RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

EZIO STAGNARO

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Rendiconti del Seminario Matematico della Università di Padova, tome 69 (1983), p. 27-35

<http://www.numdam.org/item?id=RSMUP_1983__69__27_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

A new construction of a surface of degree 5 having 31 nodes.

EZIO STAGNARO (*)

SUNTO - È noto un solo esempio di superficie di ordine 5 di $\mathbb{P}^3_{\mathbb{C}}$ avente 31 nodi (= punti doppi conici); esso è dovuto a E. G. Togliatti (cfr. [3]). Recentemente A. Beauville ha dimostrato che 31 è il massimo numero di nodi che una superficie di ordine 5 di $\mathbb{P}^3_{\mathbb{C}}$ può avere. Qui, seguendo i nostri metodi (cfr. [6], [7]), si dà una nuova costruzione di una superficie di ordine 5 di $\mathbb{P}^3_k$ con 31 nodi. La nostra superficie è definita da un polinomio omogeneo biquadratico in X_0: $\alpha X_0^4 + 2\beta X_0^2 + \gamma$; $\alpha, \beta, \gamma \in k[X_1, X_2, X_3]$, ed è costruita considerando la curva di diramazione $(\beta^2 - \alpha\gamma)^2\gamma$ del piano X_0 nella proiezione della superficie dal punto (1, 0, 0, 0). Il risultato vale anche se la caratteristica di k è $p > 0$, purchè $p \neq 2, 3, 5$ e p diverso dagli eventuali valori che danno nodi coincidenti sulla superficie.

Introduction.

The largest number of conical double points, shortly nodes, that it was possible to assign to a surface F_5 of degree 5 in $\mathbb{P}^3_{\mathbb{C}}$, $\mathbb{C}$ complex field, is 31. This result is due to E. G. Togliatti (cfr. [3]) who constructs F_5 as branch locus of a cubic hypersurface H_3 in $\mathbb{P}^5_{\mathbb{C}}$ having 15 nodes, in the projection from a generic line of H_3 over a $\mathbb{P}^3_{\mathbb{C}}$. The best limitation for the maximum number of nodes for a surface of degree 5

(*) Indirizzo dell'A.: Istituto di Matematica Applicata, Via Belzoni, 7 - 35100 Padova (Italy).

Lavoro eseguito nell'ambito del Gruppo Nazionale per le Strutture Algebriche e Geometriche e loro Applicazioni.

in $\mathbb{P}^3_{\mathbb{C}}$ was 34. Such a limit is due to A. B. Basset (cfr. [1], [2]; for a proof of Basset's limitation cfr. [8]). Then it is natural to search for examples of surfaces of degree 5 with a number of nodes > 31. Despite several attempts I never was able to find a surface of degree 5 with 32 nodes. Only recently I understood that the motivation of this failure was not in the method, as I believed, but in the non existence of such surface. In fact A. Beauville proved (cfr. [9]) that a surface of degree 5 in $\mathbb{P}^3_{\mathbb{C}}$ cannot have a number of nodes > 31; so the surface of Togliatti has the maximum number of nodes.

Since Togliatti's example is the unique example well-known, in the complex case, of a surface F_5 with 31 nodes and since 31 is just the maximum for surfaces of degree 5, it seems to me useful to show that also with our method (substantially contained in [6] and [7]) we are able to construct, not only a surface of degree 5 with 31 nodes, but also to check that such a construction works even if the characteristic of k is $p > 0$, except for a finite number of values of p.

The idea of our construction is the following. Let F_5 be a surface of degree 5 defined by a homogeneous polynomial in the variables X_0, X_1, X_2, X_3, biquadratic in X_0,

$$F_5 \colon \alpha X_0^4 + 2\beta X_0^2 + \gamma$$

and let us project F_5 from the point $(1, 0, 0, 0)$ over the plane X_0. The branch locus on the plane X_0 is $(\beta^2 - \alpha\gamma)^2\gamma$. A singularity on F_5 is projected in a singularity on $\beta^2 - \alpha\gamma$ or on γ and conversely a singularity on $\beta^2 - \alpha\gamma$ not belonging to α or a singularity on γ is the projection of two or one singularities on F_5. Then to impose a large number of isolated singularities on F_5 it is equivalent to impose a large number of singularities on $\beta^2 - \alpha\gamma$ or on γ. Now in order to impose singularities on $\beta^2 - \alpha\gamma$ we consider a curve δ having a large number of singularities and we show that it is possible to write δ in the form $\beta^2 - \alpha\gamma$. We choose as δ a curve split in three conics $\delta = \delta_1\delta_2\delta_3$ and moreover we want the quintic γ to split into a quartic φ and a line r. In other words, want to write the following equality (identity)

$$\delta_1\delta_2\delta_3 = \beta^2 - \alpha r \varphi \,. \tag{1}$$

For this we remark that if (1) holds the sextic $\delta_1\delta_2\delta_3$ is tritangent to the two lines α and r. But this condition is also sufficient in order that (1) holds; in particular if δ_i is tangent to α and r, we have

$\delta_i = \alpha r + l_i^2$ where l_i is a linear form ($i = 1, 2, 3$) and consequently

$$(2) \qquad \delta_1\delta_2\delta_3 = (\alpha r + l_1^2)(\alpha r + l_2^2)(\alpha r + l_3^2) = (l_1 l_2 l_3)^2 - \\ - \alpha r(-\alpha^2 r^2 - \alpha r(l_1^2 + l_2^2 + l_3^2) - l_1^2 l_2^2 - l_1^2 l_3^2 - l_2^2 l_3^2)$$

which gives (1) with

$$\beta = l_1 l_2 l_3 , \qquad \varphi = -\alpha^2 r^2 - \alpha r(l_1^2 + l_2^2 + l_3^2) - l_1^2 l_2^2 - l_1^2 l_3^2 - l_2^2 l_3^2 .$$

At this point in correspondence to the choosen polynomials α, β, $\gamma = r\varphi$ the surface F_5: $\alpha X_0^4 + 2\beta X_0^2 + \gamma$ has (at least) 28 nodes. Next if we were able to impose three nodes on the quartic φ which appears in (2), then the corresponding surface F_5 would have 31 nodes. Unfortunatly we were not able to find a quartic with three nodes. To overcome this difficulty we add and substract in (2) the term $\alpha^2 r^2 l^2 + 2\alpha r l_1 l_2 l_3 l$, where l is an arbitrary linear form. We get

$$(3) \qquad \delta_1\delta_2\delta_3 = (l_1 l_2 l_3 + \alpha r l)^2 - \alpha r(\varphi + \alpha r l^2 + 2 l_1 l_2 l_3 l) .$$

Putting now $\beta = l_1 l_2 l_3 + \alpha r l$, $\gamma = r(\varphi + \alpha r l^2 + 2 l_1 l_2 l_3 l)$ the corresponding surface F_5 has again 28 nodes and among the quartics $\psi = \varphi + \alpha r l^2 + 2 l_1 l_2 l_3 l$ we shall be able to find one, for particular values of the parameters, with three nodes and therefore a surface F_5 with 31 nodes.

In the n. 1 of this paper we recall some results connecting the singularities on F_5 with the singularities on the branch locus; in the n. 2 we give the polynomials δ_i, α, β, $\gamma = r\psi$ and we check that they lead to a surface F_5 with 31 nodes; finally in the n. 3 we show the way followed to impose on the quartic ψ three nodes.

1. The results of this number are due substantially to D. Gallarati ([4], [5]). They are written here with a sketch of the proof for reasons of completeness and for convenience of the reader. The exposition is like that in [7] to which we refer for further details.

Let F_5 be the surface of degree 5 in $\mathbb{P}_k^3$, k algebraically closed field of characteristic $p \neq 2, 3, 5$, defined by the homogeneous polynomial

$$\alpha X_0^4 + 2\beta X_0^2 + \gamma ; \qquad \alpha, \beta, \gamma \in k[X_1, X_2, X_3] .$$

We project F_5 from the point $(1, 0, 0, 0)$ over the plane X_0 (such a projection is not defined in $(1, 0, 0, 0)$); the branch locus on X_0 is $(\beta^2 - \alpha\gamma)^2\gamma$.

LEMMA 1. *If $A = (y_0, y_1, y_2, y_3)$, with $y_0 \neq 0$, is a singular point on F_5, then the projection $A' = (0, y_1, y_2, y_3)$ is singular point on the curve $\beta^2 - \alpha\gamma$ on X_0; conversely if A' is singular on $\beta^2 - \alpha\gamma$ and if $\alpha(A') = \alpha(y_1, y_2, y_3) \neq 0$, then the two points A_1, $A_2 = \big(\pm\sqrt{-\beta(A')/\alpha(A')}, y_1, y_2, y_3\big)$, distinct if $\beta(A') \neq 0$, are singular on F_5.*

2) *If $\alpha(A') \neq 0$ more precisely we have: A is a node on F_5 if and only if A' is a node* (= double point with distinct tangents) *on $\beta - \alpha\gamma$.*

Now we consider the case $y_0 = 0$.

3) *$B = (0, y_1, y_2, y_3)$ is singular on F_5 if and only if it is singular on γ; if $\beta(B) \neq 0$ then B is a node on F_5 if and only if it is a node on γ.*

PROOF. The statement 1) follows from the equality

$$\alpha(\alpha X_0^4 + 2\beta X_0^2 + \gamma) = (\alpha X_0^2 + \beta)^2 - (\beta^2 - \alpha\gamma)\,. \tag{4}$$

Namely we have $(\partial/\partial X_0)(\alpha X_0^4 + 2\beta X_0^2 + \gamma) = 4X_0(\alpha X_0^2 + \beta)$, consequently if A is singular on F_5, since characteristic of k is $\neq 2$ and $y_0 \neq 0$, we have that $\alpha X_0^2 + \beta$ vanishes on A. Then from (4) A, equivalently A', is singular on $\beta^2 - \alpha\gamma$. Conversely from the hypotheses we have that the right-hand side of (4) is singular in A_1, A_2, hence also the left-hand side; but $\alpha(A_i) \neq 0$, therefore F_5 is singular in A_i, $i = 1, 2$.

The proof of 2) follows by considering the tangent cone Γ_A in A to F_5 and showing that in the projection from $(1, 0, 0, 0)$ the branch locus of Γ_A is just the tangent cone in A' to $\beta^2 - \alpha\gamma$ (remark that $\alpha(A') \neq 0 \Rightarrow (1, 0, 0, 0) \notin \Gamma_A$).

The first part of 3) follows from the equality $\alpha X_0^4 + 2\beta X_0^2 + \gamma = (\alpha X_0^2 + 2\beta)X_0 + \gamma$ and the second one from

$$\sum_{i,j=0}^{3} \frac{\partial^2}{\partial X_i\,\partial X_j}(\alpha X_0^4 + 2\beta X_0^2 + \gamma))_B X_i X_j = 4\beta(B)X_0^2 + \sum_{i,j=1}^{3}\left(\frac{\partial^2}{\partial X_i\,\partial X_j}\gamma\right)_B X_i X_j\,. \qquad \text{Q.E.D.}$$

2. We consider the polynomials

$$\alpha = X_1 - X_2 + \frac{\sqrt{5}}{3} X_3 \qquad l_1 = X_1 + X_3 \qquad l_3 = 3(X_1 + X_2) - \frac{2}{3} X_3$$

$$r = X_1 - X_2 - \frac{\sqrt{5}}{3} X_3 \qquad l_2 = X_2 + X_3 \qquad l = 4(X_1 + X_2) - 2X_3$$

$$\delta_1 = \alpha r + l_1^2 \qquad \delta_2 = \alpha r + l_2^2 \qquad \delta_3 = -\alpha r + l_3^2$$

$$\psi = \frac{25}{9}(9X_1^2X_2^2 + 6X_1^2X_2X_3 - X_1^2X_3^2 + 6X_1X_2^2X_3 - 6X_1X_2X_3^2 - X_2^2X_3^2).$$

The key to the existence of the surface F_5 with 31 nodes is the following equality whose checking is only a question of calculation

$$\delta_1\delta_2\delta_3 = (l_1 l_2 l_3 + \alpha r l)^2 - \alpha r \psi . \tag{5}$$

Putting $\beta = l_1 l_2 l_3 + \alpha r l$, $\gamma = r\psi$ (5) can be written

$$\delta_1\delta_2\delta_3 = \beta^2 - \alpha\gamma . \tag{6}$$

With respect to the curves on X_0 defined by the above polynomials, the following statements hold, except possibly finite many positive values of the characteristic of k:

(*a*) *the curve* $\beta^2 - \alpha\gamma$, *split in the three conics* δ_1, δ_2, δ_3 (*cfr.* (6)), *has* 12 (*distinct*) *nodes in the common points to two conics*;

(*b*) *the curve* γ, *split in* r *and in* ψ, *has* 7 (*distinct*) *nodes*: 4 *in the common points to* r *and to* ψ *and* 3 *in the three nodes on* ψ: $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$;

(*c*) *the nodes in* (*b*) *are distinct from those in* (*a*);

(*d*) *the curves* α *and* β *do not pass through any node.*

The possibly exceptional positive values of the characteristic of k are those which give coincident nodes; we are not interested in such values.

Before proving (*a*), (*b*), (*c*) and (*d*), we show how from them the existence of a surface F_5, of degree 5, with 31 nodes follows.

THEOREM. *Let α, β, γ be the polynomials defined in this number. If the characteristic of k is different from* 2, 3, 5 *and maybe from other positive values, the surface F_5, of degree* 5, *defined by the homogeneous polynomial*

$$F_5: \alpha X_0^4 + 2\beta X_0^2 + \gamma$$

has 31 (*distinct*) *nodes and F_5 has no other singularity outside the nodes.*

PROOF. From 1) and 2) of the lemma of the n. 1 and from (a) and (d) F_5 has 24 (distinct) nodes in correspondece to the 12 nodes on $\beta^2 - \alpha\gamma$. From 1) and 3) of the lemma and from (b) and (d) F_5 has 7 nodes in the nodes on γ on X_0. From (c) all the nodes are distinct. From the first part of 1) and (3) of the lemma F_5 has no other singularity outside the $24 + 7 = 31$ nodes. Q.E.D.

Now we prove (a), (b), (c) and (d).

PROOF OF (a). (a) can be proved directly calculating the common points to two conics δ_j, $j = 1, 2, 3$:

$$\delta_1 \cap \delta_2 = \{(-4, -2, 3),\ (-2, -4, 3),\ (-3 \pm \sqrt{5}, -3 \pm \sqrt{5}, 3)\};$$

$$\delta_1 \cap \delta_3 = \text{common points to the lines } 36(11 \pm i3)X_1^2 + \\ + 24(3 \pm i4)X_1X_3 - (-31 \pm i12)X_3^2, \quad l_1^2 + l_3^2;$$

$$\delta_2 \cap \delta_3 = \text{common points to the lines } 36(11 \pm i3)X_2^2 + \\ + 24(3 \pm i4)X_2X_3 - (-31 \pm i12)X_3^2, \quad l_2^2 + l_3^2;$$

where $i^2 = -1$, and checking that all these points are distint in characteristic zero, hence in positive characteristic except, maybe, finite many values.

PROOF OF (b). It is enough to check that the 4 common points to r and to ψ are distinct in characteristic zero; this can be done with the discriminant of r and ψ with respect to X_1/X_3 or X_2/X_3 otherwise drawing on the real plane the curve ψ and seeing that r meets ψ in 4 distint real points.

PROOF OF (c). The three points (1, 0, 0), (0, 1, 0), (0, 0, 1) are clearly distinct from the nodes on $\delta_1\delta_2\delta_3$ because none of them belongs to δ_j, $j = 1, 2, 3$; moreover the remaining nodes on $r\psi$ are distinct from those on $\delta_1\delta_2\delta_3$, for otherwise if a point P belongs to r, δ_i, δ_j ($i \neq j$), then P belongs to r, l_i, l_j which is impossible.

PROOF OF (d). α does not pass through any node P on $\delta_1\delta_2\delta_3$, for otherwise P belongs to α, l_i, l_j $(i \neq j)$ which is impossible. Since α meets r in the point $(1, 1, 0)$, it follows that α does not pass through any node on $r\psi$. Also β does not pass through any node on $\delta_1\delta_2\delta_3$ otherwise if Q is such a node, from the equality $\delta_1\delta_2\delta_3 = \beta^2 - \alpha\gamma$, being Q singular on $\beta^2 - \alpha\gamma$ and on β^2, Q would be singular on $\alpha\gamma = \alpha r\psi^n$ since Q does not belong to α, then Q is a node on $r\psi$, against (c). Finally β does not pass through any node on $r\psi$ otherwise if R is such a node, R is singular on β^2 and on γ, hence R is singular on $\beta^2 - \alpha\gamma$, again contradicting (c).

3. In this number we want to show shortly the way followed to impose three double points on the quartic $\psi = \varphi + \alpha r l^2 + 2l_1 l_2 l_3 l$ of the introduction, in such way that ψ does not split in a conic counted twice, otherwise in correspondence on F_5 we have a double conic, so non isolated singularities. The more natural thing is to impose to the ψ the three singular points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. For this we must annihilate the coefficients of X_i^4, $X_i^3 X_j$, $i, j = 1, 2, 3$, $i \neq j$, in the ψ. We get an algebraic system of 9 equations. Since is pratically impossible to resolve this system, the problem is to make positions among the parameters in such a way as to resolve the system and not lose the non trivial solution. For this we impose on the ψ to be symmetric in the affine plane to the line $X_1 - X_2$, choosing

$$\alpha = \lambda(X_1 - X_2) + \mu X_3 \qquad l_1 = a_1 X_1 + a_2 X_2 + a_3 X_3$$

$$l_2 = a_2 X_1 + a_1 X_2 + a_3 X_3 \qquad r = \lambda(X_1 - X_2) - \mu X_3$$

$$l_3 = c_1(X_1 + X_2) + c_2 X_3 \qquad l = b_1(X_1 + X_2) + b_2 X_3$$

$$\delta_1 = a\alpha r + l_1^2 \qquad \delta_2 = a\alpha r + l_2^2 \qquad \delta_3 = c\alpha r + l_3^2$$

λ, μ, a_i, b_i, c_i, a, $c \in k$ (the parameters a, c are superflous but useful to get semplifications in the calculations).

With this choice we have

$$\psi = \{a^2 c\lambda^4 + a\lambda^2[c(a_1^2 + a_2^2) + ac_1^2] + ca_1^2 a_2^2 + ac_1^2(a_1^2 + a_2^2) -$$

$$- \lambda^2 b_1^2 - 2a_1 a_2 b_1 c_1\} X_1^4 + \{- 4a^2 c\lambda^4 - 2ac\lambda^2(a_1 - a_2)^2 +$$

$$+ 2ca_1 a_2(a_1^2 + a_2^2) + 2ac_1^2(a_1 + a_2)^2 - 2(a_1 + a_2)^2 b_1 c_1\} X_1^3 X_2 +$$

$$+ \{2a\lambda^2[ca_3(a_1 + a_2) + ac_1c_2] + 2a_3(a_1 + a_2)(ca_1a_2 + ac_1^2) +$$
$$+ 2ac_1c_2(a_1^2 + a_2^2) - 2\lambda^2 b_1 b_2 - 2a_1a_2(b_1c_2 + b_2c_1) -$$
$$- 2a_3(a_1 + a_2) b_1 c_1\} X_1^3 X_3 + \dots .$$

The coefficients to annihilate are now five instead of nine. Anyhow we get very complicated equations. A good simplification is got on putting $a_2 = 0$; even if with this choice we lose a parameter we shall see that it is possible to have non trivial solutions. We remark that with other choices, for instance $b_1 = 0$, we have only the trivial solution: ψ split in a conic counted twice. Choosing then $a_2 = 0$ and annihilating the coefficient of X_1^4 and of $X_1^3X_2$ in the ψ, we may write

$$(7) \qquad b_1 = \pm \frac{2a(a\lambda^2 + a_1^2)}{a_1^2}\sqrt{-c} \qquad c_1 = \pm \frac{(2a^2 + a_1^2)}{a_1^2}\sqrt{-c}\,.$$

Annihilating the coefficient of $X_1^3X_3$ in the ψ, we have

$$(8) \qquad c_2 = \frac{- aa_1a_3c_1^2 + a_1a_3b_1c_1 - ac\lambda^2a_1a_3 + \lambda^2b_1b_2}{a(a\lambda^2 + a_1)c_1}$$

where it is possible to replace b_1 and c_1 with the value given by (7).

We have still to annihilate the last two coefficient: those of $X_1X_3^3$ and X_3^4 in ψ. At this point we can try some attempts with particular numerical values of the parameters. Namely choosing $a = \lambda = a_1 = a_3 = 1$, $c = -1$, « $+$ » = « $\pm$ » from (7) and (8) we have $b_1 = 4$, $c_1 = 3$, $c_2 = (2b_2 + 2)/3$; annihilating the coefficient of $X_1X_3^3$ and X_3^4 we have $\mu^2 = (b_2 + 7)/9$, $(b_2 - 2)^2(b_2 + 2) = 0$. We cannot choose $b_2 = 2$ otherwise δ_1 and δ_2 pass through $(0, 0, 1)$ against (c), n. 2. Consequently we have $b_2 = -2$, hence $c_2 = -2/3$, $\mu^2 = 5/9$. Therefore the polynomials α, r, l_i, l, δ_i, ψ are those of n. 2 and the verifications made there assure that, with our choice, we have resolved the problem.

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Manoscritto pervenuto in redazione il 21 settembre 1981.