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A Generalized Theory of Classical Mechanics for the Two Body Problem.

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1. Introduction.

A well known result of Classical Mechanics consists of the following assertion about the Two Body (1) Problem in the reduced mass frame:

(a) if P_2 is subject to a gravity field with centre at P_1, and the orbits of its possible motions relative to P_1 are conic sections with a focus at P_1, then P_2 fulfils Newton's law: $U = -\gamma/q$.

Because of the large range of validity for Newton's law, it may asked whether it is possible to build a mechanical theory in which this law, or a physically relevant generalization of it, can be deduced from hypotheses weaker than these in (*a*).

In the present paper I try to give a *classical*, that is, non relativistic, answer to the above question. The basic change with respect to the usual Classical Mechanics consists of a generalization of Mach's axiom for mass. Moreover, in [4] it is shown that a natural application of this theory leads to a physically acceptable description of the perihelion's precession of planets. The last result is very similar to the well known approximation of the motion of a particle under Schwarzschild's solution for the metric tensor within General Relativity.

In connection with the Two Body Problem, the present theory is interesting especially because

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(1) Material points P_1 and P_2.

(i) its results are obtained within a *classical* conception of the physical world, hence

(ii) the critical reviews of the concepts of space, time, and matter, which are essential for General Relativity, are not necessary in the present theory. Therefore, the latter is much *simpler* than General Relativity.

In sections 2, 3 the axioms of a theory $\mathscr{T}$ of Classical Mechanics, restricted for shortness of treatment to the Two Body Problem, are presented in a way slightly different from [2] but in conceptual agreement with that paper. In theorems T1 to T4 some well known results are deduced from hypotheses more general than the usual ones.

In section 4 we formulate a weaker version, A2*, of axiom A2 for mass in section 2. This generalization is suggested by a general theorem of representation, T2, for the force in $\mathscr{T}$. We study the resulting theory $\mathscr{T}^*$ and some important consequences of it. In detail, in (4.15) and (4.16), we offer the most general expression for the force and, in section 5, in the mass reduced frame $\mathscr{R}_{\mathscr{T}\mathscr{B}}$, we show that the orbit belongs to a certain conic surface (T2*). As consequence, assuming that P_1 coincides with the origin of a frame $\mathscr{R}_{\mathscr{T}\mathscr{B}}$, we prove that *the motion of P_2 is plane if and only it its orbit is a conic section.*

In section 6 we show that within $\mathscr{T}^*$ any plane motion is generated only by a force $\mathscr{F}$ of a quasi-Newtonian kind, that is, by a Newtonian central force to which we add a force of little magnitude, depending on the velocity also. The force $\mathscr{F}$ reduces to Newton's when (in connection with plane motions) $\mathscr{T}^*$ reduces to $\mathscr{T}$.

Thus in $\mathscr{T}^*$ the following alternative assertion to (a) holds in the mass reduced frame:

(a*) *if P_2 is subject to a force field which admits a generalized potential and describes a plane orbit, then this orbit is a conic section and the quasi-Newtonian force law* (6.8) *holds.*

2. Classical axioms for the Two Body Problem.

We consider classical physics and regard the motions of *inertial spaces* and *inertial frames* as known. Let $\mathscr{R}_{\mathscr{I}}$ such a frame. We assume that only the particles P_1 and P_2 exist, so that they constitute an *isolated system.*

Before listing the axioms of Classical Mechanics, here enunciated briefly only for our system, we state in advance an axiom of *physical possibility* whose use is essential in several proofs.

A1 (*of Phys. Poss.*). *Let $R_{\mathscr{I}}$ be an arbitrary inertial frame. Then*

(i) *if $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are four vectors of $\mathbb{R}^3$ with $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, it is phys. poss.* (2) *that at some instant $t \in \mathbb{R}$ the two particles P_1 and P_2 have in $R_{\mathscr{I}}$ the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ respectively,*

(ii) *it is phys. poss. for P_1 and P_2 to have non-vanishing accelerations parallel with $x_2 - x_1$, in $\mathscr{R}_{\mathscr{I}}$ and hence in any other inertial frame, at some instant t.*

A2 (*mass existence*). *There are $\mu_1, \mu_2 \in \mathbb{R}^+$ for which, calling $\boldsymbol{a}_i$ the acceleration of P_i $(i = 1, 2)$ with respect to $\mathscr{R}_{\mathscr{I}}$ at the instant t, we necessarily have*

$$\mu_1 \boldsymbol{a}_1 + \mu_2 \boldsymbol{a}_2 = \boldsymbol{0}. \tag{2.1}$$

This axiom is in harmony with Mach's paper [6] and with more recent axiomatization of classical particle mechanics—cf. [2]. If (2.1) holds necessarily, then $i \mapsto \mu_i$ is called a *mass distribution*. If also $i \to \mu_i'$ is a mass distribution, then for some $b \in \mathbb{R}^+$ $\mu_i' = b\mu_i$ $(i = 1, 2)$ as is easy to prove on the basis of A1(ii). Thus we have ∞^1 mass distributions mutually connected by changes of the unit mass.

A3 (*dynamic law*). *In connection with an arbitrary choice of $\mathscr{R}_{\mathscr{I}}$ there is a function* (*force*) $\boldsymbol{f}$

$$\boldsymbol{f}\colon [\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \in \mathbb{R}^3 \times \mathbb{R}^3 \colon \boldsymbol{\xi}_1 = \boldsymbol{\xi}_2\}] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \tag{2.2}$$

$$(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4) \mapsto \boldsymbol{f}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4),$$

such that if $i \mapsto \mu_i$ is a mass distribution and with respect to $\mathscr{R}_{\mathscr{I}}$, at the instant t, P_i has the position $\boldsymbol{x}_i$ and velocity $\boldsymbol{v}_i$ $(i = 1, 2)$, then

$$\mu_1 \boldsymbol{a}_1 = \boldsymbol{f}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{v}_2, \boldsymbol{v}_1), \quad \mu_2 \boldsymbol{a}_2 = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2). \tag{2.3}$$

(2) Physically Possible means ideally realizable, i.e. technically possible for ideal experimenters—cf. [3].

The function $\boldsymbol{f}$ that fulfils the condition above is called force function, and depends only on the choice of units in the well known way. Further on by $\boldsymbol{f}$ such a function will be understood.

By A1(i) and A2 we have the following theorem:

T1 (*Action and Reaction principle for the resultant*). *If* $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{R}^3$ *and* $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$, *then*

$$\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) + \boldsymbol{f}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{v}_2, \boldsymbol{v}_1) = \mathbf{0}. \tag{2.4}$$

Usually one also assumes the following *Action and Reaction principle for the moment*: Under the assumptions in A2 we necessarily have that

$$\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) \wedge (\boldsymbol{x}_2 - \boldsymbol{x}_1) = \mathbf{0}. \tag{2.5}$$

This principle is not included in the present theory. Instead, in sect. 3, (2.5) will be deduced from the assumption that a *generalized energy integral* holds for the system $\{P_1, P_2\}$.

Well known *homogeneity* and *isotropy* properties of inertial spaces —cf. [1], § 2—restrict the form of $\boldsymbol{f}$ according to the following axiom

A4. *There is a function* $\boldsymbol{F}$: $[\mathbb{R}^3 \setminus \{\mathbf{0}\}] \times \mathbb{R}^3 \to \mathbb{R}^3$ *for which*

$$\boldsymbol{F}(\boldsymbol{x}_2 - \boldsymbol{x}_1, \boldsymbol{v}_2 - \boldsymbol{v}_1) = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{v}_1, \boldsymbol{v}_2) \tag{2.6}$$

and, for all $\boldsymbol{u}, \boldsymbol{w} \in \mathbb{R}^3$ *with* $\boldsymbol{u} \neq \mathbf{0}$ *and all proper orthogonal matrices* Q ($QQ^T = \mathbf{1}$, $\det Q = 1$).

$$Q\boldsymbol{F}(\boldsymbol{u}, \boldsymbol{w}) = \boldsymbol{F}(Q\boldsymbol{u}, Q\boldsymbol{w}). \tag{2.7}$$

By a theorem of Cauchy — cf. [7], p. 60 —, one has the following representation theorem for the above function $\boldsymbol{F}$.

T2. *Let* $\boldsymbol{F}$ *fulfil* (2.7). *Then there are three mappings* $\mathscr{A}$, $\mathscr{B}$, *and* $\mathscr{C}$ *of* $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ *into* $\mathbb{R}$, *for which*

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{u}, \boldsymbol{w}) = {} & \mathscr{A}(|\boldsymbol{u}|, |\boldsymbol{w}|, \boldsymbol{u} \times \boldsymbol{w})\, \boldsymbol{u} + \\ & + \mathscr{B}(|\boldsymbol{u}|, |\boldsymbol{w}|, \boldsymbol{u} \times \boldsymbol{w})\, \boldsymbol{w} + \mathscr{C}(|\boldsymbol{u}|, |\boldsymbol{w}|, \boldsymbol{u} \times \boldsymbol{w})\, \boldsymbol{u} \wedge \boldsymbol{w}. \end{aligned} \tag{2.8}$$

Through the weak Action and Reaction theorem T1, the mass axiom A2 implies a further restriction on the form of $\boldsymbol{F}$. Under the definitions

$$(2.9)\qquad \boldsymbol{q}=\boldsymbol{x}_2-\boldsymbol{x}_1\,,\qquad \dot{\boldsymbol{q}}=\boldsymbol{v}_2-\boldsymbol{v}_1\,,\qquad q=|\boldsymbol{q}|\,,\qquad \dot{q}=|\dot{\boldsymbol{q}}|\,,$$

the following theorem can be easily proved:

T3. *The most general function $\boldsymbol{F}$, that has the form* (2.8) *and under condition* (2.6) *fulfils* (2.4), *is given by*

$$(2.10)\qquad \boldsymbol{F}(\boldsymbol{q},\dot{\boldsymbol{q}})=\mathscr{A}(q,\dot{q},\boldsymbol{q}\times\dot{\boldsymbol{q}})\boldsymbol{q}+\mathscr{B}(q,\dot{q},\boldsymbol{q}\times\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\qquad(\mathscr{C}\equiv 0)\,.$$

3. Assumption that a certain generalized energy integral exists.

From the physical point of view, it appears natural to assume that *generalized energy integral* exists for our isolated system—cf. axiom A5 below. As a preliminary let us note that the mass axiom A2, and hence the Action and Reaction principle for the resultant, implies the following theorem, proved in textbooks after having stated the full Action and Reaction principle—cf. also [5], p. 352—but using only the part T1 of it.

In the reference frame $\mathscr{R}_{\mathscr{T}\mathscr{B}}$, whose origin is (*always*) *in P_1, and is rotationless with respect to inertial spaces, the dynamic equation of P_2 reads*—cf. (2.3)

$$(3.1)\qquad \mu^*\ddot{\boldsymbol{q}}=\boldsymbol{F}\,,\qquad \textit{where } \mu^*=\frac{\mu_1\mu_2}{\mu_1+\mu_2}\ (\textit{reduced mass}).$$

If a function $V(\boldsymbol{q},\dot{\boldsymbol{q}})$, $V\in\mathscr{C}^{(2)}\big([\mathbb{R}^3\diagdown\{\boldsymbol{0}\}]\times\mathbb{R}^3;\mathbb{R}\big)$, exists, for which

$$(3.2)\qquad F_h=F_h(\boldsymbol{q},\dot{\boldsymbol{q}})=\frac{\partial V}{\partial q_h}-\frac{d}{dt}\frac{\partial V}{\partial \dot{q}_h}\qquad(h=1,2,3),$$

then (since $\partial\boldsymbol{F}/\partial\ddot{q}_h\equiv\boldsymbol{0}$) V has the following form (linear in $\dot{\boldsymbol{q}}$):

$$(3.3)\qquad V(\boldsymbol{q},\dot{\boldsymbol{q}})=U(\boldsymbol{q})+\boldsymbol{\alpha}(\boldsymbol{q})\times\dot{\boldsymbol{q}}\,,$$

where $U\in\mathscr{C}^{(2)}(\mathbb{R}^3\diagdown\{\boldsymbol{0}\};\mathbb{R})$, $\boldsymbol{\alpha}\in\mathscr{C}^{(2)}(\mathbb{R}^3\diagdown\{\boldsymbol{0}\};\mathbb{R}^3)$.

For the kinetic energies

$$(3.4)\qquad T=\frac{1}{2}\sum_{i=1}^{2}\mu_i\dot{\boldsymbol{x}}_i^2\,,\qquad T_{\mathcal{T}\mathcal{B}}=\frac{1}{2}\mu^*\dot{q}^2\,,$$

$$T_{\mathcal{M}\mathcal{C}}=\frac{1}{2}(\mu_1+\mu_2)\,\boldsymbol{v}_{\mathcal{M}\mathcal{C}}^2\qquad\left(\boldsymbol{v}_{\mathcal{M}\mathcal{C}}=\frac{\mu_1\dot{\boldsymbol{x}}_1+\mu_2\dot{\boldsymbol{x}}_2}{\mu_1+\mu_2}\right)$$

we have the identity

$$(3.5)\qquad T=T_{\mathcal{T}\mathcal{B}}+T_{\mathcal{M}\mathcal{C}}\,.$$

The Lagrangian functions of $\{P_1, P_2\}$ in $\mathcal{R}_{\mathcal{I}}$ and $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ are

$$(3.6)\qquad \begin{aligned} &L(\boldsymbol{x}_1,\boldsymbol{x}_2,\dot{\boldsymbol{x}}_1,\dot{\boldsymbol{x}}_2)=T+V(\boldsymbol{x}_2-\boldsymbol{x}_1,\dot{\boldsymbol{x}}_2-\dot{\boldsymbol{q}}_1)\,,\\ &L_{\mathcal{T}\mathcal{B}}(\boldsymbol{q},\dot{\boldsymbol{q}})=T_{\mathcal{T}\mathcal{B}}+V(\boldsymbol{q},\dot{\boldsymbol{q}})\end{aligned}$$

respectively, so that their corresponding Hamiltonian functions read

$$(3.7)\qquad \begin{aligned} &\mathcal{H}=\sum_{h=1}^{3}\left(\frac{\partial L}{\partial\dot{x}_{1h}}\dot{x}_{1h}+\frac{\partial L}{\partial\dot{x}_{2h}}\dot{x}_{2h}\right)-L=T-U\,,\\ &\mathcal{H}_{\mathcal{T}\mathcal{B}}=\sum_{h=1}^{3}\frac{\partial L_{\mathcal{T}\mathcal{B}}}{\partial\dot{q}_h}\dot{q}_h-L_{\mathcal{T}\mathcal{B}}=T_{\mathcal{T}\mathcal{B}}-U\,.\end{aligned}$$

Axiom A2 implies $\dot{\boldsymbol{v}}_{\mathcal{M}\mathcal{C}}=\boldsymbol{0}$, hence $T_{\mathcal{M}\mathcal{C}}=\text{const.}$ *Then $\mathcal{H}$ is a first integral of the motion of $\{P_1, P_2\}$ iff $\mathcal{H}_{\mathcal{T}\mathcal{B}}$ is such an integral.* Since by (3.6) $\partial L/\partial t=0=\partial L_{\mathcal{T}\mathcal{B}}/\partial t$, *in our case $\mathcal{H}$ and $\mathcal{H}_{\mathcal{T}\mathcal{B}}$ are first integrals.*

It is clear that if A2 is excluded from the theory $\mathcal{T}$ being considered, both (3.1) and the above equivalence assertion on $\mathcal{H}$ and $\mathcal{H}_{\mathcal{T}\mathcal{B}}$ are no longer theorems. With a view to weakening A2 and considering the material reference frames which e.g. the motions of planets are referred to, not to be inertial but choices of $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ with the origin in the sun, I propose the following version of the afore-mentioned existence assumption for a generalized energy integral.

A5. *If $\mathcal{R}_{\mathcal{T}\mathcal{B}}$ has the origin in P_1 and has the same orthonormal basis $\{\boldsymbol{e}_h\}_{h=1,2,3}$ as $\mathcal{R}_{\mathcal{I}}$, the function $\mathcal{F}(\boldsymbol{q},\dot{\boldsymbol{q}})$ for which the motion of P_2 necessarily fulfils the equation $\mu^*\ddot{\boldsymbol{q}}=\mathcal{F}(\boldsymbol{q},\dot{\boldsymbol{q}})$, is afforded by a generalized* *potential* $V(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $V \in C^{(2)}([\mathbb{R}^3 \setminus A] \times \mathbb{R}^3; \mathbb{R})$, *for some set* A *without internal points, such that the functions*

$$(3.8) \qquad \mathscr{F}_h = \frac{\partial V}{\partial q_h} - \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_h} \quad (h = 1, 2, 3), \qquad \mathscr{F}_h \in \mathscr{C}([\mathbb{R}^3 \setminus A] \times \mathbb{R}^3; \mathbb{R}),$$

have continuous extensions onto $[\mathbb{R}^3 \setminus \{\boldsymbol{0}\}] \times \mathbb{R}^3$.

On the basis of some preceding considerations—cf. (3.1)—A2 implies

$$(3.9) \qquad \mathscr{F} = \boldsymbol{F}.$$

From (3.2) and (3.3) we deduce

$$(3.10) \qquad \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}} \wedge \operatorname{rot} \boldsymbol{\alpha}(\boldsymbol{q}) + \operatorname{grad} U(\boldsymbol{q}).$$

Now we are going to characterize (the form of) the functions $\mathscr{A}$ and $\mathscr{B}$. Axiom A1(i) of phys. poss. tells us that for all $\boldsymbol{q} \neq \boldsymbol{0}$ and all $\dot{\boldsymbol{q}}$, some motion of $\{P_1, P_2\}$ is possible for which, at some instant t, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$ represent the position and velocity of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$. Then (2.10) and (3.10) imply that

$$(3.11) \qquad \boldsymbol{F}(\boldsymbol{q}, \boldsymbol{0}) = \mathscr{A}(q, 0, 0)\boldsymbol{q} = \operatorname{grad} U(\boldsymbol{q}).$$

Since $\partial q / \partial q_h = q_h / q$,

$$(3.12) \qquad \frac{\partial U}{\partial q_h}(\boldsymbol{q}) = \tilde{\mathscr{A}}(q) q_h = \tilde{\mathscr{A}}(q) q \frac{\partial q}{\partial q_h}, \quad \text{where } \tilde{\mathscr{A}}(q) = \mathscr{A}(q, 0, 0),$$

hence

$$(3.13) \qquad U(\boldsymbol{q}) = \tilde{U}(q) = \int \mathscr{A}(q) q \, dq.$$

By (3.11), equation (3.10) becomes

$$(3.14) \qquad \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{q}} \wedge \operatorname{rot} \boldsymbol{\alpha}(\boldsymbol{q}) + \mathscr{A}(q)\boldsymbol{q}.$$

The term $\operatorname{rot} \boldsymbol{\alpha}(\boldsymbol{q})$ in (3.14) is isotropic iff for some function $\mathscr{D}(q)$, $\mathscr{D}: \mathbb{R}^+ \to \mathbb{R}$,

$$(3.15) \qquad \operatorname{rot} \boldsymbol{\alpha}(\boldsymbol{q}) = \mathscr{D}(q)\boldsymbol{q}.$$

By (3.14), (3.15), and (2.10)

$$\mathscr{D}(q)\,\dot{\boldsymbol{q}}\wedge\boldsymbol{q}=[\mathscr{A}(q,\dot{q},\boldsymbol{q}\times\dot{\boldsymbol{q}})-\tilde{\mathscr{A}}(q)]\boldsymbol{q}+\mathscr{B}(q,\dot{q},\boldsymbol{q}\times\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}\,, \tag{3.16}$$

$\forall\boldsymbol{q}(\neq\boldsymbol{0})$, $\forall\dot{\boldsymbol{q}}$ (see A1(i)); it is true iff

$$\mathscr{A}(q,\dot{q},\boldsymbol{q}\times\dot{\boldsymbol{q}})=\tilde{\mathscr{A}}(q)\,,\qquad \mathscr{B}\equiv 0\qquad (\mathscr{D}\equiv 0,\ \text{rot}\,\boldsymbol{\alpha}\equiv\boldsymbol{0})\,. \tag{3.17}$$

Calling $\mathscr{T}$ the theory based on axioms A1 to A5, let us summarize the preceding results by the following theorem

T4. (i) *The most general force* $\boldsymbol{F}$ *in* $\mathscr{T}$ *is positional and admits a (universal) potential* $\tilde{U}(q)$:

$$\boldsymbol{F}(\boldsymbol{q})=\text{grad}\,\tilde{U}(q)\,; \tag{3.18}$$

(ii) *also the Action and Reaction principle for the moment holds*

$$\boldsymbol{F}(\boldsymbol{q})\wedge\boldsymbol{q}=\boldsymbol{0}\,; \tag{3.19}$$

(iii) *in the above system* $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ *the motion of* P_2 *is central (with respect to the origin* P_1*) and hence plane.*

A natural specialization of $\mathscr{T}$, through a suitable choice of $\tilde{U}$, leads to Newton's theory of gravitation. The validity of this choice can be proved when e.g. new facts are postulated, e.g. that the possible orbits of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ are *conic*. In the remaining sections a theory $\mathscr{T}^*$ is studied, which can be obtained from $\mathscr{T}$ by replacing the mass axiom A2 with a weaker axiom, A2*. Under the additional assumption that the motion of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ should be plane—see axiom A6 below—(which is a theorem in $\mathscr{T}$, T4(iii)), $\mathscr{T}^*$ will be shown to admit Newtonian gravitation as a limit theory and to foresee (in a slightly approximated version) a precession of the apsidal points of quasi conic orbits, in substantial agreement with the corresponding results of General Relativity.

4. A theory $\mathscr{T}^*$ with a mass axiom A2* weaker than A2.

The general form (2.8) for $\boldsymbol{F}$ (together with the dynamical axiom A3) suggests to weaken the mass axiom A2 into the following

A2*. *For some* $\mu_1, \mu_2 \in \mathbb{R}^+$, *if* $\boldsymbol{x}_i$ $[\boldsymbol{a}_i]$ *is the position* [*acceleration*] *of* P_i *in the inertial frame* $\mathscr{R}_{\mathscr{I}}$ *at some instant* t $(i = 1, 2)$, *then we necessarily have*

$$(\mu_1 \boldsymbol{a}_1 + \mu_2 \boldsymbol{a}_2) \times (\boldsymbol{x}_2 - \boldsymbol{x}_1) = 0 \,. \tag{4.1}$$

Let A1 and A3 to A5 keep holding. Then, as is easily checked, theorem T2 keeps holding unlike theorems T1, T3, and T4. In spite of this the theory $\mathscr{T}^*$, based on A1, A2*, and A3 to A5, appears to substantially belong to classical mechanics by the notions of space, time, and mass. In fact it is easy to prove also in $\mathscr{T}^*$, with an essential use of the axiom A1(ii) (of phys. poss.), that ∞^1 mass distributions $i \mapsto \mu_i$ exist.

The replacement of axiom A2 (in $\mathscr{T}$) with A2* enlarges the class of the motions of $\{P_1, P_2\}$ compatible with the axioms of the theory being considered. In order to deal with gravitation, let us restrict this class by means of some conditions on the motions phys. poss. for $\{P_1, P_2\}$. The following axiom on the orbits of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ is reached by qualitative observations, it is a theorem, T4(iii), in $\mathscr{T}$ and certainly weaker than the requirement that, for exemple, these orbits should be conic.

A6. *In* $\mathscr{R}_{\mathscr{T}\mathscr{B}}$, *which has its origin in* P_1, *the motion of* P_2 *is plane.*

This axiom will be exploited only from section 6 on.

Let $\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ the force, relative to $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ above, exterted by the particle P_1 of mass μ_1 on the particle P_2 of mass μ_2. Hence $(4.2)_1$ below holds

$$\mu^* \ddot{\boldsymbol{q}} = \mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \equiv \dot{\boldsymbol{q}} \wedge \operatorname{rot} \boldsymbol{\alpha}(\boldsymbol{q}) + \operatorname{grad} U(\boldsymbol{q}) \,. \tag{4.2}$$

By A5 $(4.2)_2$ holds for some functions $\boldsymbol{\alpha}(\boldsymbol{q})$ and $U(\boldsymbol{q})$.

Let $\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ be the effective force exerted by P_1 on P_2, that is, the one relative to an inertial frame $R_{\mathscr{I}}$ having the same orthonormal basis $\{\boldsymbol{e}_k\}$ as $\mathscr{R}_{\mathscr{T}\mathscr{B}}$. Then

$$\mu_1 \ddot{\boldsymbol{x}}_1 = \boldsymbol{F}(-\boldsymbol{q}, -\dot{\boldsymbol{q}}) \,, \qquad \mu_2 \ddot{\boldsymbol{x}}_2 = \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \,; \tag{4.3}$$

hence by $(4.2)_1$, $(2.9)_1$, and $(3.1)_2$,

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) &= \frac{\mu_1 \mu_2}{\mu_1 + \mu_2} (\ddot{\boldsymbol{x}}_2 - \ddot{\boldsymbol{x}}_1) = \\ &= \frac{\mu_1}{\mu_1 + \mu_2} \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \frac{\mu_2}{\mu_1 + \mu_2} \boldsymbol{F}(-\boldsymbol{q}, -\dot{\boldsymbol{q}}) \,. \end{aligned} \tag{4.4}$$

Therefore the analogue of (2.7) holds for $\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, so that the same can be said of the consequence (2.8) of (2.7). Thus $\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ has the form

$$\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathscr{A}(q, \dot{q}, \boldsymbol{q}\times\dot{\boldsymbol{q}})\boldsymbol{q} + \mathscr{B}(q, \dot{q}, \boldsymbol{q}\times\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}} + \mathscr{C}(q, \dot{q}, \boldsymbol{q}\times\dot{\boldsymbol{q}})\boldsymbol{q}\wedge\dot{\boldsymbol{q}}\,. \tag{4.5}$$

In order to characterize $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ on the basis of (4.2), let us now reason like in the proof of theorem T4 in section 3. By an essential use of axiom A1(i) (of phys. poss.), from (4.2) and (4.5) with $\dot{\boldsymbol{q}} = \boldsymbol{0}$ we deduce that

$$U(\boldsymbol{q}) = \tilde{U}(q) = \int \mathscr{A}(q, 0, 0)\,q\,dq\,. \tag{4.6}$$

Furthermore let us note that the term $\operatorname{rot}\boldsymbol{\alpha}(\boldsymbol{q})$ in (4.2) is isotropic iff for some function $\mathscr{D}(q)$, $\mathscr{D}\colon \mathbb{R}^+ \to \mathbb{R}$,

$$\operatorname{rot}\boldsymbol{\alpha}(\boldsymbol{q}) = \mathscr{D}(q)\boldsymbol{q}\,. \tag{4.7}$$

Under the definition

$$\Phi(q) = \int \mathscr{D}(q)\,q\,dq\,, \tag{4.8}$$

equality (4.7) becomes

$$\operatorname{rot}\boldsymbol{\alpha}(\boldsymbol{q}) = \operatorname{grad}\Phi(q)\,, \qquad \text{hence } \Delta\Phi = 0\,. \tag{4.9}$$

The general solution of $(4.9)_2$ is

$$\Phi(q) = -\frac{K}{q} + K_1 \qquad (K, K_1 \in \mathbb{R})\,, \tag{4.10}$$

so that $(4.9)_1$ becomes

$$\operatorname{rot}\boldsymbol{\alpha}(\boldsymbol{q}) = \frac{K}{q^3}\boldsymbol{q}\,. \tag{4.11}$$

Some solutions of (4.11) can be found easily in spherical co-ordinates

r, θ, φ $(r \equiv q)$. In fact the system

$$(4.12)\qquad \begin{cases} (\operatorname{rot}\boldsymbol{\alpha})_r \equiv \dfrac{1}{r\sin\theta}\left[\dfrac{\partial}{\partial\theta}(\sin\theta\alpha_\varphi) - \dfrac{\partial\alpha_\theta}{\partial\varphi}\right] = \dfrac{K}{r^2} \\ (\operatorname{rot}\boldsymbol{\alpha})_\theta \equiv \dfrac{1}{r\sin\theta}\dfrac{\partial\alpha_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\alpha_\varphi) = 0 \\ (\operatorname{rot}\boldsymbol{\alpha})_\varphi \equiv \dfrac{1}{r}\left[\dfrac{\partial}{\partial r}(r\alpha_\theta) - \dfrac{\partial\alpha_r}{\partial\theta}\right] = 0 \end{cases}$$

$$(\boldsymbol{\alpha} = \alpha_r\hat{r} + \alpha_\theta\hat{\theta} + \alpha_\varphi\hat{\varphi})$$

is solved, for esample, by

$$(4.13)\qquad \boldsymbol{\alpha} = \left(0, 0, -\frac{K\operatorname{ctg}\theta}{r}\right) = -\frac{K\operatorname{ctg}\theta}{r}\hat{\varphi},$$

and in cartesian co-ordinates $\left(\boldsymbol{\alpha} = \sum_{h=1}^{3}\alpha_h\boldsymbol{e}_h\right)$ we have that

$$(4.14)\qquad \boldsymbol{\alpha} = \left(\frac{Kq_3q_2}{q(q_1^2+q_2^2)}, -\frac{Kq_3q_1}{q(q_1^2+q_2^2)}, 0\right) \in \mathscr{C}^2(\mathbb{R}^3\setminus A; \mathbb{R}^3),$$

where $A = \mathbb{R}\boldsymbol{e}_3$. Then for some function $\tilde{U}(q)$ of class $\mathscr{C}^{(2)}$ and some $K \in \mathbb{R}$ the force function (4.2) has the form

$$(4.15)\qquad \mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{K}{q^3}\boldsymbol{q}\wedge\dot{\boldsymbol{q}} + \operatorname{grad}\tilde{U}(q),$$

so that it obviously has a continuous extension onto $[\mathbb{R}^3\setminus\{\boldsymbol{0}\}]\times\mathbb{R}^3$; hence it is compatible with A5.

By (4.4)

$$(\mu_1 + \mu_2)\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mu_1\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mu_2\boldsymbol{F}(-\boldsymbol{q}, -\dot{\boldsymbol{q}}),$$

so that by (4.15) we easily obtain

$$(4.16)\qquad \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{\bar{K}}{q^3}\boldsymbol{q}\wedge\dot{\boldsymbol{q}} + \operatorname{grad}\tilde{U}(q),$$

$$\text{where } \bar{K} = K\frac{\mu_1+\mu_2}{\mu_1-\mu_2} = \bar{K}(\mu_1, \mu_2).$$

Remark that the presence of the factor $(\mu_1 - \mu_2)^{-1}$ in (4.16) does not imply a singularity for $\mu_1 = \mu_2$. In fact the integration constant K can depend on μ_1 and μ_2. The assumptions that

$$(4.17)\qquad \lim_{\mu_i \to 0^+} \bar{K} = 0 \quad (i = 1, 2)\,, \qquad \bar{K}(\xi_1 + \xi_2, \eta_1 + \eta_2) = \sum_{i,j=1}^{2} \bar{K}(\xi_i, \eta_j)$$

on $\bar{K}$—cf. $(4.16)_{2,3}$—are natural and imply that

$$(4.18)\qquad \bar{K}(\mu_1, \mu_2) = h\mu_1\mu_2\,, \qquad K(\mu_1, \mu_2) = h\mu_1\mu_2 \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\,,$$

where $h \in \mathbb{R}$ is a universal constant.

Note that the force law (4.16) is compatible with axiom A2*, in that (4.16) yields

$$(4.19)\qquad (\mu_1 \ddot{\boldsymbol{x}}_1 + \mu_2 \ddot{\boldsymbol{x}}_2) \times (\boldsymbol{x}_2 - \boldsymbol{x}_1) = [\boldsymbol{F}(-\boldsymbol{q}, -\dot{\boldsymbol{q}}) + \boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})] \times \boldsymbol{q} = 0\,.$$

5. Orbits of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ in the theory $\mathscr{T}^*$.

The dynamical equation of P_2 in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ is

$$(5.1)\qquad \mu^* \ddot{\boldsymbol{q}} = \mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{K}{q^3} \boldsymbol{q} \wedge \dot{\boldsymbol{q}} + \operatorname{grad} \tilde{U}(q)\,.$$

Let us call *Reduced Moment of Momentum* of P_2 the vector $\boldsymbol{L}^*$:

$$(5.2)\qquad \boldsymbol{L}^* = \boldsymbol{q} \wedge \mu^* \dot{\boldsymbol{q}}\,.$$

Then

$$(5.3)\qquad \mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = -\frac{K}{\mu^* q^3} \boldsymbol{L}^* + \operatorname{grad} \tilde{U}(q)\,,$$

so that

$$(5.4)\qquad \dot{\boldsymbol{L}}^* = \boldsymbol{q} \wedge \mu^* \ddot{\boldsymbol{q}} = \boldsymbol{q} \wedge \mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\Omega} \wedge \boldsymbol{L}^*\,, \qquad \text{where } \boldsymbol{\Omega} = -\frac{K\boldsymbol{q}}{\mu^* q^3}\,.$$

T1*. *In the theory $\mathscr{T}^*$, $|\boldsymbol{L}^*|$ is a first integral of the motion of P_2 with respect to $\mathscr{R}_{\mathscr{T}\mathscr{B}}$:*

$$(5.5)\qquad \frac{d}{dt} |\boldsymbol{L}^*| = 0\,, \quad \textit{i.e.}\ |\boldsymbol{L}^*| = \textit{const.}$$

Let $\mathscr{R}_{\mathscr{D}}$ be a generally non-inertial frame $\{P_1, \boldsymbol{J}_1, \boldsymbol{J}_2, \boldsymbol{J}_3\}$ in whose motion with respect to $\mathscr{R}_{\mathscr{T}\mathscr{B}}$, to be regarded as the dragging motion, the plane $(P_1, \boldsymbol{J}_1, \boldsymbol{J}_2)$ is rolling on P_2's trajectory in $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ without sliding. The index $A[R]$ will be used for quantities referred to $\mathscr{R}_{\mathscr{T}\mathscr{B}}[\mathscr{R}_{\mathscr{D}}]$ (and regarded therefore as absolute [relative]). For P_2 we have

$$(5.6)\qquad \begin{cases} \boldsymbol{v}^A = \dot{\boldsymbol{q}} = \boldsymbol{v}^R + \boldsymbol{v}^D \qquad (\boldsymbol{v}^D = \boldsymbol{0})\,, \\ \boldsymbol{a}^A = \ddot{\boldsymbol{q}} = \boldsymbol{a}^R + \boldsymbol{a}^D + 2\boldsymbol{\omega}^D \wedge \boldsymbol{v}^R\,. \end{cases}$$

(α) *Determination of* $\boldsymbol{\omega}^D$. Since $\mathscr{R}_{\mathscr{D}}$ is rotating with respect to $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ around an axis through P_1, we have that

$$(5.7)\qquad \boldsymbol{v}^D = \boldsymbol{\omega}^D \wedge \boldsymbol{q}\,,$$

and by $(5.6)_3$, $\boldsymbol{\omega}^D \| \boldsymbol{q}$. Furthermore, by (5.2) $\boldsymbol{L}^*$ is orthogonal to the plane $(P_1, \boldsymbol{J}_1, \boldsymbol{J}_2)$, so that for suitable orientation of $\boldsymbol{J}_1$, $\boldsymbol{J}_2$ and for $\boldsymbol{q} \wedge \dot{\boldsymbol{q}} \neq \boldsymbol{0}$

$$(5.8)\qquad \boldsymbol{J}_3 = \frac{\boldsymbol{L}^*}{|\boldsymbol{L}^*|} = \frac{\mu^*}{L^*}\, \boldsymbol{q} \wedge \dot{\boldsymbol{q}} \qquad (L^* = |\boldsymbol{L}^*|)\,.$$

Poisson's equation for $\boldsymbol{J}_3$ reads

$$(5.9)\qquad \frac{d}{dt} \boldsymbol{J}_3 = \boldsymbol{\omega}^D \wedge \boldsymbol{J}_3\,,$$

and by (5.8), (5.5), and (5.3) we also have that

$$(5.10)\qquad \frac{d}{dt} \boldsymbol{J}_3 = \frac{\mu^*}{L^*}\, \boldsymbol{q} \wedge \ddot{\boldsymbol{q}} = \frac{1}{L^*}\, \boldsymbol{q} \wedge \left(- \frac{K}{\mu^* q^3}\, \boldsymbol{L}^* \right) = - \frac{K}{\mu^* q^3}\, \boldsymbol{q} \wedge \boldsymbol{J}_3\,,$$

hence, by $(5.4)_4$, $(\boldsymbol{\omega}^D - \boldsymbol{\Omega}) \wedge \boldsymbol{J}_3 = \boldsymbol{0}$, i.e. $\boldsymbol{\omega}^D - \boldsymbol{\Omega} = \chi \boldsymbol{J}_3$ for some $\chi \in \mathbb{R}$. Furthermore $\boldsymbol{\omega}^D \| \boldsymbol{q} \| \boldsymbol{\Omega}$. Hence $\chi = 0$ and

$$(5.11)\qquad \boldsymbol{\omega}^D = \boldsymbol{\Omega} = - \frac{K}{\mu^* q^3}\, \boldsymbol{q}\,.$$

(β) *Determination of the angular velocity* $\boldsymbol{\omega}^A[\boldsymbol{\omega}^R]$ *of the plane* $(P_1, \boldsymbol{J}_3, \boldsymbol{q}/q)$ *with respect to* $\mathscr{R}_{\mathscr{T}\mathscr{B}}[\mathscr{R}_{\mathscr{D}}]$. Since $P_2 = P_1 + \boldsymbol{q}$ always belongs to the plane $(P_1, \boldsymbol{J}_1, \boldsymbol{J}_2)$, we have $\boldsymbol{\omega}^R \| \boldsymbol{J}_3$. Hence

$$\boldsymbol{\omega}^R = \dot{\Theta}\boldsymbol{J}_3 , \quad \text{where } \Theta = \widehat{\boldsymbol{J}_1 P_1 P_2} . \tag{5.12}$$

Furthermore by (5.2) and (5.5), the vector $\boldsymbol{L}^*$ is constant in $\mathcal{R}_{\mathcal{D}}$, the area swept out in unit time is constant, and

$$\boldsymbol{L}^* = \mu^* \dot{\Theta} q^2 \boldsymbol{J}_3 ; \tag{5.13}$$

by (5.12) we have $\boldsymbol{L}^* = \mu^* q^2 \boldsymbol{\omega}^R$, that is

$$\boldsymbol{\omega}^R = \frac{L^*}{\mu^* q^2} \boldsymbol{J}_3 . \tag{5.14}$$

By the composition theorem for angular velocities

$$\boldsymbol{\omega}^A = \boldsymbol{\omega}^R + \boldsymbol{\omega}^D = \frac{L^*}{\mu^* q^2} \boldsymbol{J}_3 - \frac{K}{\mu^* q^3} \boldsymbol{q} . \tag{5.15}$$

Hence

$$\frac{d}{dt} \hat{q} = \boldsymbol{\omega}^A \wedge \hat{q} , \quad \text{where } \hat{q} = \boldsymbol{q}/q . \tag{5.16}$$

(γ) *Expression of the time derivative of* $\boldsymbol{\omega}^A$ *with respect to* $\mathcal{R}_{\mathcal{T}\mathcal{B}}$. By (5.15), (5.11) and (5.16)

$$\begin{aligned}\frac{d}{dt} \boldsymbol{\omega}^A &= \frac{d}{dt}\left(\frac{L^*}{\mu^* q^2} \boldsymbol{J}_3 - \frac{K}{\mu^* q^2} \hat{q}\right) = \frac{d}{dt}\left(\frac{1}{\mu^* q^2}\right)(L^* \boldsymbol{J}_3 - K\hat{q}) + \\ &+ \frac{1}{\mu^* q^2} \frac{d}{dt}(L^* \boldsymbol{J}_3 - K\hat{q}) = -\frac{2(dq/dt)}{\mu^* q^3}(L^* \boldsymbol{J}_3 - K\hat{q}) + \\ &+ \frac{1}{\mu^* q^2}(L^* \boldsymbol{\omega}^D \wedge \boldsymbol{J}_3 - K\boldsymbol{\omega}^A \wedge \hat{q}) = -2\frac{dq/dt}{q} \boldsymbol{\omega}^A + \\ &+ \frac{1}{\mu^* q^2}\left[L^*\left(-\frac{K\boldsymbol{q}}{\mu^* q^3}\right) \wedge \boldsymbol{J}_3 - K\left(\frac{L^*}{\mu^* q^2} \boldsymbol{J}_3 - \frac{K}{\mu^* q^3} \boldsymbol{q}\right) \wedge \hat{q}\right],\end{aligned}$$

hence

$$\frac{d}{dt} \boldsymbol{\omega}^A = \frac{d}{dt}(\ln q^{-2}) \boldsymbol{\omega}^A , \tag{5.17}$$

Thence we lastly obtain that

$$\text{(5.18)} \qquad \frac{d}{dt}\hat{\omega}^A = \mathbf{0}\,, \quad \frac{d}{dt}(\hat{q}\times\hat{\omega}^A) = 0\,, \quad \text{where } \hat{\omega}^A = \boldsymbol{\omega}^A/|\boldsymbol{\omega}^A|\,.$$

Some results obtained in this section within $\mathscr{T}^*$ are summarized in the following theorem

T2*. *According to the theory* $\mathscr{T}^*$, *based on axioms* A1, A2*, *and* A3 *to* A5, (i) *the most general force function* $\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ $[\boldsymbol{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})]$ *relative to* $\mathscr{R}_{\mathscr{T}\mathscr{B}}[\mathscr{R}_{\mathscr{I}}]$ *is expressed by* (4.15) [(4.16)], (ii) *the motions of* P_2 *compatible with these forces have trajectories in* $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ *lying* (*each*) *on a fixed conic surface in* $\mathscr{R}_{\mathscr{T}\mathscr{B}}$ *that has the vertex at* P_1, *the axis parallel with* $\hat{\omega}^A$, *and semi-aperture* σ *where*

$$\text{(5.19)} \qquad \sigma = \arccos|\hat{\omega}^A \times \hat{q}| = \arccos\left\{\frac{|K|}{\sqrt{K^2 + (L^*)^2}}\right\}.$$

As was expected, for $K \to 0$ we have $\sigma \to \pi/2$, i.e. *the conic surface reduces to a plane.* It can be asserted that $\mathscr{T}$ is the special case of $\mathscr{T}^*$ obtained for $K = 0$—cf. (4.16).

It is evident that P_2's *plane orbits in* $\mathscr{R}_{\mathscr{T}\mathscr{B}}$—cf. A6—*are necessarily conics*; however it must be remarked that the plane through P_2's trajectory l_2 can contain P_1 only when l_2 belongs to a straight line.

6. On the theory $\mathscr{T}^* +$ A6. On plane motions.

We want to determine the most general expression of $\mathscr{F}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ —cf. (4.15)—in the theory $\mathscr{T}^*$+A6 obtained from $\mathscr{T}^*$ by the addition of A6 as an axiom. It suffices to require the following identity

$$\text{(6.1)} \qquad \dot{\boldsymbol{b}}\wedge\boldsymbol{b} = \mathbf{0}\,, \quad \text{where } \boldsymbol{b} = \dot{\boldsymbol{q}}\wedge\ddot{\boldsymbol{q}}\,,$$

i.e. that the about vector $\boldsymbol{b}$, which is orthogonal to the osculatory plane for P_2's trajectory, should have an invariant direction. By (4.15) and the following definition of $A = A(q)$

$$\text{(6.2)} \qquad A(q) = \mathscr{A}(q, 0, 0) \quad \big(A(q)\boldsymbol{q} = \operatorname{grad}\tilde{U}(q)\text{—cf. (4.6)}\big)\,,$$

we have that

$$(6.3)\qquad \boldsymbol{b} = \dot{\boldsymbol{q}}\wedge\frac{\mathscr{F}(\boldsymbol{q},\dot{\boldsymbol{q}})}{\mu^*} = \dot{\boldsymbol{q}}\wedge\left(-\frac{K}{\mu^* q^3}\boldsymbol{q}\wedge\dot{\boldsymbol{q}} + \frac{A}{\mu^*}\boldsymbol{q}\right) =$$

$$= \left(-K\frac{\dot{\boldsymbol{q}}\times\dot{\boldsymbol{q}}}{\mu^* q^3}\right)\boldsymbol{q} + \left(\frac{K}{\mu^* q^3}\boldsymbol{q}\times\dot{\boldsymbol{q}}\right)\dot{\boldsymbol{q}} + \left(\frac{A}{\mu^*}\right)\dot{\boldsymbol{q}}\wedge\boldsymbol{q}\,.$$

Remembering that $dq/dt = \boldsymbol{q}\times\dot{\boldsymbol{q}}/q$ and writing A' for dA/dq, we have that

$$\dot{\boldsymbol{b}} = \dot{\boldsymbol{q}}\wedge\dddot{\boldsymbol{q}} = \dot{\boldsymbol{q}}\wedge\left[\frac{3K\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q^5}\boldsymbol{q}\wedge\dot{\boldsymbol{q}} - \frac{K}{\mu^* q^3}\boldsymbol{q}\wedge\left(-\frac{K}{\mu^* q^3}\boldsymbol{q}\wedge\dot{\boldsymbol{q}} + \frac{A}{\mu^*}\boldsymbol{q}\right) + \right.$$

$$\left. + \frac{A'\,\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q}\boldsymbol{q} + \frac{A}{\mu^*}\dot{\boldsymbol{q}}\right] = \frac{3K\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q^5}(\dot{\boldsymbol{q}}\times\dot{\boldsymbol{q}}\boldsymbol{q} - \boldsymbol{q}\times\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}) +$$

$$+ \left(\frac{K}{\mu^* q^3}\right)^2 \boldsymbol{q}\times\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}\wedge\boldsymbol{q} + \frac{A'\,\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q}\dot{\boldsymbol{q}}\wedge\boldsymbol{q}\,,$$

that is,

$$(6.4)\qquad \dot{\boldsymbol{b}} = \left[\frac{3K\boldsymbol{q}\times\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}\times\dot{\boldsymbol{q}}}{\mu^* q^5}\right]\boldsymbol{q} + \left[-\frac{3K(\boldsymbol{q}\times\dot{\boldsymbol{q}})^2}{\mu^* q^5}\right]\dot{\boldsymbol{q}} +$$

$$+ \left[\left(\frac{K}{\mu^* q^3}\right)^2 \boldsymbol{q}\times\dot{\boldsymbol{q}} + \frac{A'\,\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q}\right]\dot{\boldsymbol{q}}\wedge\boldsymbol{q}\,.$$

For $\boldsymbol{q}\wedge\dot{\boldsymbol{q}} \neq \boldsymbol{0}$ the vectors $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and $\dot{\boldsymbol{q}}\wedge\boldsymbol{q}$ are linearly independent. Hence (6.1) holds iff the corresponding components of the vectors $\boldsymbol{b}$ and $\dot{\boldsymbol{b}}$, put in evidence by $(6.3)_4$ and (6.4), are proportional, i.e.

$$(6.5)\qquad \frac{-\dfrac{K\dot{\boldsymbol{q}}\times\dot{\boldsymbol{q}}}{\mu^* q^3}}{\dfrac{3K}{\mu^* q^5}\boldsymbol{q}\times\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}\times\dot{\boldsymbol{q}}} = \frac{\dfrac{K\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q^3}}{-\dfrac{3K(\boldsymbol{q}\times\dot{\boldsymbol{q}})^2}{\mu^* q^5}} = \frac{\dfrac{A}{\mu^*}}{\left(\dfrac{K}{\mu^* q^3}\right)^2\boldsymbol{q}\times\dot{\boldsymbol{q}} + \dfrac{A'\,\boldsymbol{q}\times\dot{\boldsymbol{q}}}{\mu^* q}}\,.$$

While $(6.5)_1$ holds identically, $(6.5)_2$ is equivalent with equation

$$(6.6)\qquad A + \frac{q}{3}A' = -\frac{K^2}{3\mu^* q^4}\,,$$