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Majorizing-Injectivity in Abelian Lattice-Ordered Groups.

ANTHONY W. HAGER - JAMES J. MADDEN (*)

Introduction.

$\mathcal{L}A$ stands for the category of abelian l-groups with l-homomorphisms and Arch the full subcategory with archimedean objects. It is well-known that neither $\mathcal{L}A$ nor Arch has any nontrivial injectives. In contrast, we shall describe here a quite nice theory of « majorizing-injectivity », as follows (with most definitions given in the text):

The main theorem is that, in $\mathcal{L}A$, K is majorizing-injective iff K is (conditionally Dedekind) complete and divisible. The necessity in this theorem is new here; the proof is *via* a simple ultrapower construction. The sufficiency was known, but the proof presented below is new and most natural, being a model of Banaschewski's abstract scheme of relative injectivity; the existential element of the proof is provided by Pierce's theorem that $\mathcal{L}A$ has the Amalgamation Property.

For Arch, the Amalgamation Property fails (also shown by Pierce), but the theorem above implies that Arch is « majorizing-injectively complete », and it follows that majorizing embeddings amalgamate.

We note here (in the introduction only) that the results specialize to the category, say $\mathcal{U}$, of archimedean l-groups with distinguished strong order unit and unit-preserving l-homomorphisms: Here, every morphism is majorizing, and the theorem yields Popa's result that $\mathcal{U}$-injective = complete divisible; so $\mathcal{U}$ is injectively-complete, and the Amalgamation Property follows.

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1. The $\mathcal{M}$-injectives.

All objects and morphisms will be in $\mathcal{LA}$, the category of abelian l-groups with l-group homomorphisms. This section, containing the main theorem, takes place « within $\mathcal{LA}$ » and is reasonably self-contained. Most of what we need about $\mathcal{LA}$ is standard, and can be found in [BKW].

A morphism $A \xrightarrow{\mu} B$ is called: *majorizing* if $\forall b \in B\ \exists a \in A$ with $|b| \leqslant \mu(a)$; *large* if each ideal $I \neq (0)$ of B has $I \cap \mu(A) \neq (0)$; an *embedding* if μ is one-to-one; an *essential* morphism if μ is a large embedding.

Let $\mathcal{M}$ be the class of majorizing embeddings. An $\mathcal{M}$-*essential* morphism is an essential morphism which is in $\mathcal{M}$—a large majorizing embedding. An object $K \in |\mathcal{LA}|$ is called $\mathcal{M}$-*injective* if given $G \xrightarrow{\varphi} K$ and $G \xrightarrow{\mu} H \in \mathcal{M}$ there is $H \xrightarrow{\varepsilon} K$ with $\varepsilon\mu = \varphi$.

1.1 THEOREM. K is $\mathcal{M}$-injective iff K is complete and divisible.

1.1 is the main theorem. The proof will use (c) of the following.

1.2 THEOREM. These are equivalent conditions on K.

(a) K is $\mathcal{M}$-injective.

(b) Any $K \to A \in \mathcal{M}$ has a left inverse.

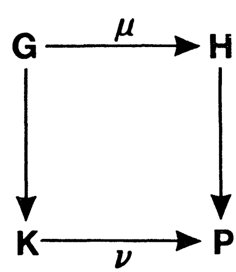
(c) Any $\mathcal{M}$-essential $K \to A$ is an isomorphism.

1.2 is one of the important *desiderata* for injectivity. The proof is quite « formal » from the following properties of $\mathcal{M}$ and the $\mathcal{M}$-essential morphisms.

1.3 LEMMA. (a) Let $\mu \in \mathcal{M}$. Then, μ is $\mathcal{M}$-essential iff $\alpha\mu \in \mathcal{M}$ implies $\alpha \in \mathcal{M}$.

(b) If $\mu \in \mathcal{M}$, then there is α with $\alpha\mu$ $\mathcal{M}$-essential.

(c) $\mathcal{LA}$ has pushouts, and for a pushout diagram



$\mu \in \mathcal{M}$ implies $\nu \in \mathcal{M}$.

The fact that, for a pushout as in 1.3 (c), μ embedding $\Rightarrow$ ν embedding, will be derived from the following (essentially equivalent) deep fact:

1.4 LEMMA (Pierce [P_1]). $\mathcal{L}A$ has the « Amalgamation Property »: Given embeddings $G \xrightarrow{\sigma_i} H_i$ $(i = 1, 2)$, there are embeddings $H_1 \xrightarrow{\tau_i} P$ $(i = 1, 2)$ with $\tau_1\sigma_1 = \tau_2\sigma_2$.

Before giving the proofs, we make some remarks about the related literature:

[V] shows, in essence, that in archimedean vector lattices, the complete objects are $\mathcal{M}$-essentially injective. [A] shows that in archimedean f-algebras with identity, the complete objects are $\mathcal{M}$-essentially injective. [L_1], [LS], [L_2] show that in vector lattices, the complete objects are $\mathcal{M}$-injective. [AHM] shows that in $\mathcal{L}A$, the complete vector lattices are $\mathcal{M}$-injective.

The proofs in [A] and [AHM] use representations of the algebras or groups as algebras or groups of extended real-valued functions, then the Gleason projectivity theorem in topology. The proofs in [V], [L_1], and [LZ] use the Kantorovich generalization of the Hahn-Banach Theorem. The proof in [L_2] is a direct argument by Zorn's Lemma (as Kantorovich/Hahn-Banach).

The connection between these various results are discussed in [AHM], including the fact that the result for Arch implies the result for $\mathcal{L}A$; this point is adumbrated in § 2 below, particularly 2.4.

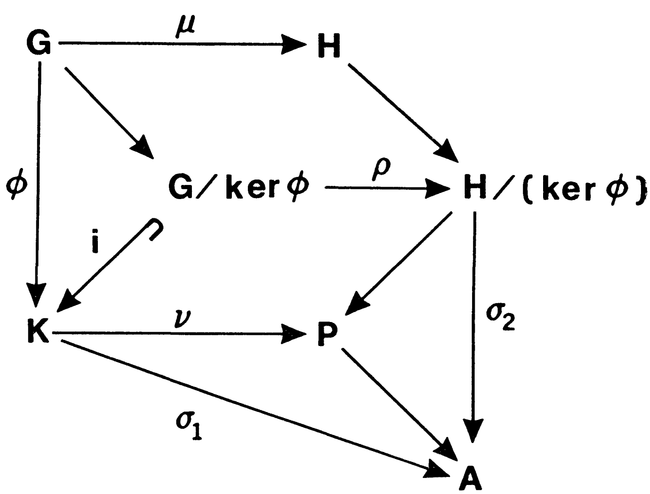
It will be clear that everything said here is true for vector lattices. The complete situation for f-rings is unclear to us.

PROOF OF 1.3. (a) $\Rightarrow$. Clearly, $\alpha\mu$ majorizing implies μ majorizing. If $\alpha\mu$ is an embedding, then $\ker(\alpha) \cap \operatorname{im}(\mu) = (0)$, and $\ker[\alpha)$ will be (0) when μ is large.

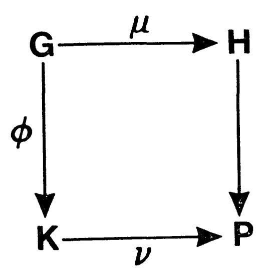
$\Leftarrow$. If $A \xrightarrow{\mu} B$ is not large, then $I \cap \mu(A) = (0)$ for some ideal $I \neq (0)$. Then for the canonical projection $B \xrightarrow{\alpha} B/I$, we have $\alpha \notin \mathcal{M}$, while $\alpha\mu \in \mathcal{M}$ whenever $\mu \in \mathcal{M}$.

(b) Given $A \xrightarrow{\mu} B$, let I be an ideal of B maximal for $I \cap \mu(A) = (0)$. The projection $B \xrightarrow{\alpha} B/I$ has $\alpha\mu$ large, and $\alpha\mu \in \mathcal{M}$ whenever $\mu \in \mathcal{M}$.

(c) $\mathcal{L}A$ is a variety (equational class) by [B_4], and hence has free sums and pushouts, by [J]. We first show, indirectly, that pushouts preserve embeddings. Consider



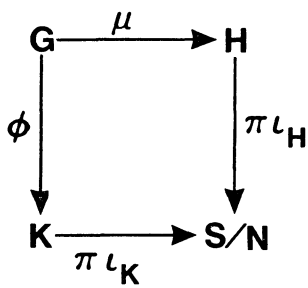
where φ and the embedding μ are given, then a homomorphism theorem produces the square with ϱ on the bottom, which is clearly a pushout square, with ϱ clearly an embedding. We now push out ϱ and i to the square with terminus P. Then, the square



is a pushout, being made from the two interfacing pushout squares in the above diagram. We are to show ν is an embedding. To do this, apply 1.4 to the embeddings ϱ and i: there are *embeddings* σ_1 and σ_2 with $\sigma_1 i = \sigma_2 \varrho$. But since P came from pushing out ϱ and i, there is $P \xrightarrow{\sigma} A$ with $\sigma\nu = \sigma_1$; thus ν is an embedding.

We now show that for a pushout square as above, μ majorizing $\Rightarrow \nu$ majorizing. This can be done directly: A pushout square as in 1.3 (c)

« is » (see [J]) really



where

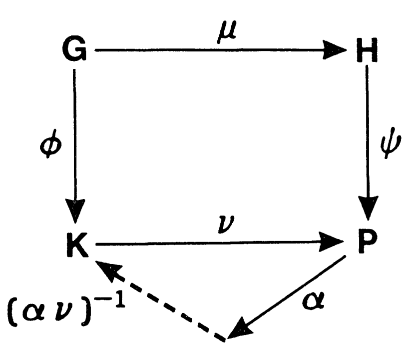
$$H \xrightarrow{\iota_H} S \xleftarrow{\iota_K} K$$

is the free sum, N is the ideal generated by all $\iota_H(\mu(g)) - \iota_K(\varphi(g))$ $(g \in G)$, and $S \xrightarrow{\pi} S/N$ is the projection. The point is that $\iota_H(H) \cup \cup\, \iota_K(K)$ generates S as an ideal [M]. Thus, with μ majorizing, so does $\iota_H(\mu(G)) \cup \iota_K(K)$, and so $\pi\big(\iota_H(\mu(G))\big) \cup \pi(\iota_K(K))$ generates S/N as an ideal, which means that $\pi\iota_K$ is majorizing.

Proof of 1.2. $(a) \Rightarrow (b)$. Since $K \xrightarrow{id} K$ lifts over μ.

$(b) \Rightarrow (c)$. Let $K \xrightarrow{\mu} A$, be $\mathcal{M}$-essential. By (b), there is $K \xleftarrow{\alpha} A$ with $\alpha\mu = id_K$. By 1.3 (a), $\alpha \in \mathcal{M}$, so α is an embedding and μ is an isomorphism.

$(c) \Rightarrow (a)$. Suppose given $G \xrightarrow{\mu} H \in \mathcal{M}$ and $G \xrightarrow{\varphi} K$, and consider the diagram



in which: The square is the pushout from 1.3 (*c*), so that $\nu \in \mathcal{M}$. Then 1.3 (*b*) provides α with $\alpha\nu$ $\mathcal{M}$-essential. Assuming (*c*), $\alpha\nu$ is an isomorphism, and $(\alpha\nu)^{-1}\,\alpha\psi$ is the desired lifting of φ over μ.

PROOF OF 1.1. We shall use 1.2 (*c*).

Suppose K is complete and divisible, and let $K \xrightarrow{\mu} A$ be $\mathcal{M}$-essential. We suppress μ and write $K \subseteq A$. Since K is divisible and large, K is dense. Since K is archimedean (being complete, by 11.2.2 of [BKW]), A is archimedean by 1.5 below. We now have $a = \bigvee_A\{k \in K | k \leqslant a\}$ for each $a \in A^+$, because K is dense in archimedean A (11.3.6 of [BKW]). Now $\bigvee_K\{k \in K | k \leqslant a\}$ exists, because K is complete and majorizes A. But $\bigvee_A(\ ) = \bigvee_K(\ )$ whenever the latter exists, because K is dense (11.3.5 of [BKW]). Thus $a^+ \in K$; hence $A^+ \subseteq K$ and $A = K$.

Conversely, suppose K satisfies 1.2 (*c*). Lemma 1.6 below shows that K must be archimedean. So let $K \xrightarrow{\mu} A$ be the embedding of K into the completion of the divisible hull of K (see [BKW], pp. 31 and 237). Since μ is large and majorizing, it is an isomorphism, hence K is complete and divisible.

This completes the proof of 1.1. modulo 1.5 and 1.6 below.

1.5 LEMMA [LZ]. Let $G \to H$ be $\mathcal{M}$-essential: If G is archimedean, then H is archimedean.

PROOF. $A \in \mathfrak{L}A$ is archimedean iff $I_A = (0)$, where $I_A = \{a | \exists b \text{ with } n|a| \leqslant b\ \forall n \in Z\}$ is the ideal of infinitely small elements. Writing $G \subseteq H$, we have here that $I_A = I_H \cap G$ because G majorizes, $I_A = (0)$ because G is archimedean, whence $I_H = (0)$ because G is large.

1.6 LEMMA. If G is not archimedean, there is $\mathcal{M}$-essential $G \xrightarrow{\mu} H$ which is not onto.

PROOF. In an *l*-group, we write $a \ll b$ to mean that $n|a| \leqslant |b|$ $\forall n \in Z^+$. Then, $G \in \text{Arch}$ iff $a \ll b \Rightarrow a = 0$. We shall prove the following below:

1.7 LEMMA. Let G be divisible, let $b \in G$, and suppose that $\{s \in G | 0 < s \ll b\} \neq \emptyset$. Then there is $E \in \mathfrak{L}A$, an embedding $G \hookrightarrow E$, and $h \in E$ such that (1) $s \in G$ and $s \ll b \Rightarrow s \ll h$; (2) $h \ll b$.

We prove 1.6: Suppose G is not archimedean. We can assume G divisible, by replacing G by its divisible hull. Take b as in 1.7 and apply 1.7 to produce $G \hookrightarrow E$ and h, etc. We can assume that G majorizes E, by replacing E by the ideal generated by G (which ideal contains h, since $h < b$). By 1.3 (b), choose an ideal I of E maximal for $I \cap G = (0)$, and let $G \to H$ be the embedding $G \hookrightarrow E \to E/I \equiv H$; this is majorizing, and *large*. Let $h' = h + I$. Since homomorphisms preserve $\ll$, we find that $s \in G$ and $s \ll b \Rightarrow s \ll h'$; and $h' \ll b$. Thus $h' \notin G$ (since $h' \in G$ would imply that $h' \ll h'$, which is impossible).

To prove 1.7, we use an ultraproduct construction. See [BS], but we indicate the basics:

If G is a set, S is a set, and $\mathcal{F}$ is an ultrafilter on S, the ultrapower $G^S/\mathcal{F}$ is G^S modulo $f \sim g \equiv \{s | f(x) = g(s)\} \in \mathcal{F}$. Then $G \to G^S/\mathcal{F}$ *via* $g \mapsto \bar{g} \equiv$ the constant function with value g. If G is partially ordered, then so is $G^S/\mathcal{F}$ *via* $[f] \leqslant [g] \equiv \{s | f(s) \leqslant g(s)\} \in \mathcal{F}$. If $G \in \mathfrak{L}\mathcal{A}$, one defines $+$ in a natural way so that $G^S/\mathcal{F} \in \mathfrak{L}\mathcal{A}$, and G divisible $\Rightarrow G^S/\mathcal{F}$ divisible.

1.8 LEMMA (Weinberg [W]). Let G be a partially ordered set, S a nonvoid strictly up-directed subset of G. Then there is an ultrafilter $\mathcal{F}$ on S and $h \in G^S/\mathcal{F}$ such that

(1) $\bar{g} < h$ iff $g <$ some $s \in S$; and

(2) $S < g \Rightarrow h < \bar{g}$.

PROOF. For $s \in S$, let $U_s = \{t | s < t\}$ and let $\mathcal{F}$ be any ultrafilter with $\mathcal{F} \supseteq \{U_s | s \in S\}$. Let h be the equivalence class of the inclusion function $S \to G$. (1) and (2) can be verified without difficulty.

We prove 1.7: Let G and b be as in 1.7, let $S = \{s \in G | 0 < s \ll b\}$, and apply 1.8 to produce $G^S/\mathcal{F} = E$ and $h \in E$, etc. Instead of $\bar{g}$ *per* 1.8, we just write g. (1) If $s \in S$ and $n \in Z^+$, then $ns < (n+1)s \in S$, so that $ns < h$ by 1.8 (1). Thus $s \ll h$. (2) If $n \in Z^+$, then $nS = S < b$, when $S < (1/n)b$ (G is divisible, recall), whence by 1.8 (2), $h < (1/n)b$, or $nh < b$. Thus, $h \ll b$.

This concludes the proof of 1.7, and so 1.6 is proved.

2. Behavior of $\mathcal{M}$-injectivity.

We describe the place of $\mathcal{M}$-injectivity in $\mathfrak{L}\mathcal{A}$ in Banaschewski's formal framework for relative injectivity given in [B_2, B_3]. The reader may not find it totally necessary to have [B_3] in hand, but this would help.

The setting is a category $\mathcal{C}$ with a distinguished class $\mathcal{E}$ of morphisms. $K \in |\mathcal{C}|$ is $\mathcal{E}$-injective if each $G \to K$ «lifts» over any $\mathcal{E}$-morphism out of G. An $\mathcal{E}$-essential morphism is a $\mu \in \mathcal{E}$ for which $\alpha\mu \in \mathcal{E} \Rightarrow \alpha \in \mathcal{E}$. An $\mathcal{E}$-injective hull of $G \in |\mathcal{C}|$ is an $\mathcal{E}$-essential $G \to K$ with K $\mathcal{E}$-injective. $\mathcal{E}$-injectivity is called «properly behaved» if (A) the analogue of 1.2 holds, and (B) each object has an essentially unique $\mathcal{E}$-injective hull, and $(\mathcal{C})$ every $\mathcal{E}$-injective hull satisfies certain conditions spelled out for $(\mathcal{L}A, \mathcal{M})$ in 2.3 below.

$[B_2, B_3]$ present axioms $E1$-$E6$ on $(\mathcal{C}, \mathcal{E})$ ensuring proper behavior, and, in particular, show that $E1$-$E4 \Rightarrow (A)$. Our 1.3 (b) is $E3$ for $(\mathcal{L}A, \mathcal{M})$. $E4$ is that $G \xrightarrow{\mu} H \in \mathcal{E}$ and $G \xrightarrow{\varphi} K$ embed into a square with μ on top and with bottom $\in \mathcal{E}$; this follows from existence of pushouts «preserving $\mathcal{E}$» as in 1.3 (c). Our proof that 1.3 $\Rightarrow$ 1.2 is essentially the proof that $E1$-$E4 \Rightarrow (A)$.

$(\mathcal{L}A, \mathcal{M})$ doesn't satisfy (B), of course, but 1.1 and 1.2 easily yield

2.1 For $(\mathcal{L}A, \mathcal{M})$: $\mathcal{M}$-injective hulls are unique, and G has an $\mathcal{M}$-injective hull (the completion of the divisible hull) iff G is archimedean.

2.1 has its place in abstract $\mathcal{E}$-injectivity, as well: $E5$ is this: Each well-ordered direct system in $\mathcal{E}$ has an upper bound in $\mathcal{E}$. This holds for $(\mathcal{L}A, \mathcal{M})$ because $\mathcal{L}A$ is a variety. And, « $E6$ at $G \in |\mathcal{C}|$ » is that the class of all $\mathcal{E}$-essential $G \to H$ is *small*. Then, with $E1$-$E5$ assumed, $E6$ at G holds iff G has a unique $\mathcal{E}$-injective hull. ($\Rightarrow$ is Prop. 3 § 1 of $[B_2]$ or in Prop. 4 of $[B_3]$. $\Leftarrow$ is easy using (A).) Thus

2.2 For $(\mathcal{L}A, \mathcal{M})$: $E6$ holds at G iff G is archimedean.

Proof (Sketch). $\Leftarrow$. For G archimedean and $G \subseteq H$ $\mathcal{M}$-essentially, for $h \in H^+$, $h = \bigvee G_h$ for some $G_h \subseteq G$ as in the proof of 1.1, so that $|H| \leqslant 2^{|G|}$, and the condition follows. $\Rightarrow$ can be argued from 1.5.

So 2.2 and Banaschewski's results imply 2.1. Looking more closely at *how* yields this novel construction of the completion of the divisible hull of archimedean G, as a «maximal $\mathcal{M}$-essential extension»: let $G \xrightarrow{\mu_\alpha} H_\alpha$, or (H_α, μ_α), be a *set* representing all $\mathcal{M}$-essential extensions of G (by $E6$ at G), made into a partially ordered set $\mathcal{I}$ and a direct system by $(H_\alpha, \mu_\alpha) \leqslant (H_\beta, \mu_\beta)$ if there is $H_\alpha \xrightarrow{\iota} H_\beta$ (the bonding morphism) with $\iota\mu_\alpha = \mu_\beta$. $E5$ permits application of a maximality principle to infer a maximal member of $\mathcal{I}$.

The following asserts (C) for our circumstance:

2.3 In $\mathcal{L}\mathcal{A}$, the following conditions on $G \xrightarrow{\mu} K \in \mathcal{M}$ are equivalent.

(*a*) $G \xrightarrow{\mu} K$ is an $\mathcal{M}$-injective hull of G.

(*b*) μ is $\mathcal{M}$-essential, and if $\alpha\mu$ is $\mathcal{M}$-essential, then α is an isomorphism.

(*c*) K is $\mathcal{M}$-injective, and if $\mu = \alpha\beta$ for $\alpha, \beta \in \mathcal{M}$ and domain α $\mathcal{M}$-injective, then α is an isomorphism.

Proof. According to Cor. 2, § 1 of $[B_2]$, 2.3 will hold if we show that (*c*) $\Rightarrow$ G has an $\mathcal{M}$-injective hull, that is, by 2.1, G is archimedean. Assume (*c*). By 1.1, K is complete, hence archimedean, and so is G.

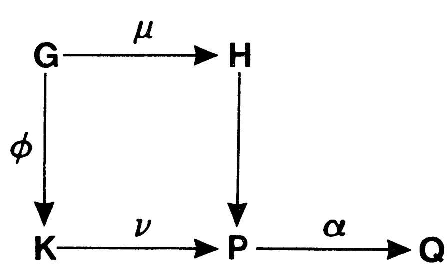
Finally, let Arch be the full subcategory of $\mathcal{L}\mathcal{A}$ with archimedean objects. The situation in Arch and its relation to $\mathcal{L}\mathcal{A}$ is a model (as is seen easily now) of these more general observations:

Suppose $\mathcal{C}$ is a variety with $\mathcal{E}$ a class of embeddings satisfying $E1$-$E3$ and $E4'$: Pushouts preserve $\mathcal{E}$ (in the sense of 1.3). Here, $E5$ holds. Let $\mathcal{S}$ be the full subcategory with objects the G's where $E6$ holds. So $G \in |\mathcal{S}|$ iff G has an $\mathcal{E}$-injective hull iff G is a subobject of an $\mathcal{E}$-injective object.

2.4 (*a*) In $\mathcal{S}$, $E1$-$E6$ hold and $\mathcal{E}$-injectivity is properly behaved.

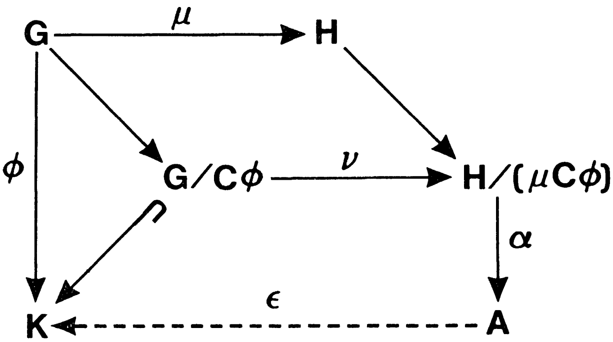
(*b*) Let $K \in |\mathcal{C}|$. K is $\mathcal{E}$-injective in $\mathcal{C}$ iff K is $\mathcal{E}$-injective in $\mathcal{S}$.

Proof. (*a*) $E4$ is the only thing to verify: We construct



from $\mu \in \mathcal{E}$ and φ (both assumed in $\mathcal{S}$) by pushing out to get the $\mathcal{C}$-square with $\nu \in \mathcal{E}$ (by $E4'$). $E3$ (cf. 1.3 (*b*)) provides α with $\alpha\nu$ ε-essential. This implies that $Q \in |\mathcal{S}|$: We have the $\mathcal{E}$-injective hull $K \xrightarrow{\iota} \hat{K}$ and there is $Q \xrightarrow{\varepsilon} \hat{K}$ with $\varepsilon(\alpha\nu) = \iota$. By $\mathcal{E}$-essentiality of $\alpha\nu$, $\varepsilon \in \mathcal{E}$ hence is an embedding.

(*b*). $\Rightarrow$ is clear. Conversely, consider



where $\mu \in \mathcal{E}$ and φ (both in $\mathcal{C}$) are given and K is $\mathcal{M}$-injective in $\mathcal{S}$. Let $C\varphi$ be the congruence for φ, and $(\mu\mathcal{C}\varphi)$ the generated congruence in H. We create ν as shown and ν is the bottom of a pushout square; whence $\nu \in \mathcal{E}$ by $E4'$. $E3$ provides α with $\alpha\nu$ $\mathcal{E}$-essential. Since $G/\mathcal{C}\varphi \in |\mathcal{S}|$, $A \in |\mathcal{S}|$ follows as in (*a*). Then ε exists, hence also the desired « lift » of φ over μ.

3. Amalgamations in Arch.

A pair of embeddings $G \xrightarrow{\sigma_i} H_i$ $(i = 1, 2)$ is said to amalgamate if there are embeddings $H_i \xrightarrow{\tau_i} P$ $(i = 1, 2)$ with $\tau_1\sigma_1 = \lambda_2\sigma_2$. A category $\mathcal{C}$ is said to have the Amalgamation Property (AP) if, in $\mathcal{C}$, each pair of embeddings amalgamates. As noted earlier, Pierce has shown that $\mathcal{L}A$ has AP [P_1], while AP fails in Arch [P_2]. Also, there is strong connection between types of injective completeness and versions of AP: On one hand, under simple hypotheses on $\mathcal{C}$, relative injective completeness implies AP (see [P_1] for a discussion); on the other hand, § 2 explains how the version of « AP for $\mathcal{E}$ » embodied in $E4$ implies, under further hypotheses, $\mathcal{E}$-injective completeness. This motivates us to the following observations about amalgamating in Arch.

3.1 Theorem. In Arch, the embeddings $G \xrightarrow{\sigma_i} H_i$ $(i = 1, 2)$ will amalgamate if one of the following holds:

(*a*) The σ_i are majorizing.

(*b*) The σ_i are large.

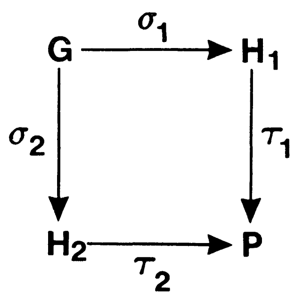
(*c*) σ_1 is majorizing and large.

3.1 (*b*) is a special case of, but also the core of, Theorem 3 of [P_2]. In any event, the following yields a proof immediately.

3.2 Conrad [C]). In Arch, each G has a maximum essential extension $G \xrightarrow{\varepsilon} G^{ess}$: If $G \xrightarrow{\sigma} H$ is essential, then there is $H \xrightarrow{\tau} G^{ess}$ with $\tau g = \varepsilon$.

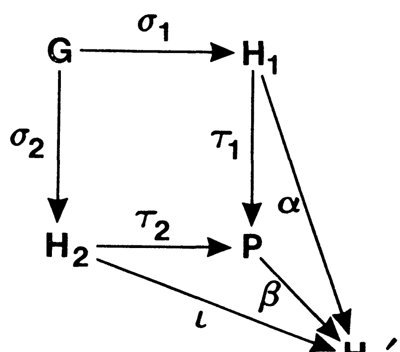
The rest of 3.1 can be proved using either products or pushouts. We choose the latter:

3.3 LEMMA. In a category, suppose

(*) 

is a pushout square, that σ_1 is an embedding, and that H_2 embeds in a σ_1-injective. Then, τ_2 is an embedding.

PROOF. Consider the diagram



arising from (*) as follows: First, $H_2 \xrightarrow{\iota} H_2'$ is the hypothesized embedding into the σ_1-injective H_2'. Then, there is α with $\alpha\sigma_1 = \iota\sigma_2$. Then, because (*) is a pushout, there is β with $\beta\tau_1 = \alpha_1$ and $\beta\tau_2 = \iota$. Thus, $\beta\tau_2$ is an embedding, hence τ_2 is an embedding.

3.4 Corollary. Suppose the category has pushouts. Let $\mathcal{E}$ be a class of embeddings for which each object embeds into an $\mathcal{E}$-injective. Then, the pair of embeddings $G \xrightarrow{i} H_i$ $(i = 1, 2)$ will amalgamate if either of the following holds:

(1) Each $\sigma_i \in \mathcal{E}$.

(2) $\sigma_1 \in \mathcal{E}$ and σ_1 is embedding-essential for the category (*i.e.*, $\varphi\sigma_1$ an embedding $\Rightarrow \varphi$ embedding).

Proof. Pushout $G \xrightarrow{\sigma_i} H_i$ to a diagram like (*) in 3.2.

If each $\sigma_i \in \mathcal{E}$, we apply 3.2 twice to conclude that each τ_i is an embedding. This proves (1). If just $\sigma_1 \in \mathcal{E}$, then τ_2 is an embedding by 3.2; so $\tau_2\sigma_2 = \tau_1\sigma_1$ is an embedding also. Thus, if σ_1 essential, then τ_1 is an embedding. This proves (2).

3.4 and the following prove 3.1 (*a*) and (*c*).

3.5 Arch has pushouts.

Proof. Any variety $\mathcal{V}$ (like $\mathfrak{L}A$) has pushouts [J]. Any SP-subclass $\mathfrak{R}$ (like Arch) is reflective in $\mathcal{V}$, that is, to each $V \in \mathcal{V}$ corresponds $rV \in \mathfrak{R}$ and a map $V \xrightarrow{\varrho} rV$ with the property: If $R \in \mathfrak{R}$, and $\varphi \in \mathrm{Hom}(V, R)$ then there is unique $\bar{\varphi} \in \mathrm{Hom}(rV, R)$ with $\bar{\varphi}\varrho = \varphi$. Then, if $\mathcal{V}$ has pushouts and $\mathfrak{R}$ is reflective, then $\mathfrak{R}$ has pushouts (obtained by «reflecting» $\mathcal{V}$-pushouts).

Finally, consider, in Arch, the class $\mathcal{E}$ of majorizing embeddings which are also epimorphisms of Arch (*i.e.*, right cancellable morphisms). With reference to the Banaschewski scheme discussed in § 2, one can verify that $\mathcal{E}$-essential $= \mathcal{E}$, whence $E6$ holds, and that the analogue of 1.3 (*b*) holds (from 3.1 (1), or more directly, from 3.3), whence $E3$ holds. The other axioms present no trouble, and it follows (from $[B_2, B_3]$; see § 2) that:

3.6 In Arch, majorizing-epi-injectivity is properly behaved.

Let $\mathfrak{R}$ be the full subcategory of Arch whose objects are $\mathcal{E}$-injective. Since $\mathcal{E} \subseteq$ epics, it follows that $\mathfrak{R}$ is epireflective (a reflection morphism $G \to rG$ being the embedding into the $\mathcal{E}$-injective hull) and evidently, the smallest reflective subcategory for which each reflection morphism is a majorizing epic embedding.

This seems quite interesting, but we have been unable to determine what the objects of $\mathfrak{R}$ are. The basic problem is that we don't know what the Arch-epimorphisms are, nor even if Arch is co-well-powered. The strongest working conjecture is that an embedding $G \hookrightarrow H$ is epic iff H is a subobject of the vector lattice hull of G. It would then follow that Arch is co-well-powered, that every epic is majorizing so that $\mathcal{E}$-injective = epi-injective, and $\mathfrak{R}$ = vector lattices. The author's opinions on this differ.

The situation for $\mathcal{L}A$ is that $G \hookrightarrow H$ is epic iff H is a subobject of the divisible hull of G [AC], and hence that epi-injective = divisible. See also [HM].

Bacsich has given an elegant discussion of epi-injectivity for universal theories in [B_1], which see.

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