RENDICONTI
*del*
SEMINARIO MATEMATICO
*della*
UNIVERSITÀ DI PADOVA

FABIO ZANOLIN

**On forced periodic oscillations in dissipative Liénard systems**

*Rendiconti del Seminario Matematico della Università di Padova*, tome 69 (1983), p. 51-62

<http://www.numdam.org/item?id=RSMUP_1983__69__51_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 69 (1983)

# On Forced Periodic Oscillations in Dissipative Liénard Systems.

FABIO ZANOLIN (*)

## 1. Introduction.

In this paper we study the problem of the existence of periodic solutions for second order differential systems of the Liénard type:

$$(L) \qquad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t) \qquad (' = d/dt)$$

where $h(t)$ is a periodic forcing term and $\varphi$ and $g$ are gradient functions. Our main result considers the case when the amplitude of the dissipative term $\varphi$, in the direction of the restoring force $g$, overcomes the $L^1$-norm of $h$.

Moreover, we suppose the field $g$ satisfies a suitable geometric condition (see $(k)$ below) which generalizes the usual assumptions required in the literature. Then, as a corollary, we obtain an extension to the systems of some classical and recent results like the theorems of Lefschetz [5], Reuter [14], Reissig [12], Mawhin [8], Bebernes-Martelli [1], Ward [17].

Our main tool in the proof is the following theorem by J. Mawhin [9], [10], recalled here in a simpler but less general form (adapted for second order systems) for the reader's convenience.

(*) Indirizzo dell'A.: Istituto Matematico, Università, Piazzale Europa 1, 34100 Trieste.

Work announced at the Conference on Differential Equations and Applications, Retzhof (Graz), 1-6 June 1981.

LEMMA 1 ([10, Theorem 1]). *Let* $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ *be continuous and p-periodic in t.*

($\alpha$) *Assume that for all possible p-periodic solutions* $x(t)$ *of class* $C^2$ *of*

$$(D_\lambda) \qquad x''(t) = \lambda f(t, x(t), x'(t)), \qquad \lambda \in (0, 1],$$

*there exist two constants* $K_0$ *and* $K_1$, *independent of* $\lambda$ *and* $x(t)$ *such that*

$$\sup_{[0,p]} |x(t)| < K_0, \qquad \sup_{[0,p]} |x'(t)| < K_1.$$

($\beta$) $\deg_B(S, \Omega, 0) \neq 0$ *where*

$$S\colon z \mapsto \frac{1}{p}\int_0^p f(t, z, 0)\, dt, \qquad z \in \mathbb{R}^n$$

*and* $\Omega = B(0, r_0)$ *for* $r_0$ *sufficiently great. Then the differential system* $(D_1)$ $(\lambda = 1)$ *has at least one p-periodic solution of class* $C^2$.

## 2. Notations.

In what follows, $|\cdot| = (\cdot|\cdot)^{\frac{1}{2}}$ is the euclidean norm in $\mathbb{R}^n$ induced by the inner product $(\cdot|\cdot)$. If $u, v\colon \mathbb{R} \to \mathbb{R}^n$ are continuous and $p$-periodic functions $(p > 0)$, we define $|u|_\infty = \sup_{[0,p]} |u(t)|$, $(u, v)_2 = \int_0^p \big(u(s)|v(s)\big) ds$ (the $L^2$-scalar product of $u$ and $v$), $|v|_q = \left(\int_0^p |v(s)|^q\, ds\right)^{1/q}$, $q \geqslant 1$ (the $L^q$-norm of $v$). Moreover, $Qu = (1/p)\int_0^p u(s)\, ds$ is the mean value of $u(t)$ is a period. If $x \in \mathbb{R}^n$, $r > 0$, then $B(x, r)$, $\overline{B(x,r)}$ are the open and the closed ball of center $x$ and radius $r$; the sphere $S^{n-1}$ is the boundary of $B(0,1)$. Finally we recall from Krasnosel'skii [4] the following definition: let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-function; we say that $V$ is nondegenerate with nonzero index, shortly $\gamma(V, \infty) \neq 0$, if there exists $r_0 > 0$ such that: grad $V(x) \neq 0$ for every $|x| \geqslant r_0 > 0$ and $\deg_B(\operatorname{grad} V, B(0, r), 0) \neq 0$ for every $r \geqslant r_0$, where $\deg_B(\cdot, \cdot, \cdot)$ is the Brouwer degree (see [15], [6]).

## 3. The main result and its consequences.

Let us consider the differential system

$$(L) \qquad x''(t) + \frac{d}{dt}\varphi(x(t)) + g(x(t)) = h(t)$$

where it is assumed once for all:

(i) $\varphi = \operatorname{grad} F$, with $F: \mathbb{R}^n \to \mathbb{R}$ a $C^2$-function;

(ii) $g = \operatorname{grad} G$, with $G: \mathbb{R}^n \to \mathbb{R}$ a $C^1$-function

(ii) $h: \mathbb{R} \to \mathbb{R}^n$ continuous and $p$-periodic ($p > 0$).

For any $z \in S^{n-1}$, we consider $V_z = \{w \in \mathbb{R}^n: (g(w)|z) = 0\}$. Let $M > 0$ be a real number and define

$$W(M) = \{x \in \mathbb{R}^n: \overline{B(x, M)} \cap V_z \neq \emptyset, \text{ for every } z \in S^{n-1}\}.$$

Remark that $x \in W(M)$ if and only if, for every $z \in S^{n-1}$ there exist $|y_z| \leqslant M$ such that $(g(x + y_z)|z) = 0$.

Then the following result holds true.

THEOREM 1. *Let us assume* (i), (ii) *and* (iii). *Let* $a > 0$, $b \geqslant 0$, *exist such that*

$$(j) \qquad (\varphi(x)|g(x)) \geqslant a|g(x)| - b \qquad (x \in \mathbb{R}^n)$$

*or*

$$(j') \qquad (\varphi(x)|g(x)) \leqslant -a|g(x)| + b \qquad (x \in \mathbb{R}^n)$$

*holds. Finally, let $G$ be nondegenerate with nonzero index and let us assume*

($k$) *For any $M > 0$, either $W(M)$ is bounded, or*

$$\lim_{\substack{|x| \to +\infty \\ x \in W(M)}} |g(x)| = +\infty .$$

*Then equation* ($L$) *has a $p$-periodic solution, for any forcing term $h$,* *such that*

$$Qh = 0 \qquad \textit{and} \qquad \sup_t \left| \int_0^t h(s)\, ds \right| < a\,.$$

REMARK 1. It is easy to see that $(k)$ is satisfied if

$$(s) \qquad \begin{aligned} &x_i g_i(x) > 0 \quad \textit{if } |x_i| \geqslant R \textit{ for } i = 1, \ldots, k \\ &x_i g_i(x) < 0 \quad \textit{if } |x_i| \geqslant R \textit{ for } i = k+1, \ldots, n \end{aligned}$$

holds, where $x = (x_1, \ldots, x_n)$, $g = (g_1, \ldots, g_n)$.

In fact, $(s)$ implies that, for any $M$, $W(M)$ is bounded: it is sufficient to observe that the set $\{x : \overline{B(x, M)} \cap V_{e_i} \neq \emptyset,\ i = 1, \ldots, n\}$, with $\{e_1, \ldots, e_n\}$ the canonical basis in $\mathbb{R}^n$, is bounded by $(R + M) n^{\frac{1}{2}}$. Moreover, from known results on topological degree, $(s)$ also implies that $\gamma(G, \infty) = \deg_B(g, B(0, r), 0) \neq 0$ for $r > Rn^{\frac{1}{2}}$ ($g = \operatorname{grad} G$) and hence $G$ is nondegenerate with nonzero index.

Condition $(s)$ was used in [11], [7], [19] in order to get periodic solutions in Liénard equations.

Let there exist $N : \mathbb{R}^n \to \mathbb{R}^n$ continuous and such that $\deg_B(N, B(0, r), 0) \neq 0$, for $r \geqslant r_0 > 0$. If

$(w)$ *for any* $M > 0$, *there exists* $r_M > 0$ *such that* $(g(x + y) | N(x)) > 0$, *for every* $|x| \geqslant r_M$, $|y| \leqslant M$,

holds, then it is not difficult to see that $\gamma(G, \infty) \neq 0$ and $(k)$ is satisfied. In fact, through a homothopy, $\gamma(G, \infty) = \deg_B(N, B(0, r), 0) \neq 0$ (for $r$ sufficiently large) and $W(M)$ is bounded because it is contained in the ball $\overline{B(0, r_M)}$. The case $N(x) = x$ (or $= -x$) has been considered in [18] for Rayleigh equations.

Another simple condition on $g$ which ensures $(k)$ is

$$(m) \qquad \lim_{|x| \to +\infty} |g(x)| = +\infty\,.$$

Observe that if $g$ is a homeomorphism, then $(m)$ is trivially satisfied ($g$ is proper) and $G$ is also nondegenerate with index different from zero (see [6, Th. 3.3.3]).

Moreover, recall (Hadamard Theorem [3]) that $(m)$, together with

$(m')$ $G$ *of class* $C^2$, $\det(\operatorname{Hess} G(x)) \neq 0$ *for all* $x \in \mathbb{R}^n$

implies that $g = \operatorname{grad} G$ is a homeomorphism (diffeomorphism). Finally, recall from Krasnosel'skii [4, Lemma 6.5] that $(m)$ together with

$$(m'') \qquad \lim_{|x|\to+\infty} G(x) = +\infty \quad (-\infty)\,,$$

implies that $G$ is nondegenerate with nonzero index.

Actually, the degree condition on $g$ holds true if $G$ satisfies $(m'')$ and has a finite number of critical points [2, Th. 2].

The following simple example shows how condition $(k)$ can be easily proved (without make use of $(s)$, $(m)$, $(w)$).

EXAMPLE. Let $n = 2$, $g(x) = (g_1, g_2)(x) = \big(x_2 - q(x_1), x_1 - x_2\big)$, $x = (x_1, x_2)$; where $q$ is a continuously differentiable function such that

$$(e) \qquad q'(0) \neq 1\,;$$

$$(ee) \qquad q(s) = s \quad \textit{iff } s = 0.$$

*Under* $(e)$ *and* $(ee)$, $g = \operatorname{grad} G$ *with* $G$ *nondegenerate with nonzero index and* $(k)$ *holds.*

PROOF. Obviously, $g$ admits a potential $G$, moreover, by $(ee)$, $g(x) = 0$ only for $x = 0$. Evaluating the jacobian of $g$ at $0$, we see that $\det g'(0) = q'(0) - 1$ and therefore $(e)$ implies $\deg\big(g, B(0,r), 0\big) \neq 0$ for every $r > 0$ (see [6, p. 3]).

Take $z_1 = (1,1)/\sqrt{2} \in S^1$. Then $\big(g(x)|z_1\big) = 0$ iff $x_1 = q(x_1)$, iff $x_1 = 0$ (for $(ee)$). Hence $V_{z_1} = \{(0, x_2): x_2 \in \mathbb{R}\}$ ($x_2$ — axis).

Take $z_2 = e_2 = (0,1)$. Then $\big(g(x)|z_2\big) = 0$ iff $x_1 = x_2$ and $V_{z_2} = \{(t,t): t \in \mathbb{R}\}$ ($x_2 = x_1$ line).

Since $w \in W(M)$ implies $\overline{B(w, M)} \cap V_{z_i} \neq \emptyset$ for $i = 1, 2$, one easily realizes that, for any $M$, $W(M)$ is bounded.

Observe that, in this example, $(s)$ fails and $(m)$ holds only if

$$\lim_{|s|\to+\infty} |q(s) - s| = +\infty$$

(which is not assumed here). Condition $(w)$ is not so easily reached.

From Theorem 1 we have, as an immediate consequence:

COROLLARY 1. *Let* (i), (ii), (iii) *hold and let* $(s)$, *either* $(w)$, *either* $(m)$ *and* $(m')$, *or* $(m)$ *and* $(m'')$ *be satisfied. Finally, let us assume* $Qh = 0$

*and*

$$(\circ) \qquad \liminf_{|x|\to+\infty} (\varphi(x)|g(x))/|g(x)| \geqslant K > \sup_t \left| \int_0^t h(s)\,ds \right|$$

*or*

$$(\circ') \qquad \limsup_{|x|\to+\infty} (\varphi(x)|g(x))/|g(x)| \leqslant -K < -\sup_t \left| \int_0^t h(s)\,ds \right| .$$

*Then equation* $(L)$ *has a* $p$*-periodic solution.*

PROOF OF COROLLARY 1. It is easy to see that $(\circ)$, (respectively $(\circ')$) implies $(j)$ (respectively $(j')$) with $a = K - \varepsilon$, $\varepsilon > 0$ sufficiently small, and $b = \sup_{|x|\leqslant r_\varepsilon} a|g(x)| + \sup_{|x|\leqslant r_\varepsilon} |(\varphi(x)|g(x))|$, for a suitable $r_\varepsilon$ depending on $\varepsilon$. Then, from Remark 1 and Theorem 1, the thesis follows. Q.E.D.

Theorem 1 also extends a theorem of Lefschetz [5] to the systems (see [13, Satz 5.3.4, p. 227]); in fact:

COROLLARY 2. *Assume* (i), (ii) *and* (iii). *Let* $G$ *be nondegenerate with nonzero index and let*

$$(a') \qquad \lim_{|x|\to+\infty} |g(x)|/|x| = +\infty ,$$

$$(b') \qquad |\varphi(x) - C\cdot g(x)| \leqslant B|x| , \qquad \textit{for } |x| \geqslant R,$$

*with* $C$ *a positive* (*negative*) *definite matrix and* $B$, $R$ *positive constants. Then equation* $(L)$ *has a* $p$*-periodic solution.*

PROOF OF COROLLARY 2. Set $\tilde{G}(x) = G(x) - (Qh|x)$ and $\tilde{h}(t) = h(t) - Qh$. Then equation $(L)$ is equivalent to $x'' + (d/dt)\varphi(x) + \tilde{g}(x) = \tilde{h}$; $\tilde{g} = \operatorname{grad} \tilde{G}$. Clearly, $\tilde{G}$ is nondegenerate with index $\neq 0$ $(\gamma(\tilde{G}, \infty) = \gamma(G, \infty))$ and satisfies $(m)$ of Remark 1. Moreover, from $(b')$ we have:

$$\begin{aligned} -|C\cdot Qh||\tilde{g}(x)| - B|x||\tilde{g}(x)| + (C\cdot\tilde{g}(x)|\tilde{g}(x)) &\leqslant (\varphi(x)|\tilde{g}(x)) \leqslant \\ \leqslant (C\cdot\tilde{g}(x)|\tilde{g}(x)) + B|x||\tilde{g}(x)| + |C\cdot Qh||\tilde{g}(x)| ; &\qquad \text{for } |x| \geqslant R \end{aligned}$$

and hence, using $(a')$, we easily obtain $(\circ)$ or $(\circ')$ of Corollary 1, with $K = +\infty$, according to the fact that $C$ be positive or negative definite. Then we apply Theorem 1 and the thesis follows. Q.E.D.

Remark that, for the scalar case, hypothesis ($a$) in [13, p. 227] implies ($a'$) and also the condition on the index of $G$.

In the case of the scalar differential equation

$$(L') \qquad x''(t) + f(x(t))\cdot x'(t) + g(x(t)) = h(t)\,,$$

with $f$, $g$, $h\colon \mathbb{R} \to \mathbb{R}$ continuous functions and $h$, $p$-periodic in $t$, we immediately obtain:

Corollary 3 (scalar case). *Let us assume*

$$(s') \qquad g(x)\cdot \operatorname{sign} x > 0 \;\; (<0) \qquad \textit{for } |x| \geqslant d > 0,$$

$$(s'') \qquad \left(\int_0^x f(t)\,dt\right)\cdot \operatorname{sign} x \geqslant K \;\; (\leqslant -K) \qquad \textit{for } |x| \geqslant d > 0.$$

*Then the scalar Liénard equation* ($L'$) *has a* $p$*-periodic solution for any* $h$ *such that*

$$Qh = 0 \qquad \textit{and} \qquad \sup_t \left| \int_0^t h(s)\,ds \right| < K\,.$$

Proof of Corollary 3. Set $\varphi(x) = \int_0^x f(s)\,ds$ and observe that

$$(\varphi(x)|g(x)) = \varphi(x)\cdot g(x) = (\varphi(x) \operatorname{sign} x)(g(x) \operatorname{sign} x)$$

and $|g(x)| = g(x) \operatorname{sign} x$ (resp. $= -g(x) \operatorname{sign} x$) for $|x| \geqslant d$. Then, apply Corollary 1 with sign condition ($s$). Q.E.D.

Remark 2. Using a standard perturbation argument based on Ascoli-Arzelà Theorem (see [11, Remark 1]) and the a-priori bounds reached in the proof of Theorem 1 (see the next section), it is not difficult to see that the thesis of Corollary 3 holds true even if in ($s'$) the inequalities are not strict.

Let us observe also that, if

$$\lim_{|x| \to +\infty} \left(\int_0^x f(t)\,dt\right) \operatorname{sign} x = +\infty\,(-\infty)\,,$$

then no bound is required on $\left|\int_0^t h(s)\,ds\right|$. If, moreover,

$$\lim_{|x|\to+\infty} g(x)\cdot \operatorname{sign} x = +\infty\,(-\infty)\,,$$

then also the assumption $Qh = 0$ can be dropped. In fact, passing to the equivalent equation $x'' + f(x)x' + \tilde{g}(x) = \tilde{h}$, where $\tilde{g}(x) = g(x) - Qh$, $\tilde{h}(t) = h(t) - Qh$, we see that all the hypotheses in Corollary 3 are satisfied.

Therefore, we obtain as corollaries of our result: the theorem of Reuter [14] (see also [16, p. 509], [13, Satz 5.4.3]) in the part concerning existence of periodic solutions, as well as other classical results of the same type (see [16, p. 487-488], [13, Satz 5.3.7, p. 234; Satz 5.3.8-9, p. 236-9] and [12, Satz I]), a theorem by Mawhin [8, Th. 5.4, p. 378] when $f$ and $g$ are polinomials (with even and odd degree respectively) and theorems by Bebernes-Martelli [1, Th. 2, $c \neq 0$] and Ward [17].

## 4. Proofs.

In the proof of our main result, the following easily estabilished inequality is used.

LEMMA 2. *Let $y = (y_1, \dots, y_n)\colon \mathbb{R} \to \mathbb{R}^n$ be a $p$-periodic $C^1$-function and let $t_1, \dots, t_n \in [0, p]$. Then, for every $t \in \mathbb{R}$,*

$$|y(t) - w| \leqslant (p/2)^{\frac{1}{2}} |y'|_2\,, \qquad \textit{where } w = \operatorname{col}(y_i(t_i))\,.$$

Idea of the proof: use Jensen's Inequality on each component.

PROOF OF THEOREM 1. First of all, let us observe that

$$S(z) := \frac{1}{p}\int_0^p \left\{-\frac{d}{dt}\varphi(z) - g(z) + h(t)\right\} dt = -g(z)\,,$$

for all $z \in \mathbb{R}^n$. Thus, as $G$ is nondegenerate with nonzero index, we immediately see that condition $(\beta)$ of Mawhin's Lemma is satisfied.

Let $x(t)$ be a $p$-periodic solution of

$$(L_\lambda) \qquad x''(t) + \lambda \frac{d}{dt}\varphi(x(t)) + \lambda g(x(t)) = \lambda h(t), \qquad \lambda \in (0, 1].$$

Passing to the mean value in $(L_\lambda)$ and recalling $Qh = 0$, we obtain (after a division by $\lambda > 0$)

$$(1) \qquad \int_0^p g(x(t))\, dt = 0$$

and hence it follows that

$$Z(x)(t) := \int_0^t g(x(s))\, ds, \qquad \text{is } p\text{-periodic.}$$

We take now the $L^2$-scalar product of $(L_\lambda)$ by $Z(x)$ and observe:

$$(x'', Z(x))_2 = -(g(x), x')_2 = G(x(0)) - G(x(p)) = 0,$$

$$\left(\frac{d}{dt}\varphi(x), Z(x)\right)_2 = -(\varphi(x), g(x))_2,$$

$$(g(x), Z(x))_2 = \tfrac{1}{2}|Z(x)(p)|^2 - \tfrac{1}{2}|Z(x)(0)|^2 = 0,$$

$$(h, Z(x))_2 = -(H, g(x))_2, \qquad \text{where } H(t) := \int_0^t h(s)\, ds.$$

Thus we immediately obtain (Hölder inequality)

$$(2) \qquad |(\varphi(x), g(x))_2| \leqslant |g(x)|_1 |H|_\infty.$$

Then, from $(j)$ or $(j')$ and (2), a bound for $|g(x)|_1$ is reached:

$$(3) \qquad |g(x)|_1 \leqslant C_1 = bp/(a - |H|_\infty).$$

Moreover, from (3), it easily follows

$$(4) \qquad \min_{[0,p]} |g(x(t))| \leqslant C_1/p.$$

Let $z \in S^{n-1}$ and let us take the inner product of (1) by $z$. Then, thanks to the mean value theorem, we have that $\big(g(x(t_z))|z\big) = 0$ for some $t_z \in [0, p]$.

So, for every $z \in S^{n-1}$, there exists $x_z = x(t_z) \in \mathbb{R}^n$, such that

$$x_z \in V_z\,. \tag{5}$$

Let us set $u_z(t) = x(t) - x_z$ and take the $L^2$-scalar product of $(L_\lambda^{\sim})$ by $u_z$. Observe that

$$(x'', u_z)_2 = -|x'|_2^2\,,$$

$$\left(\frac{d}{dt}\varphi(x), u_z\right)_2 = \left(\frac{d}{dt}\varphi(x), x\right)_2 = -(\varphi(x), x')_2 = F\big(x(0)\big) - F\big(x(p)\big) = 0$$

and so, with easy computations (using Hölder inequality and $\lambda \leqslant 1$) one has

$$|x'|_2^2 = |u_z'|_2^2 \leqslant |g(x)|_1 |u_z|_\infty + |h|_1 |u_z|_\infty\,. \tag{6}$$

Moreover, taking into account that $|u_z|_\infty^2 \leqslant (p/2)|u_z'|_2^2$ (Lemma 2) we obtain also (from (6))

$$|u_z|_\infty \leqslant (p/2)\big(|g(x)|_1 + |h|_1\big)\,. \tag{7}$$

Hence, from (7) and (3), we have a bound for $|u_z|_\infty$:

$$|u_z|_\infty = |x - x_z|_\infty \leqslant M\,, \tag{8}$$

with $M = (p/2)\big(C_1 + |h|_1\big)$.

Finally, from (8) and (6), also a bound for $|x'|_2$ (independent of $\lambda$ and $x$) is given:

$$|x'|_2 \leqslant C_2\,. \tag{9}$$

Observe that (5) and (8) mean that

$$x(t) \in W(M) \qquad \text{for every } t \in [0, p]\,. \tag{10}$$

Then, condition $(k)$, together with (4) and (10), implies that there exists a constant $C_3$ (independent of $\lambda$ and $x$) such that

$$|x(t)| \leqslant C_3 \quad \text{for some } t \in [0, p]. \tag{11}$$

From (9) and (11) we easily obtain a bound for $|x|_\infty$:

$$|x|_\infty < K_0 . \tag{12}$$

From (9) and (12) and equation $(L_\lambda)$ we see that $|x''|_2$ is also bounded (in virtue of the continuity of $g$ and $\varphi' = \operatorname{Hess} F$) and so, with easy inequality (Lemma 2), we obtain

$$|x'|_\infty < K_1 . \tag{13}$$

Therefore, condition ($\alpha$) in Mawhin's Lemma holds true and the thesis follows. Q.E.D.

At last, we observe that (from the proof) it is clear that the assumption $|H|_\infty < a$ can be changed into $|\bar{H}|_\infty < a$, where $\bar{H}$ is any other primitive of $h$ $(\bar{H}'(t) = h(t))$.

## REFERENCES

[1] J. W. Bebernes - M. Martelli, *Periodic solutions for Liénard systems*, Boll. Un. Mat. Ital., (5), **16**-A (1979), pp. 398-405.

[2] A. Castro - A. C. Lazer, *Critical point theory and the number of solutions of a nonlinear Dirichlet problem*, Ann. Mat. Pura Appl., (IV), **70** (1979), pp. 113-137.

[3] W. B. Gordon, *On the diffeomorphisms of euclidean space*, Amer. Math. Monthly, **79** (1972), pp. 755-759.

[4] M. A. Krasnosel'skii, *The operator of translation along the trajectories of differential equations*, Amer. Math. Soc. Providence, R.I. (1968).

[5] S. Lefschetz, *Existence of periodic solutions for certain differential equations*, Proc. Nat. Acad. Sci. USA, **29** (1943), pp. 29-32.

[6] N. G. Lloyd, *Degree theory*, Cambridge University Press, Cambridge (1978).

[7] M. Martelli, *On forced nonlinear oscillations*, J. Math. Anal. Appl., **69** (1979), pp. 496-504.

[8] J. Mawhin, *Degré topologique et solutions périodiques des systèmes différentielles non linéaires*, Bull. Soc. Roy. Sci. Liège, **38** (1969), pp. 308-398.

[9] J. Mawhin, *Equations intégrales et solutions périodiques des systèmes différentielles non linéaires*, Bull. Acad. Roy. Belge Cl. Sci., (5), **55** (1969), pp. 934-947.

[10] J. Mawhin, *Existence of periodic solutions for higher order differential systems that are not of class D*, J. Differential Equations, **8** (1970), pp. 523-530.

[11] J. Mawhin, *An extension of a theorem of A. C. Lazer on forced nonlinear oscillations*, J. Math. Anal. Appl., **40** (1972), pp. 20-29.

[12] R. Reissig, *Über eine nichtlineare Differentialgleichung* 2. *Ordnung*, Math. Nachrichten, **13** (1955), pp. 313-318.

[13] R. Reissig - G. Sansone - R. Conti, *Qualitative theorie nichtlinearer differentialgleichungen*, Ed. Cremonese, Roma (1963).

[14] G. E. H. Reuter, *A boundedness theorem for nonlinear differential equations of the second order*, Proc. Cambridge Philos. Soc., **47** (1951), pp. 49-54.

[15] N. Rouche - J. Mawhin, *Equations Differentielles Ordinaires* (2 vol.), Masson, Paris (1973).

[16] G. Sansone - R. Conti, *Equazioni differenziali non lineari*, Ed. Cremonese, Roma (1956).

[17] J. R. Ward, *Periodic solutions for a class of ordinary differential equations*, Proc. Amer. Math. Soc., **78** (1980), no. 3, pp. 350-352.

[18] F. Zanolin, *Periodic solutions for second order differential systems with damping*, Rend. Sem. Mat. Univ. Padova, **65** (1981) (in print).

[19] F. Zanolin, *Remarks on multiple periodic solutions for nonlinear ordinary differential systems of Liénard type*, Boll. Un. Mat. Ital. (6), **1**-B (1982). (in print).

Manoscritto pervenuto in redazione il 28 dicembre 1981.