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On the homology groups of $q$-complete spaces

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On the Homology Groups of q-Complete Spaces.

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SUNTO - Sia \( X \) uno spazio complesso q-completo n-dimensionale; allora \( H_k(X, \mathbb{Z}) = 0 \) per ogni \( k > n + q \). Sia poi \( (X, Y) \) una q-coppia di Runge di spazi q-completi e \( Y \) privo di singularità; allora \( H_k(X \text{ mod } Y, \mathbb{Z}) = 0 \) per ogni \( k > n + q \).

It is known (Sorani [8]) that if \( X \) is a q-complete manifold then \( H_k(X, \mathbb{Z}) = 0 \) for \( k > n + q \) and \( H_{n+q}(X, \mathbb{Z}) \) is a free group. The proof of this theorem comes from ideas of Serre, Thom and Andreotti-Frankel; but it does seem to be easily generalizable to the singular case. In this paper we prove that if \( X \) is a q-complete n-dimensional complex space then \( H_k(X, \mathbb{Z}) = 0 \) if \( k > n + q \). We don't know if \( H_{n+q}(X, \mathbb{Z}) \) is torsion free or free. We use a lemma (furuncle-lemma) of Andreotti-Grauert and a theorem of Coen which extends the results of Sorani to the case of an open subset of a Stein space. Moreover we apply our theorem to obtain a vanishing theorem for the relative homology of q-Runge pairs.

§ 0. We consider throughout this paper analytic complex spaces countable at the infinity. A complex space \( X \) is said to be q-complete when there exists a \( C^\infty \)-function \( h: X \to \mathbb{R} \) such that \( X(c) = \{ x \in X | h(x) < c \} \) is relatively compact in \( X \) for every \( c \in \mathbb{R} \), and every \( x \in X \) has a neighborhood \( V \) with the following property: there exist an

isomorphism $\chi$ of $V$ onto an analytic subset $A$ of an open subset $U$ of $\mathbb{C}^n$ and a $C^\omega$-function $\phi: U \to \mathbb{R}$ such that $h = \phi \circ \chi$ and the Levi form

$$L(\phi, y)(u) = \sum_{i,j=1}^n \left( \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j} \right) u_i \overline{u}_j$$

has at least $n - q$ positive eigenvalues at every point $y \in U$; the function $h$ is said to be strongly $q$-plurisubharmonic.

If $X$ is a complex space and $Y$ an open subset of $X$, the pair $(X, Y)$ is said to be a $q$-Runge pair if the natural homomorphism

$$q^Y_!: H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$$

has dense image for every $p = 0, 1, \ldots, n$, where $\Omega^p_X$ is the sheaf of holomorphic $p$-forms (see for instance [5]).

We recall the following theorem that we will use in the proof of our result:

**Theorem 0.1** (Coen, [4]). Let $X$ be a $q$-complete open subspace of a Stein space $S$; let $\dim X = n$. Then

$$H_k(X, \mathbb{Z}) = 0 \quad \text{if } k > n + q \quad \text{and}$$

$$H_{n+q}(X, \mathbb{Z}) \quad \text{is torsion free.}$$

A similar theorem was known for manifolds:

**Theorem 0.2** (Sorani, [8]). Let $X$ be a $q$-complete manifold, and let $\dim X = n$. Then

$$H_k(X, \mathbb{Z}) \quad \text{if } k > n + q \quad \text{and}$$

$$H_{n+q}(X, \mathbb{Z}) \quad \text{is free.}$$

By means of the results of Ferrari ([5] and [6]) and Le Potier [7] we know something else about these groups:

**Theorem 0.3.** Let $X$ be a $q$-complete complex space, and let $n = \dim X$. Then $H_k(X, \mathbb{C}) = 0$ and $H_k(X, \mathbb{Z})$ is a torsion group for each $k > n + q$. 

§ 1. In order to prove the theorem we need the following

**LEMMA 1.1** (Benedetti, [2]). *Let X be a reduced q-complete complex space. Then the function h defining the q-completeness of X can be chosen such that the set \{local minima of h in X\} is discrete in X.*

The proof of our theorem requires, besides this result, the Mayer-Vietoris sequence and the furuncle-lemma ([1], p. 237).

**THEOREM 1.2.** *Let X be a q-complete complex space, and let \( \dim X = n \). Then \( H_k(X, \mathbb{Z}) = 0 \) if \( k > n + q \).*

**Proof.** Without loss of generality we can suppose \( X \) reduced. Let \( h \) be a non-negative function chosen as in 1.1. For every \( t \in \mathbb{R} \) we put \( X(t) = \{x \in X | h(x) < t\} \) and \( B(t) = \{x \in X | h(x) = t\} \). Every open set \( X(t) \) is a q-complete space. Let \( t_0 = \min h(x) \); it follows that \( B(t_0) \) is finite and then, thanks to the property of \( h \), it is possible to find \( d \in \mathbb{R}, d > t_0 \), such that \( X(d) \) is contained in an open Stein set. Therefore (theorem 0.1.) \( H_k(X(t), \mathbb{Z}) = 0 \) if \( k > n + q \) and \( t < d \).

Then let us consider the set \( A = \{t \in \mathbb{R} | \forall r < t \text{ and } \forall k > n + q \text{ } H_k(X(r), \mathbb{Z}) = 0 \} \neq \emptyset \).

We will see that \( A = [t_0, +\infty[ \) by means of the furuncle-lemma. Let \( t \in A \); we claim that there exists \( \epsilon > 0 \) such that \( t + \epsilon \in A \).

We cover \( \partial X(t) \) with a finite family \( \{U_i\}_{t \leq i \leq p} \) of open relatively compact Stein sets for which there exist closed embeddings \( \psi_i: U_i \to V_i \), with \( V_i \) open subset of \( \mathbb{C}^n \), and non-negative strongly q-plurisubharmonic functions \( h_i: V_i \to \mathbb{R} \) such that \( h_i \circ \psi_i = h \). Then we consider a family \( \{W_i\} \) of open sets covering \( \partial X(t) \) and such that \( W_i \subset U_i \) for every \( i \), and a family \( \{g_i\} \) of \( C^\infty \)-functions, non-negative, such that \( g_i \) has compact support in \( U_i \) and \( g_i(x) > 0 \) for every \( x \in W_i \).

It is possible to choose \( p \) constants \( c_i > 0, 1 \leq i \leq p \), such that the functions \( f_i = h - \sum_{k=1}^i c_k g_k \) are strongly q-plurisubharmonic ones and the sets \( C_i = \{x \in X | f_i(x) < 0\} \) q-complete.

Since \( B(t) \setminus \partial X(t) \) is a finite set, by lessening if necessary the constants \( c_k \) we can suppose that no point \( x \in B(t) \setminus \partial X(t) \) is in \( C_p \); then there exist an open Stein set \( V \subset X \) and an \( \epsilon > 0 \) such that \( V \cap C_p = \emptyset \) and \( X(t + \epsilon) \subset C_p \cup V \). Moreover, from the construction we see that, if we put \( C_0 = X(t), C_i \setminus C_{i-1} \subset U_i \) for \( 1 \leq i \leq p \).
Let now \( t' < t + \varepsilon \). For every \( i = 0, 1, \ldots, p \), \( C_i \cap X(t') \) is \( q \)-complete too. Indeed, \( f_i \) is constructed from \( h \) through small perturbations, and therefore the Levi forms of \( h \) and of \( f_i \) in a point \( x \) are positive definite on the same \( q \)-codimensional subspace. Then the following function determines the \( q \)-completeness of \( X(t') \cap C_i \).

\[
g(x) = \frac{1}{t - f_i(x)} + \frac{1}{t' - h(x)}.\]

Now, put \( Y_i = X(t') \cap C_i \); in particular \( Y_0 \) is \( X(t) \). We show by induction that \( H_k(Y_i, Z) = 0 \) for \( k > n + q \) for every \( i \). It is true (by assumption) for \( i = 0 \). Let now \( i > 1 \) and let us consider the Mayer-Vietoris sequence of the pair \( (Y_{i-1}, Y_i \cap U_i) \):

\[
H_k(Y_{i-1} \cap U_i, Z) \rightarrow H_k(Y_{i-1}, Z) \oplus H_k(Y_i \cap U_i, Z) \rightarrow H_k(Y_i, Z) \rightarrow H_{k-1}(Y_{i-1} \cap U, Z)
\]

\( Y_{i-1} \) and \( Y_i \) are \( q \)-complete and therefore \( Y_{i-1} \cap U_i \) and \( Y_i \cap U_i \) are \( q \)-complete open subsets of the Stein space \( U_i \). Applying 0.1. and the induction we find \( Z) = 0 \) if \( k > n + q + 1 \) and

\[
0 \rightarrow H_{n+q+1}(Y_i, Z) \rightarrow H_{n+q}(Y_{i-1} \cap U_i, Z) \quad \text{if} \quad k = n + q + 1.
\]

Thanks to 0.3 \( H_{n+q+1}(Y_i, Z) \) is a torsion group; on the other hand \( H_{n+q}(Y_{i-1} \cap U_i, Z) \) is torsion free; therefore \( H_{n+q+1}(Y_i, Z) = 0 \). Then in particular \( H_k(X(t') \cap C_i, Z) = 0 \) if \( k > n + q \); since finally \( X(t') = (X(t') \cap C_i) \cup (X(t') \cap V) \), and this union is disjoint, also \( H_k(X(t'), Z) = 0 \) for each \( k > n + q \).

Therefore \( A \) is open. If we suppose \( s = \sup A < + \infty \), we can find a sequence of points of \( A \) \( t_n \rightarrow s \). But then

\[
H_k(X(s), Z) = \lim H_k(X(t_n), Z) = 0
\]

and this is a contradiction, since \( s \notin A \). Then \( \sup A = + \infty \). In particular \( m \in A \) for every \( m \in \mathbb{N} \), and then

\[
H_k(X, Z) = \lim H_k(X(m), Z) = 0 \quad \text{for each} \quad k > n + q.
\]
Remark. This theorem allows us to remove the assumption of a Stein environment in several results; for instance, in the corollaries 2.1 and 2.4 of [4].

§ 2. We recall the following proposition:

**Proposition 2.1** (Le Potier [7]). Let $X$ be a complex space, and let $n = \dim X$. Then there exists a canonical homomorphism

$$\theta^{n,q}: H^q(X, \Omega^n_X) \rightarrow H^{n+q}(X, \mathbb{C});$$

moreover, it is surjective if $X$ is $q$-complete.

If $X$ is a complex manifold $H^{n+q}(X, \mathbb{C})$ has a natural topology, thanks to De Rham’s theorem; moreover we have the following

**Lemma 2.2** (see Le Potier [7], Remarque III, 6). Let $X$ be a complex manifold. Then $\theta^{n,q}$ is continuous with respect to the natural topologies.

**Proof.** We can factorize the map $\theta^{n,q}$, with $q > 0$ (the case $q = 0$ is similar), in the following way:

$$H^q(X, \Omega^n_X) \xrightarrow{\xi} \frac{\text{Ker} \left( \Gamma(X, \mathcal{A}^{n,q}) \rightarrow \Gamma(X, \mathcal{A}^{n,q+1}) \right)}{\text{Im} \left( \Gamma(X, \mathcal{A}^{n,q-1}) \rightarrow \Gamma(X, \mathcal{A}^{n,q}) \right)} \xrightarrow{\kappa} \frac{\text{Ker} \left( \Gamma(X, \mathcal{E}^{n,q}) \rightarrow \Gamma(X, \mathcal{E}^{n,q+1}) \right)}{\text{Im} \left( \Gamma(X, \mathcal{E}^{n,q-1}) \rightarrow \Gamma(X, \mathcal{E}^{n,q}) \right)} \xrightarrow{k} H^{n+q}(X, \mathbb{C})$$

where $\mathcal{A}^{n,q}$ is the sheaf of $C^\infty$-differential forms of type $(n, q)$ and $\mathcal{E}^k$ is the sheaf of $C^\infty$-differential forms of type $k$. The map $g$ is continuous (with respect to the Fréchet topologies on the modules of sections), since $\mathcal{A}^{n,q}$ is a fine resolution of Fréchet sheaves of $\Omega^n$ (by means of the results of [3]); $h$ is continuous since it comes from the natural inclusion of $(n, q)$-forms into $(n + q)$-forms; $k$ is continuous by definition.

**Theorem 2.3.** Let $(X, Y)$ be a $q$-Runge pair of $q$-complete spaces; let $Y$ be free of singularities. Then $H_k(X \mod Y, \mathbb{Z}) = 0$ for each $k > n + q$.

**Proof.** If $k > n + q + 1$ the theorem follows from the relative homology sequence of the pair $(X, Y)$ and from theorem 1.2.
Let now $k = n + q + 1$. We begin proving that $H_{n+q+1}(X \mod Y, \mathbb{C}) = 0$. In the following commutative diagram

\[
\begin{array}{ccc}
H^q(X, \Omega^n_f) & \xrightarrow{\partial\phi^q} & H^{n+q}(X, \mathbb{C}) \\
\downarrow v_f & & \downarrow v_f \\
H^q(Y, \Omega^n_f) & \xrightarrow{\partial\phi^q} & H^{n+q}(Y, \mathbb{C})
\end{array}
\]

$\partial\phi^q$ is continuous and surjective (applying lemmas 2.1 and 2.2) and $v_f$ has dense image by hypothesis; thus $v_f$ has dense image too. Moreover, the natural algebraic pairing $\langle H^{n+q}(X, \mathbb{C}), H_{n+q}(X, \mathbb{C}) \rangle$ is also topological (see Sorani [9]); then the natural homomorphism

\[
H_{n+q}(Y, \mathbb{C}) \xrightarrow{j} H_{n+q}(X, \mathbb{C})
\]

is injective. Thus considering the exact sequence

\[
0 \to H_{n+q+1}(X \mod Y, \mathbb{C}) \to H_{n+q}(Y, \mathbb{C}) \xrightarrow{j} H_{n+q}(X, \mathbb{C})
\]

we find $H_{n+q+1}(X \mod Y, \mathbb{C}) = 0$; then $H_{n+q+1}(X \mod Y, \mathbb{Z})$ is a torsion group. But in the natural relative exact sequence

\[
H_{n+q+1}(X, \mathbb{Z}) \to H_{n+q+1}(X \mod Y, \mathbb{Z}) \xrightarrow{j} H_{n+q}(Y, \mathbb{Z})
\]

$j$ is injective, applying theorem 1.2; moreover $H_{n+q}(Y, \mathbb{Z})$ is a torsion free group (proposition 0.2). Therefore $H_{n+q+1}(X \mod Y, \mathbb{Z}) = 0$.

REFERENCES


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