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A Remark on Certain Overdetermined Systems of Partial Differential Equations.

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SUMMARY - A connection between the surjectivity of a differential polynomial on a Gevrey space and the solvability of certain overdetermined systems is indicated.

Let $P(D)$, $D = (D_1, \ldots, D_n)$, $D_j = -i\,\partial x_j$, $j = 1, \ldots, n$, be a linear differential operator on R^n with constant coefficients and let $d_j \geqslant 1$ be rational numbers, $d = (d_1, \ldots, d_n)$. Denote by $\Gamma^d(\Omega)$, Ω an open set of R^n, the set of all C^∞ complex valued functions f on Ω such that for every compact subset K of Ω there exists a positive constant c, depending on K and f, such that

$$\sup_{\alpha \in \mathbb{Z}_+^n} \sup_{x \in K} c^{-|\alpha|} \Gamma(\langle d, \alpha \rangle + 1)^{-1} |D^\alpha f(x)| < \infty .$$

Here $\mathbb{Z}_+$ is the set of all non negative integers, Γ is the Euler gamma function and $\langle d, \alpha \rangle = \sum_{j=1}^{n} d_j \alpha_j$.

A connection had been pointed out in [1] between the surjectivity of $P(D)$ on the space $\Gamma^1(\Omega) = A(\Omega)$ of all the real analytic functions

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on Ω and the solvability of the system

$$P(D)v = w \qquad \Big(\sum_{j=1}^{n} D_j^2 + D_t^2\Big) v = 0\,, \qquad D_t = -i\,\partial_t\,,$$

in an open neighborhood of Ω in R^{n+1}. Here we show that the same type of result holds for any $\Gamma^d(\Omega)$.

Let

$$Q(D, D_t) = \sum_{\langle d,\alpha\rangle + h \leqslant m} c_{\alpha,h} D^\alpha D_t^h\,, \tag{1}$$

be a linear differential operator on R^{n+1} with constant coefficients $c_{\alpha,h}$, $(\alpha, h) \in \mathbb{Z}_+^{n+1}$, and assume that $c_{0,m} \neq 0$. If f_h, $h = 0, \ldots, m-1$, are given functions in $\Gamma^d(\Omega)$, then according to a theorem by G. Talenti [4], there exist an open neighborhood U of Ω in R^{n+1} and one and only one function $u \in \Gamma^{(d,1)}(U)$ such that

$$\begin{aligned} &Q(D, D_t)\,u = 0 \qquad \text{in } U, \\ &\partial_t^h u(x, 0) = f_h(x)\,, \qquad x \in \Omega,\ h = 0, \ldots, m-1. \end{aligned} \tag{2}$$

Consider the problem of finding a solution v of the system

$$P(D)\,v = w\,, \qquad Q(D, D_t)\,v = 0 \tag{3}$$

in an open neighborhood V of Ω in R^{n+1}, assuming that $Q(D, D_t)$, of the form (1), be $(d, 1)$-hypoelliptic on R^{n+1} (1). From this assumption it follows that every distribution solution of (3) on V is in $\Gamma^{(d,1)}(V)$. Note also that a necessary condition for the solvability of (3) in V is that

$$Q(D, D_t)\,w = 0 \qquad \text{in } V. \tag{4}$$

Hence $w \in \Gamma^{(d,1)}(V)$.

(1) This implies that $c_{0,m} \neq 0$, if m is the order of Q. If $d_j = r/s_j$, $j = 1, \ldots, n$; r, s_j positive integers, an example of operator of the form (1) which is $(d, 1)$-hypoelliptic on R^{n+1} is given by

$$Q(D, D_t) = \sum_{j=1}^{n} D_j^{2s_j} + D_t^{2r}\,.$$

Suppose now that the equality $P(D)\Gamma^d(\Omega) = \Gamma^d(\Omega)$ holds for the given $P(D)$ and $\Omega \subset R^n$ and that w is a given function on V satisfying (4). Put

$$f_h(x) = \partial_t^h w(x, 0)\,, \qquad x \in \Omega,\ h = 0, \dots, m-1,$$

and let $u_h \in \Gamma^d(\Omega)$ be such that $P(D)u_h = f_h$ in Ω. By the theorem quoted above there exist an open neighborhood $U \subset V$ of Ω in R^{n+1} and one and only one function $v \in \Gamma^{(d,1)}(U)$ such that

$$Q(D, D_t)v = 0 \quad \text{in } U,$$

$$\partial_t^h v(x, 0) = u_h(x)\,, \qquad x \in \Omega,\ h = 0, \dots, m-1.$$

This implies that

$$Q(D, D_t)\big(P(D)v\big) = 0 \quad \text{in } U,$$

$$\partial_t^h\big(P(D)v\big)(x, 0) = f_h(x)\,, \qquad x \in \Omega,\ h = 0, \dots, m-1.$$

Hence $P(D)v = w$ in U by the uniqueness of the solution of the problem (2) in $\Gamma^{(d,1)}(U)$. Thus we conclude that v is a solution of the system (3) in U.

On the other hand, given $f \in \Gamma^d(\Omega)$, there exists a solution $u \in \Gamma^{(d,1)}(U)$ of the problem (2) when $f_0 = f$, $f_h = 0$, $h = 1, \dots, m-1$. If the system (3), with $w = u$, has a solution v in an open neighborhood V of Ω in R^{n+1}, then the function $z(x) = v(x, 0) \in \Gamma^d(\Omega)$ is a solution of the equation $P(D)z = f$.

So we have proved the following result.

THEOREM. Let $P(D)$ be a linear differential operator with constant coefficients and let $d_j \geq 1$, $j = 1, \dots, n$, be rational numbers, $d = (d_1, \dots, d_n)$.

Then for every open set Ω contained in R^n and every $(d, 1)$-hypoelliptic differential operator of the form (1), the following statements are equivalent:

i) $P(D)\Gamma^d(\Omega) = \Gamma^d(\Omega)$,

ii) for every w satisfying (4) there exists a solution of the system (3) in an open neighborhood of Ω in R^{n+1}.

When $\Omega = R^n$, sufficient conditions on $P(D)$ for i) to hold are proved in [3]. For the case when all the d_j's are equal and $\Omega = R^n$, see [2].

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