1}(\xi, j\theta) \, d\xi \right| = \text{by definition of } (\mathcal{G}_j)
\]

\[
= \left| \int_0^{x_j} G(x_{j+1}, t; \xi, j\theta) u_z^j(\xi, j\theta) \, d\xi \right| < P \int_0^{x_j} |G(x_{j+1}, t; \xi, j\theta)| \, d\xi
\]

\[
< \frac{P}{\sqrt{\pi}} \int_0^{\infty} \exp \left[ -\eta^2 \right] d\eta = \frac{P}{2}.
\]

To estimate \( I_2, I_3 \) we recall the following elementary estimates on \( N_z \)

\[
|N_z(x_{j+1}, t; 0, \tau)| < \frac{4}{\sqrt{\pi b^2}}, \quad (x_{j+1} \geq b > 0)
\]

\[
|N_z(x_{j+1}, t; x_{j+1}, \tau)| < \frac{1}{2\sqrt{\pi b^2}}.
\]
Therefore

$$|I_3| < \frac{4G}{\sqrt{\pi b^2}} (t - j\theta) < 4G \frac{\theta}{\sqrt{\pi b^2}},$$

$$|I_3| < \frac{1}{2\sqrt{\pi b^2}} \int_{j\theta}^{t} |u_{x_{j+1}}(x, \tau)| d\tau.$$

Putting together these estimates as parts of (3.6) we have

$$|u_{x_{j+1}}(x, t)| < P + 8G \frac{\theta}{\sqrt{\pi b^2}} + \frac{1}{\sqrt{\pi b^2}} \int_{j\theta}^{t} |u_{x_{j+1}}(x, \tau)| d\tau$$

for all \(t \in (j\theta, (j + 1)\theta)\).

Consequently by Gronwall's inequality (3.5) follows at once.

Consider now the problem (3), \(j = 1\). Since \(x \rightarrow h(x)\) is Lipschitz continuous in \([0, b]\), \(h'(x)\) exists for a.e. \(x \in [0, b]\) and

$$\text{ess sup} |h'(x)| \leq H < \tilde{H} = \max \{H, G\}.$$

Therefore by the previous argument

$$|u_1(x, t)| < \tilde{H} e^x + 8Gxe^x, \quad t \in (0, \theta]$$

where for simplicity of notation we have set \(x = \theta/\sqrt{\pi b^2}\). The function \((x, t) \rightarrow u_1(x, t) \equiv v(x, t)\), will satisfy the Dirichlet problem

$$Lv = 0 \quad \text{in } R_1,$$

$$v(0, t) = g(u_0(0, t), t); \quad |g(u_0(0, t), t)| < G, \quad t \in (0, \theta]$$

$$|v(x, t)| < \tilde{H} e^x + 8Gxe^x, \quad t \in (0, \theta]$$

$$v(x, 0) = h'(x), \quad |h'(x)| < H \quad \text{a.e. } x \in (0, b].$$

Consequently by the maximum principle

$$|u_1(x, \theta)| < \tilde{H} e^x + 8Gxe^x$$
and by (3.5)
\[ |u_2^i(x, t)| < \tilde{H} e^{2\alpha} + 8G e^{2\alpha} = \tilde{H} e^{2\alpha} + 8G(2\alpha) e^{2\alpha}, \quad t \in (\theta, 2\theta]. \]

Proceeding in this fashion we obtain
\[ |u_2^i(x, t)| < \tilde{H} e^{4\alpha} + 8G(j\alpha) e^{4\alpha}. \]

Now \( j\alpha = (1/\sqrt{\pi \beta}) j\theta < \bar{T} \), and therefore the lemma is proved. Next we introduce the function \( t \rightarrow \tilde{s}_\theta(t) \) defined by
\[ \tilde{s}_\theta(t) = x_j - u_2^i(x_j, j\theta)(t - (j - 1)\theta), \quad t \in ((j - 1)\theta, j\theta]. \]

For \( t = j\theta \), \( \tilde{s}_\theta(t) = x_{j+1} \), \( \tilde{s}_\theta(0) = b \), so that the graph of \( \tilde{s}_\theta(\cdot) \) is obtained by connecting the points
\[ (b, 0), \ (x_2, \theta), \ldots, \ (x_n, (n - 1)\theta) \quad \text{for} \ t \in (0, (n - 1)\theta], \]
and by the graph of (3.7) for \( t \in [(n - 1)\theta, \ T] \). The points \( (x_j, (j - 1)\theta) \), \( j = 1, 2, \ldots, n \), are the lower vertices at the right side of the \( R_j \)’s.

By Lemma 3.2
\[ b < \tilde{s}_\theta(t) < b + C_1 \bar{T}, \quad t \in [0, \bar{T}] \]

therefore the sequence \( \{\tilde{s}_\theta(\cdot)\} \) is equibounded and equilipschitz, so that by Ascoli-Arzelà theorem a subsequence, relabeled with \( \theta \), converges uniformly to some nondecreasing, Lipschitz continuous curve \( t \rightarrow s^*(t) \), with Lipschitz constant bounded by \( C_1 \). Since \( \|s_\theta - \tilde{s}_\theta\|_\infty < C_1 \theta \), also \( s_\theta(t) \) converges uniformly to \( s^*(t) \).

Let \( D_{s^*} \) be the domain defined by
\[ D_{s^*} = \{0 < x < s^*(t)\} \times (0, \bar{T}], \]
and let \( u^* \) be the unique solution of

\[
\begin{align*}
Lu^* &= 0 \quad \text{in} \ D_{s^*}, \\
u_2^i(0, t) &= g(u^*(0, t), t), \quad t \in (0, \bar{T}], \\
u^*(x, 0) &= h(x), \quad x \in (0, b], \\
u^*(s(t), t) &= 0, \quad t \in (0, \bar{T}].
\end{align*}
\]
We will show that \( u_\theta(x, t) \to u^*(x, t) \) uniformly in \( S \) and that the pair \((u^*, s^*)\) so obtained is actually the unique solution of \((S\Phi)\) in the introduction.

We remark that as a consequence, in view of the uniqueness for \((S\Phi)\), the selection of subsequences is superfluous.

The following lemma will be needed.

**Lemma 3.3.** Let \((x, t) \to v(x, t)\) be the unique solution of

\[
L v = 0 \quad \text{in} \ D_*,
\]

\[
v_x(0, t) = g(u_\theta(0, t), t), \quad t \in (0, T],
\]

\[
v(x, 0) = h(x), \quad x \in (0, b],
\]

\[
v(s^*(t), t) = 0, \quad t \in (0, T].
\]

Then

\[
0 < v(x, t) < C_1(s^*(t) - x), \quad (x, t) \in D_*.
\]

**Proof.** The lemma is proved by standard barrier techniques and the maximum principle [10].

### 4. Convergence of the scheme.

**Lemma 4.1.** \( u_\theta(x, t) \to u^*(x, t) \) uniformly in \( S \), as \( \theta \to 0 \).

**Proof.** By the triangle inequality

\[
\|u_\theta - u^*\|_{\infty, S} \leq \|w_1\|_{\infty, S} + \|w_2\|_{\infty, S}
\]

where \( w_1(x, t) = u_\theta(x, t) - v(x, t) \) and \( w_2(x, t) = v(x, t) - u^*(x, t) \), and \( v \) is defined in Lemma 3.3.

Set

\[
\alpha(t) = \min \{s_\theta(t), s^*(t)\}; \quad \beta(t) = \max \{s_\theta(t), s^*(t)\},
\]

\[
\delta(t) = \beta(t) - \alpha(t), \quad t \in [0, T].
\]

We already know that \( \delta(t) \to 0 \) uniformly in \([0, T]\), as \( \theta \to 0 \).
Consider the rectangle $R_j$, $1 \leq j < n$. We claim that if

$$|w_1(x, (j-1) \theta)| \leq C_1\|\delta\|_{(j-1)\theta}, \quad x \in (0, \infty),$$

then

$$|w_1(x, t)| \leq C_1\|\delta\|_t, \quad (x, t) \in (0, \infty) \times ((j-1) \theta, j\theta].$$

If for $t \in [(j-1) \theta, j\theta]$, $x_j < s^*(t)$ then $w_1$ solves the problem

$$Lw_1 = 0 \quad \text{in } \{0 < x < x_j\} \times ((j-1) \theta, j\theta],$$

$$w_{1x}(0, t) = 0, \quad t \in ((j-1) \theta, j\theta],$$

$$|w_1(x, (j-1) \theta)| \leq C_1\|\delta\|_{(j-1)\theta}, \quad x \in (0, x_j],$$

$$w_1(x_j, t) = -v(x_j, t), \quad t \in ((j-1) \theta, j\theta].$$

Hence $|w_1(x, t)| \leq \max \{C_1\|\delta\|_{(j-1)\theta}, \max_{(j-1)\theta < t \leq j\theta} v(x_j, t)\}$, $(x, t) \in R_j$. By Lemma 3.3 we obtain

$$|w_1(x, t)| \leq C_1\|\delta\|_t, \quad (x, t) \in [0, \infty) \times [(j-1) \theta, j\theta].$$

If for $t \in [(j-1) \theta, j\theta]$, $x_j > s^*(t)$, then $w_1$ solves the problem

$$Lw_1 = 0 \quad \text{in } \{0 < x < s^*(t)\} \times ((j-1) \theta, j\theta],$$

$$w_{1x}(0, t) = 0, \quad t \in ((j-1) \theta, j\theta],$$

$$|w_1(x, (j-1) \theta)| \leq C_1\|\delta\|_{(j-1)\theta}, \quad 0 < x < s^*((j-1) \theta],$$

$$w_1(s^*(t), t) = u_0(s^*(t), t), \quad t \in ((j-1) \theta, j\theta].$$

By Lemma 3.2 we have $0 < u_0(x, t) \leq C_1(x_j - x)$, $(x, t) \in R_j$ and therefore

$$|w_1(x, t)| \leq C_1\|\delta\|_t, \quad (x, t) \in (0, \infty) \times ((j-1) \theta, j\theta].$$

If $t^* \in ((j-1) \theta, j\theta)$ such that $x_j = s^*(t^*)$, then we repeat analogous arguments in the domains so determined.

Now since for $t = 0$, $w_1(x, 0) = 0$, $x \in (0, \infty)$, an inductive argument gives

$$|w_1(x, t)| \leq C_1\|\delta\|_t, \quad x \in (0, \infty), \quad t \in [0, T].$$
As for $w_2$, since it solves the problem

$$Lw_2 = 0 \quad \text{in } D_*,$$

$$w_{2x}(0, t) = g(u(0, t), t) - g(u^*(0, t), t), \quad t \in (0, T]$$

$$w_2(x, 0) = 0, \quad x \in (0, b]$$

$$w_2(t^*(t), t) = 0, \quad t \in (0, T]$$

it can be dominated [2, 3] by the function

$$\tilde{w}_2(x, t) = \int_0^t N(x, t; 0, \tau)|g(u(0, \tau), \tau) - g(u^*(0, \tau), \tau)|d\tau$$

the unique solution of

$$L\tilde{w}_2 = 0 \quad \text{in } S,$$

$$\tilde{w}_{2x}(0, t) = -|g(u(0, t), t) - g(u^*(0, t), t)|, \quad t \in (0, T],$$

$$\tilde{w}_2(x, 0) = 0, \quad x \in (0, \infty).$$

Hence

$$|w_2(x, t)| \leq G_1\int_0^t N(x, t; 0, \tau)|u(0, \tau) - u^*(0, \tau)|d\tau.$$

We deduce that

$$|u(x, t) - u^*(x, t)| \leq |w_1(x, t)| + G_1 \frac{1}{\sqrt{\tau}} \int_0^t \frac{|u(0, \tau) - u^*(0, \tau)|}{\sqrt{t - \tau}} d\tau$$

$$\leq C_1 \|\delta\|_t + G_1 \frac{1}{\sqrt{\pi}} \int_0^t \sup_{x \in [0, \infty)} \frac{|u(x, \tau) - u(x, \tau)|}{\sqrt{t - \tau}} d\tau.$$

And by Gronwall’s inequality

$$\sup_{x \in [0, \infty)} |u(x, t) - u(x, t)| \leq C_1 \|\delta\|_t \exp \left[ \frac{2G_1}{\sqrt{\pi}} \sqrt{T} \right] = C_2 \|\delta\|_t$$

for all $t \in [0, T]$. The lemma is proved.
LEMMA 4.2. The pair \((u^*, s^*)\) coincides with the unique solution of \((S\tilde{f})\).

PROOF. The only thing that remains to be proved is the Stefan condition \(u_s(s(t), t) = -\dot{s}(t), \ t \in (0, T]\). Such a condition has been shown to be equivalent to the integral identity [2, page 85]

\[
(4.2) \quad s(t) = b - \int_0^t g(u(0, \tau), \tau) \, d\tau + \int_0^b h(x) \, dx - \int_0^{s(t)} u(x, t) \, dx,
\]

and hence it will be sufficient to prove that \(s^*, u^*\), satisfy \((4.2)\).

Integrating the equation \(Lu^j = 0\) over \(R_j\), we obtain

\[
(4.3) \quad \int_{\theta}^{\theta} u_2^j(x, \tau) \, d\tau = -\int_{\theta}^{\theta} g(u(0, \tau), \tau) \, d\tau + \int_0^z u_1^j(x, (j-1)\theta) \, dx - \int_0^z u_1^j(x, j\theta) \, dx.
\]

Also for \((x, t) \in R_p, 1 < p < n\), integrate \(Lu^p = 0\) over the rectangle \(\{0 < x < x_p\} \times ((p-1)\theta, t]\). It gives

\[
(4.4) \quad -\int_{\theta}^{\theta} u_2^p(x, \tau) \, d\tau = -\int_{\theta}^{\theta} g(u^p(0, \tau), \tau) \, d\tau + \int_0^{x_p} u_1^p(x, (p-1)\theta) \, dx - \int_0^{x_p} u_1^p(x, t) \, dx.
\]

By the definition of \((\mathfrak{F}_j)\) we have

\[
\int_0^{z_{j+1}} u_1^{j+1}(x, j\theta) \, dx = \int_0^{z_j} u_1^j(x, j\theta) \, dx,
\]

therefore adding the identities \((4.3)\) for \(j = 1, 2, ..., p - 1\) and \((4.4)\)
we obtain

\( (4.5) \quad -\sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} u^i_j(x, \tau) \, d\tau \int_{(p-1)\theta}^{t} u^p_k(x, \tau) \, d\tau = \)

\[ = -\int_{0}^{b} g(u_\theta(0, \tau), \tau) \, d\tau + \int_{0}^{b} h(x) \, dx - \int_{0}^{\varphi^p} w(x, t) \, dx . \]

We rewrite the left hand side of (4.5) as follows:

\[ -\sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} u^i_j(x, \tau) \, d\tau - \int_{(p-1)\theta}^{t} u^p_k(x, \tau) \, d\tau = -\sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} u^i_j(x, j\theta) \, d\tau - \]

\[ -\int_{(p-1)\theta}^{t} w^p_k(x, \tau) \, d\tau - \sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} e^i_j(\tau) \, d\tau - \int_{(p-1)\theta}^{t} e^p_a(\tau) \, d\tau , \]

where

\[ e^i_j(t) = u^i_j(x_i, t) - u^i_j(x_i, i\theta) , \quad (i-1)\theta < t < i\theta, \quad i = 1, 2, \ldots, n. \]

We observe that the numbers \( -u^i_j(x, j\theta) \) are the slopes of the Lipschitz continuous polygonal \( t \rightarrow \bar{\theta}_\theta(t) \) for \( (j-1)\theta < t < j\theta \), consequently

\[ -\sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} u^i_j(x, j\theta) \, d\tau - \int_{(p-1)\theta}^{t} w^p_k(x, \tau) \, d\tau = \int_{0}^{t} \frac{d}{d\tau} \bar{\theta}_\theta(\tau) \, d\tau = \bar{\theta}_\theta(t) - b . \]

Carrying this in (4.5) gives

\[ (4.6) \quad \bar{\theta}_\theta(t) = b - \int_{0}^{t} g(u_\theta(0, \tau), \tau) \, d\tau + \int_{0}^{b} h(x) \, dx - \int_{0}^{\varphi^p} w(x, t) \, dx + \]

\[ + \sum_{j=1}^{n-1} \int_{(j-1)\theta}^{j\theta} e^i_j(\tau) \, d\tau + \int_{(p-1)\theta}^{t} e^p_a(\tau) \, d\tau . \]
By virtue of Lemma 4.1 and the uniform convergence \( \bar{s}_\theta(t) \to s^*(t) \), letting \( \theta \to 0 \) in (4.6) gives

\[
(4.7) \quad s^*(t) = b - \int_0^t g(\bar{s}(0, \tau), \tau) \, d\tau + \int_0^b h(x) \, dx - \int_0^t \bar{s}^*(x, t) \, dx + \lim_{\theta \to 0} \left\{ \sum_{i=1}^{\frac{p-1}{\theta}} \int_{(j-1)\theta}^{j\theta} e_{\theta}(\tau) \, d\tau + \int_{(v-1)\theta}^{t} e_{\theta}(\tau) \, d\tau \right\}.
\]

Therefore the lemma will be proved if we show that the limit in (4.7) is zero.

In order to estimate the \( e_{\theta}(\cdot) \) we will need a representation for \( u^j_x(x_j, t), t \in ((j-1)\theta, j\theta] \).

Consider identity (3.3). By taking the derivative with respect to \( x \) and integrating by parts the first two integrals (the identity \( N_x = -G_{\xi} \) is used) we obtain

\[
(4.8) \quad \int_0^{x_k} G(x, t; \xi, k\theta) u^k_x(\xi, k\theta) \, d\xi = - \int_0^{x_k} G(x, t; \xi, (k-1)\theta) u^k_x(\xi, (k-1)\theta) \, d\xi = \]

\[
= - \int_{(k-1)\theta}^{k\theta} N_x(x, t; 0, \tau) g(u_0(0, \tau), \tau) \, d\tau + \int_{(k-1)\theta}^{k\theta} N_x(x, t; x_k, \tau) u^k_x(x_k, \tau) \, d\tau.
\]

Now by the construction of \( u^k \), the second integral in (4.8) can be rewritten as

\[
- \int_0^{x_k-1} G(x, t; \xi, (k-1)\theta) u^{k-1}_x(\xi, (k-1)\theta) \, d\xi,
\]

therefore adding the identities in (4.8) for \( k = 1, 2, \ldots, (j-1) \), chang-
ing the sign and evaluating the sum at $x = x_j$, gives

\begin{equation}
(4.9) \quad \int_0^b G(x_j, t; \xi, 0) h'(\xi) \, d\xi - \int_0^{x_{j-1}} G(x_j, t; \xi, (j-1)\theta) w_{x_j}^{j-1}(\xi, (j-1)\theta) \, d\xi =
\end{equation}

\begin{equation}
= \int_0^b N_\sigma(x_j, t; 0, \tau) g(u_\sigma(0, \tau), \tau) \, d\tau - \sum_{k=1}^{(j-1)\theta} \int_0^{b_\theta} N_\sigma(x_j, t; x_k, \tau) u_{x_k}^j(x_k, \tau) \, d\tau.
\end{equation}

Consider now the representation (3.2). By the way $u^j$ has been constructed, the first integral on the right hand side of (3.2) reads

\begin{equation}
\int_0^{x_{j-1}} G(x_j, t; \xi, (j-1)\theta) w_{x_j}^{j-1}(\xi, (j-1)\theta) \, d\xi.
\end{equation}

Finally adding (3.2) and (4.9) gives

\begin{equation}
(4.10) \quad \frac{1}{2} u^j_{x_j}(x_j, t) = \int_0^b G(x_j, t; \xi, 0) h'(\xi) \, d\xi -
\end{equation}

\begin{equation}
- \int_0^t N_\sigma(x_j, t; 0, \tau) g(u_\sigma(0, \tau), \tau) \, d\tau + \sum_{k=1}^{(j-1)\theta} \int_0^{b_\theta} N_\sigma(x_j, t; x_k, \tau) u_{x_k}^j(x_k, \tau) \, d\tau +
\end{equation}

\begin{equation}
+ \int_0^t N_\sigma(x_j, t; x_j, \tau) u^j_{x_j}(x_j, \tau) \, d\tau, \quad t \in ((j-1)\theta, j\theta] .
\end{equation}

The error terms $e_\sigma^j(t)$, $j = 2, 3, \ldots, (p-1)$, can now be written as

\begin{equation}
\frac{1}{2} e_\sigma^j(t) = \frac{1}{2} [u^j_{x_j}(x_j, t) - u^j_{x_j}(x_j, j\theta)] =
\end{equation}

\begin{equation}
= \int_0^b [G(x_j, t; \xi, 0) - G(x_j, j\theta; \xi, 0)] h'(\xi) \, d\xi -
\end{equation}

\begin{equation}
- \int_0^t [N_\sigma(x_j, t; 0, \tau) - N_\sigma(x_j, j\theta; 0, \tau)] g(u_\sigma(0, \tau), \tau) \, d\tau +
\end{equation}
By the mean value theorem and standard estimates on the exponential function we have

\[ + \int_{t}^{\theta} N_{x}(x_{j}, j\theta; 0, \tau) g(u_{0}(0, \tau), \tau) d\tau + \]

\[ + \sum_{k=1}^{j-1} \int_{(k-1)\theta}^{k\theta} [N_{x}(x_{j}; t; x_{k}, \tau) - N_{x}(x_{j}, j\theta; x_{k}, \tau)] u_{x}^{k}(x_{k}, \tau) d\tau + \]

\[ + \int_{(j-1)\theta}^{j\theta} [N_{x}(x_{j}; t; x_{j}, \tau) - N_{x}(x_{j}, j\theta; x_{j}, \tau)] u_{x}^{j}(x_{j}, \tau) d\tau - \]

\[ - \int_{t}^{j\theta} N_{x}(x_{j}, j\theta; x_{j}, \tau) u_{x}^{j}(x_{j}, \tau) d\tau = J_{1} + J_{2} + J_{2} + J_{3} + J_{4} + J_{4}. \]

By the mean value theorem and standard estimates on the exponential function we have

\[ |G(x_{j}, t; \xi, 0) - G(x_{j}, j\theta; \xi, 0)| \leq \left( K_{1} + \frac{K_{a}}{t^{\theta}} \right) \theta \leq K_{1}\theta + K_{a}(j-1)^{-\frac{3}{2}} \theta^{-\frac{1}{4}}. \]

Here and in what follows \( K_{a} \) denote constants independent of \( j \). Consequently

\[ |J_{1}| \leq Hb(K_{1}\theta + K_{a}(j-1)^{-\frac{3}{2}} \theta^{-\frac{1}{4}}). \]

As for \( J_{2} \) analogous arguments yield the estimate

\[ |N_{x}(x_{j}, t; 0, \tau) - N_{x}(x_{j}, j\theta; 0, \tau)| \leq K_{a}\theta, \]

where we have used the fact that \( x_{j} > b > 0 \). Therefore \( J_{2} \) can be estimated by

\[ |J_{2}| \leq G K_{a} T \theta. \]

Let us now turn to \( J_{3} \). First we note that for \( k = 1, 2, \ldots, (j-2) \),

\[ |N_{x}(x_{j}, t; x_{k}, \tau) - N_{x}(x_{j}, j\theta; x_{k}, \tau)| \leq (j\theta - t) \sup_{t \leq \tau \leq j\theta} |N_{x}(x_{j}, q; x_{k}, \tau)| < \]

\[ \leq K_{b}\theta + K_{a}\theta \sup_{t \leq \tau \leq j\theta} \frac{x_{j} - x_{k}}{(q - \tau)^{\frac{3}{2}}} \exp \left[ -\frac{(x_{j} - x_{k})^{2}}{4(q - \tau)} \right]. \]
Now since \((x_k, kθ), (x_j, (j - 1)θ)\) belong to the maximal extension of the graph \((t, so(t))\), we have \((x_j - x_k) ≤ 2C_1(θ - τ)\), for \(k = 1, 2, \ldots, (j - 2)\), and hence

\[
|J_3| ≤ \sum_{k=1}^{j-2} \int_{(k-1)θ}^{kθ} \left| N_ω(x_j, t; x_k, τ) - N_ω(x_j, jθ; x_k, τ) \right| \left| u_k^*(x_k, τ) \right| dτ + \int_{(j-2)θ}^{(j-1)θ} \left| N_ω(x_j, t; x_{j-1}, τ) - N_ω(x_j, jθ; x_{j-1}, τ) \right| \left| u_{x_j}^{-1}(x_{j-1}, τ) \right| dτ ≤
\]

\[
(\int_{0}^{θ} [(j - 1)θ - τ]^{-\frac{3}{4}} dτ + \int_{(j-2)θ}^{(j-1)θ} + C_1 \int_{(j-2)θ}^{(j-1)θ} \left| N_ω(x_j, t; x_{j-1}, τ) - N_ω(x_j, jθ; x_{j-1}, τ) \right| dτ.
\]

In order to estimate the last integral recall that

\[
|N_ω(x_j, τ; x_{j-1}, τ)| ≤ K_θ(θ - τ)^{-\frac{3}{4}}, \quad θ > τ.
\]

Substituting this in the estimate of \(J_3\) we obtain after some algebra

\[
|J_3| ≤ K_θ \sqrt{θ}.
\]

Analogous considerations yield \(|J_3'| + |J_4'| + |J_4'| ≤ K_θ \sqrt{θ}\). Therefore the error terms \(e_θ'(t)\) can be estimated by

\[
|e_θ'(t)| ≤ K_θ \sqrt{θ} + K_10(j - 1)^{-\frac{3}{4}} θ^{-\frac{1}{4}}.
\]

For the error \(E_θ\) in (4.6) we have

\[
|E_θ| ≤ \sum_{j=1}^{p-1} \int_{(j-1)θ}^{jθ} |e_θ(τ)| dτ + \int_{(p-1)θ}^{θ} |e_θ(τ)| dτ ≤ \int_{0}^{θ} |e_θ(τ)| dτ + \sum_{j=2}^{p} \int_{(j-1)θ}^{jθ} |e_θ(τ)| dτ.
\]
Now trivially $|e^t_\theta| < 2C_1$. Therefore

$$|E_\theta| < 2C_1\theta + K_9\sqrt{\theta} \sum_{j=1}^p \theta + K_{10}\sqrt{\theta} \sum_{j=2}^p (j-1)^{-\frac{1}{2}}.$$ 

Since $\sum_{j=2}^p \theta < p\theta < T$, and the series $\sum_{j=2}^p (j-1)^{-\frac{1}{2}}$ converges, we obtain

$$|E_\theta| < K_{11}\sqrt{\theta}.$$ 

This proves the lemma.

5. The error estimate.

The purpose of this section is to give an estimate of the speed of convergence of the approximate interface $s_\theta(t)$ to the true interface $s(t)$. In view of Lemma 4.1 and (4.1) this will also complete the proof of the theorem.

LEMMA 5.1. There exists a constant $C$ depending upon $H, b, G_1, G_2, T$ such that

$$\|s_\theta - s\|_T < C\sqrt{\theta}.$$ 

PROOF. Subtract (4.6) from (4.2) and use (4.11) to obtain

$$|s(t) - \bar{s}_\theta(t)| < \int_0^t \left| g(u(0, \tau), \tau) - g(u_\theta(0, \tau), \tau) \right| d\tau +$$

$$+ \left| \int_0^{s(t)} u(x, t) \, dx - \int_0^{\bar{s}_\theta(t)} u_\theta(x, t) \, dx \right| +$$

$$+ \left| \int_0^{\bar{s}_\theta(t)} u_\theta(x, t) \, dx - \int_0^{z_\theta} u_\theta(x, t) \, dx \right| + K_{11}\sqrt{\theta},$$
where \( \bar{u}_\phi \) is the unique solution of the problem

\[
L\bar{u}_\phi = 0 \quad \text{in } \mathcal{D}_{\bar{u}_\phi} = \{0 < x < \bar{s}_\phi(t)\} \times (0, T),
\]

\[
\bar{u}_\phi(x, 0) = h(x), \quad x \in (0, b],
\]

\[
\bar{u}_\phi(\bar{s}_\phi(t), t) = 0, \quad 0 < t < T.
\]

On the basis of the maximum principle and standard barriers estimates \([10,7]\), as in Lemma 3.3, we have

\[
0 < u(x, t) < C_1(s(t) - x), \quad (x, t) \in \mathcal{D}_T,
\]

\[
0 < \bar{u}_\phi(x, t) < C_1(\bar{s}_\phi(t) - x), \quad (x, t) \in \mathcal{D}_{\bar{u}_\phi}.
\]

**Proposition 5.1.** There exist constants \( B_1 \) and \( B_2 \) depending upon \( H, b, G_1, G_2, T, \) such that

\[
\left| \int_0^{s(t)} u(x, t) \, dx - \int_0^{\bar{s}_\phi(t)} \bar{u}_\phi(x, t) \, dx \right| \leq B_1 \int_0^t \sup_{0 \leq \tau \leq \tau} \frac{|s(\tau) - \bar{s}_\phi(\tau)|}{\sqrt{t - \tau}} \, d\tau + B_2 \int_0^t \frac{\|\delta\|_T}{\sqrt{t - \tau}} \, d\tau.
\]

**Proof of Proposition 5.1.** Set

\[
\bar{\alpha}(t) = \min \{s(t), \bar{s}_\phi(t)\}; \quad \bar{\beta}(t) = \max \{s(t), \bar{s}_\phi(t)\},
\]

\[
\bar{\delta}(t) = \bar{\beta}(t) - \bar{\alpha}(t), \quad t \in [0, T].
\]

Obviously \( \bar{\alpha}(\cdot), \bar{\beta}(\cdot) \) are non-decreasing Lipschitz continuous functions with Lipschitz constant bounded by \( C_1. \) Then

\[
\left| \int_0^{s(t)} u(x, t) \, dx - \int_0^{\bar{s}_\phi(t)} \bar{u}_\phi(x, t) \, dx \right| \leq \int_0^{\bar{\alpha}(t)} |u(x, t) - \bar{u}_\phi(x, t)| \, dx + \int_0^{\bar{\beta}(t)} y(x, t) \, dx = I_1 + I_2
\]

where

\[
y(x, t) = \begin{cases} u(x, t) & \text{if } \bar{\alpha}(t) = \bar{s}_\phi(t) \\ \bar{u}_\phi(x, t) & \text{if } \bar{\alpha}(t) = s(t). \end{cases}
\]
We dominate the integrand in $I_1$ by the sum $|v - \bar{u}_0| + |u - v| \leq v_1 + v_2$
where $v$ is defined in Lemma 3.3 and $v_1, v_2$ are the solutions of the problems

\[
\begin{aligned}
(Lv_1 &= 0, \quad \{0 < x < \bar{x}(t)\} \times \{0 < t < T\}, \\
v_{1x}(0, t) &= 0, \quad 0 < t < T, \\
v_1(x, 0) &= 0, \quad 0 < x < b, \\
v_1(\bar{x}(t), t) &= C_1 \delta(t), \quad 0 < t < T
\end{aligned}
\]

and

\[
\begin{aligned}
(Lv_2 &= 0 \quad (0, \infty) \times (0, T], \\
v_{2x}(0, t) &= -|g(u(0, t), t) - g(u_0(0, t), t)|, \quad 0 < t < T, \\
v_2(x, 0) &= 0, \quad x \in [0, \infty).
\end{aligned}
\]

As for $v_2$, it can be represented explicitly [9], by

\[
v_2(x, t) = \int_0^t N(x, t; 0, \tau)|g(u(0, \tau), \tau) - g(u_0(0, \tau), \tau)|d\tau.
\]

Therefore by virtue of the Lipschitz continuity of $g(\cdot, t)$ and (4.1) we obtain

\[
v_2(x, t) \leq \frac{G_1}{\sqrt{\pi}} C_2 \int_0^t \frac{\|\delta\|_\tau}{\sqrt{t - \tau}} d\tau.
\]

By an argument of [2] page 87, $v_1$ can be dominated by $z(x, t) + z(-x, t)$ where $z$ solves

\[
\begin{aligned}
(Lz &= 0, \quad \{-\bar{x}(t) < x < \infty\} \times \{0 < t < T\}, \\
z(-\bar{x}(t), t) &= C_1 \delta(t), \quad 0 < t < T, \\
z(x, 0) &= 0, \quad -b < x < \infty.
\end{aligned}
\]

Then there exists a constant $K_1$ depending only upon the data such
that
\[ z(x, t) < K_{12} \int_0^t \| \tilde{\delta}_\tau \| |I^s_2(x, t; -\tilde{\alpha}(\tau), \tau)| d\tau. \]

Therefore
\[ \tilde{\alpha}(t) \int_0^t v_1(x, t) dx < \tilde{\alpha}(t) \int_0^\infty z(x, t) dx < \tilde{\alpha}(t) \int_0^t \| \tilde{\delta}_\tau \| \frac{d\tau}{\sqrt{t-\tau}}. \]

The estimate of \( I_2 \) is done by exploiting again the methods of [2] page 87 and dominating the integrand by \( \tilde{z}(x, t) \), the solution of
\[
\begin{cases}
L\tilde{z} = 0, & \{\tilde{\alpha}(t) < x < \infty\} \times \{0 < t < T\}, \\
\tilde{z}(x, 0) = 0, & b < x < \infty, \\
\tilde{z}(\tilde{\alpha}(t), t) = C_1 \delta(t), & 0 < t < T.
\end{cases}
\]

It gives
\[ I_2 < K_{14} \int_0^t \| \tilde{\delta}_\tau \| \frac{d\tau}{\sqrt{t-\tau}}. \]

**Proposition 5.2.** There exist a constant \( \tilde{B} \) depending upon \( H, b, G_1, G_2, T \) such that
\[ \left| \int_0^{x_p} \bar{u}_\theta(x, t) dx - \int_0^{x_p} u_\theta(x, t) dx \right| < \tilde{B}\theta. \]

**Proof of Proposition 5.2.** For \( t > 0 \) fixed we have \( \bar{\delta}_\theta(t) > x_p \). Then
\[
\left| \int_0^{x_p} \bar{u}_\theta(x, t) dx - \int_0^{x_p} u_\theta(x, t) dx \right| < \int_0^{x_p} |\bar{u}_\theta(x, t) - u_\theta(x, t)| dx + \\
\left. \int_{x_p}^{x_p} \bar{u}_\theta(x, t) dx \right| = J_1 + J_2.
\]
Standard barrier arguments \([7, 8, 9]\) give

\[
\sup_{(x,t) \in D_{2a}} |\bar{u}_\theta(x,t)| \leq C_1(b + C_1 T).
\]

On the other hand by the construction of \(\bar{\theta}(\cdot)\), the distance \(\bar{\theta}(t) - x_p\) is at most \(C_1\theta\). Hence

\[
J_2 < K_{15}\theta.
\]

To estimate the integrand in \(J_1\) we proceed as in Lemma 4.1. In fact, in the present situation the estimates are simpler since \(\bar{u}_\theta - u_\theta\) has zero flux at the fixed face \(x = 0\). We obtain the estimate analogous to (4.1)

\[
|\bar{u}_\theta(x, t) - u_\theta(x, t)| \leq K_{18} \sup_{0 \leq \tau \leq t} |\bar{\theta}(\tau) - \theta(\tau)|.
\]

As remarked above for all \(t \in [0, T]\) we have

\[
\bar{\theta}(t) - \theta(t) < C_1\theta,
\]

and hence the proposition is proved.

We now conclude the proof of Lemma 5.1. Notice that by virtue of the Lipschitz continuity of \(g(\cdot, t)\) and (4.1) we have

\[
\int_0^t |g(u(0, \tau), \tau) - g(u_\theta(0, \tau), \tau)|\,d\tau \leq G_1 T^4 \int_0^t \frac{|u(0, \tau) - u_\theta(0, \tau)|}{\sqrt{t - \tau}}\,d\tau \leq G_1 T^4 C_2 \int_0^t \frac{\|\delta\|_{\tau}}{\sqrt{t - \tau}}\,d\tau.
\]

Putting together the various estimates so obtained we see that for all \(0 < t < T\) we have

\[
|s(t) - \bar{s}(t)| < K_{18}\sqrt{\theta} + K_{19} \int_0^t \frac{\|\delta\|_{\tau}}{\sqrt{t - \tau}}\,d\tau + K_{17} \int_0^t \frac{\|\delta\|_{\tau}}{\sqrt{t - \tau}}\,d\tau.
\]

Now

\[
|s(t) - \bar{s}(t)| > |s(t) - \theta(t)| - |\theta(t) - \bar{\theta}(t)|
\]
and
\[ \| \delta \|_T = \sup_{0 \leq \tau \leq T} |s(q) - \overline{s}_\delta(q)| + \sup_{0 \leq \tau \leq T} |\overline{s}_\delta(q) - \overline{s}_\theta(q)| < \| \delta \|_T + C_1 \theta . \]

Consequently the above implies
\[ \| \delta \|_T < K_{20} \sqrt{T} + K_{21} \int_0^T \frac{\| \delta \|_T}{\sqrt{T - \tau}} d\tau . \]

The proof of the lemma is concluded with an application of Gronwall’s inequality.

6. Modifications of the scheme.

Consider the sequence of approximating problems \((\mathcal{F}_j)\) introduced in Section 2. The theorem remains true if we modify the flux condition on the fixed face \(x = 0\) in \((\mathcal{F}_j)\) as follows

(6.i) Retarded flux.
We set \(u_1^j(0, t) = g(h(0), t), 0 < t < \theta\) and for \(j > 1\) we replace the flux condition in \((\mathcal{F}_j)\) with
\[ u_2^j(0, t) = g(u^{j-1}(0, t - \theta), t), \quad (j - 1) \theta < t < j \theta . \]

(6.ii) Piecewise constant flux.
The modification consists in freezing the flux on \(x = 0\) in the \(j\)-th rectangle at its value at the lower left corner.
Set \(u_2^j(0, t) = g(h(0), 0), 0 < t < \theta\) and then for \(j > 1\)
\[ u_2^j(0, t) = g(u^{j-1}(0, (j - 1) \theta), (j - 1) \theta), \quad (j - 1) \theta < t < j \theta . \]

The proof of the convergence and the error estimate for the scheme obtained with the above modifications, is carried out in essentially the same way as indicated in the previous sections. Some minor modifications are needed which we leave to the reader, (cf. [1]).
REFERENCES


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