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Soluble Products of Two Locally Finite Groups with Min- p for Every Prime p .

BERNHARD AMBERG (*)

Introduction.

It is known that every soluble product of two Černikov groups is a Černikov group; see (1) and (4). Also, in (3), Folgerung 5.3, p. 237, it was shown that every soluble product of two locally nilpotent π -groups with min- p for every prime p in a set of primes π , is a (locally finite) π -group with min- p for every p . In the following these results will be generalized.

If the soluble group $G = AB$ is the product of two π -subgroups A and B , one of which satisfies min- p for every prime p in the set of primes π , then G is a π -group. Thus, in particular, G is locally finite, and the set of primes p , for which there is an element of order p in G , equals the set of primes, for which there is an element of this order in A or B (Corollary 2.2). If even both, A and B , satisfy min- p for every p , and if A or B is almost locally-nilpotent, then G satisfies min- p for every p (Corollary 2.5). Further related results are discussed in section 2.

These results suggest the conjecture that every soluble product of two locally finite groups with min- p must satisfy min- p . To solve this problem one would have to search for additional «factorized» subgroups of the factorized group $G = AB$.

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Notations.

- π = set of primes;
 πG = set of primes p , for which there is an element of order p in the group G ;
 $J(G)$ = intersection of all subgroups of finite index in G ;
 $O_p(G)$ = maximal normal p -subgroup of G ;
 G_π = π -component of the locally nilpotent group G ;
factor = epimorphic image of a subgroup;
 π -group = group, in which the order of every element contains only primes from the set of primes π .

A group satisfies *min- p* for the prime p , if each of its p -subgroups satisfies the minimum condition on subgroups,

If X is a class of groups, a group is an *almost X-group* or is *almost X* if it contains a normal subgroup of finite index which is contained in the class X ,

A group is a *Černikov group* if it is almost abelian and satisfies the minimum condition on subgroups.

1. Lemmas.

The following general reduction lemma is useful for the investigation of factorized soluble groups.

LEMMA 1.1. *Let X be a class of groups closed under the forming of subgroups, epimorphic images and extensions. If there exists a soluble group $G = AB$, which is the product of two X -subgroups A and B , but is itself not an X -group, then there also exists a soluble group $F = S_1 S_2 = KS_1 = KS_2$, which is the product of two X -subgroups S_1 and S_2 and is itself not an X -group and where K is an abelian normal subgroup of F with $K \cap S_1 = K \cap S_2 = 1$. Furthermore, if the soluble group $T = T_1 T_2$ is the product of two X -subgroups T_1 and T_2 and if the sum of the derived lengths of T , T_1 and T_2 is smaller than the sum of the derived lengths of F , S_1 and S_2 , then T is an X -group.*

REMARK. $F = S_1 S_2$ is found below as a factor of a suitable « counterexample » $G = AB$, where S_1 is a factor of A and S_2 is a factor of B .

PROOF. Among all soluble products of two \mathbf{X} -groups, which themselves are no \mathbf{X} -groups, choose one $G = AB$ such that the sum of the derived lengths of G , A and B is minimal. Let C be the last nontrivial term of the derived series of G .

The group $G/C = (AC/C)(BC/C)$ is the product of two \mathbf{X} -subgroups $AC/C \simeq A/(A \cap C)$ and $BC/C \simeq B/(B \cap C)$. By induction G/C is an \mathbf{X} -group. Since \mathbf{X} is extension closed, C is not an \mathbf{X} -group.

By Amberg (1), Theorem 1.7, p. 108, the factorizer $X = X(C)$ of C in G satisfies

$$X = AC \cap BC = C(A \cap BC) = C(B \cap AC) = (A \cap BC)(B \cap AC).$$

Since C is an abelian normal subgroup of G , $A \cap C$ is a normal subgroup of $X = C(A \cap BC)$ and $B \cap C$ is a normal subgroup of $X = C(B \cap AC)$. Therefore $C^* = (A \cap C)(B \cap C)$ is normal in X . As an abelian product of two \mathbf{X} -groups, C^* is an \mathbf{X} -group.

Let $K = C/C^*$, $S_1 = (A \cap BC)C^*/C^*$ and $S_2 = (B \cap AC)C^*/C^*$. Then $C \cap (A \cap BC) \subseteq (A \cap C) \subseteq C^*$ and $C \cap (B \cap AC) \subseteq (B \cap C) \subseteq C^*$. Hence $F = S_1 S_2 = K S_1 = K S_2$ where $K \cap S_1 = K \cap S_2 = 1$. Assume that F is an \mathbf{X} -group. Then also the subgroup K of F is an \mathbf{X} -group. Since C^* is an \mathbf{X} -group, also C is an \mathbf{X} -group. This contradiction shows that F is not an \mathbf{X} -group. The other assertions are clear.

For any class of groups \mathbf{X} , let $\mathbf{R}_{\mathbf{X}}(Y)$ be the product of all normal \mathbf{X} -subgroups of the group Y , the so-called \mathbf{X} -radical of Y . If A is a subgroup and N a normal subgroup of the group G , then for any epimorphism closed class of groups \mathbf{X} , obviously the following holds: $\mathbf{R}_{\mathbf{X}}(A)N/N \subseteq \mathbf{R}_{\mathbf{X}}(AN/N)$. For, if K is a normal \mathbf{X} -subgroup of A , then $KN/N \simeq K/(K \cap N)$ is a normal \mathbf{X} -subgroup of AN/N . Some other statements of this kind are contained in the following lemma.

LEMMA 1.2. *If G is a group, A a subgroup and N a normal subgroup of G with $A \cap N = 1$, then for any class of groups \mathbf{X} the following holds:*

- (a) $\mathbf{R}_{\mathbf{X}}(A)N/N \subseteq \mathbf{R}_{\mathbf{X}}(AN/N)$,
- (b) *If for every group Y the \mathbf{X} -radical $\mathbf{R}_{\mathbf{X}}(Y)$ is an \mathbf{X} -group, then $\mathbf{R}_{\mathbf{X}}(A)N/N = \mathbf{R}_{\mathbf{X}}(AN/N)$; in particular $\mathbf{R}_{\mathbf{X}}(A)N$ is a normal subgroup of AN ,*
- (c) *If B is another subgroup of G such that $AN = BN$ and if for every group Y the \mathbf{X} -radical $\mathbf{R}_{\mathbf{X}}(Y)$ is an \mathbf{X} -group, then $\mathbf{R}_{\mathbf{X}}(A)N = \mathbf{R}_{\mathbf{X}}(B)N$ and this group is normalized by AB .*

PROOF. (a) If K is a normal X -subgroup of A , then $KN/N \simeq K/(K \cap N) \simeq K$ is a normal X -subgroup of AN/N . This shows (a).

(b) Since $\mathbf{R}_X(A)N/N \simeq \mathbf{R}_X(A)/(\mathbf{R}_X(A) \cap N) \simeq \mathbf{R}_X(A)$, the group $\mathbf{R}_X(A)N/N$ is a normal X -subgroup of AN/N . Therefore (or directly by (a))

$$\mathbf{R}_X(A)N/N \subseteq \mathbf{R}_X(AN/N).$$

Let S be the uniquely determined normal subgroup of AN such that $N \subseteq S$ and $S/N = \mathbf{R}_X(AN/N)$. By the modular law $S = S \cap AN = (A \cap S)N$. This implies

$$\mathbf{R}_X(AN/N) = S/N = (A \cap S)N/N \simeq (A \cap S)/(A \cap S) \cap N \simeq (A \cap S).$$

Thus $A \cap S$ is a normal X -subgroup of A . Therefore $(A \cap S) \subseteq \mathbf{R}_X(A)$ and thus

$$\mathbf{R}_X(AN/N) = (A \cap S)N/N \subseteq \mathbf{R}_X(A)N/N.$$

This proves (b).

(c) By (b) we have

$$\mathbf{R}_X(A)N/N = \mathbf{R}_X(AN/N) = \mathbf{R}_X(BN/N) = \mathbf{R}_X(B)N/N.$$

If $g = ab$ where $a \in A$ and $b \in B$, then

$$\begin{aligned} (\mathbf{R}_X(A)N)^g &= (\mathbf{R}_X(A))^b N = (\mathbf{R}_X(A)N)^b = (\mathbf{R}_X(B)N)^b = \mathbf{R}_X(B)N = \\ &= \mathbf{R}_X(A)N. \end{aligned}$$

Thus $\mathbf{R}_X(A)N$ is normalized by AB .

For any group Y denote by $\mathbf{J}(Y)$ the intersection of all subgroups of Y with finite index. If Y is a Černikov group, then $Y/\mathbf{J}(Y)$ is finite and $\mathbf{J}(Y)$ is a characteristic radicable abelian p -group of finite rank.

LEMMA 1.3. *If the locally soluble group $G = AB$ is the product of two torsion subgroups A and B with $\min\{p\}$ for the prime p , then the following holds:*

- (a) $\langle \mathbf{J}(\mathbf{O}_p(A)), \mathbf{J}(\mathbf{O}_p(B)) \rangle$ is a p -subgroup of G and therefore locally finite,

(b) If $\mathbf{O}_p(A)$ and $\mathbf{O}_p(B)$ are finite, then $\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \rangle$ is a finite p -subgroup of G .

PROOF. As a p -subgroup of A , the group $\mathbf{O}_p(A)$ satisfies the minimum condition on subgroups. As a locally soluble group with minimum condition on subgroups $\mathbf{O}_p(A)$ is a Černikov group; see Kegel and Wehrhritz (6), Theorem 1.E.6, p. 31. Thus $\mathbf{J}(\mathbf{O}_p(A))$ is a characteristic radicable abelian p -subgroup of finite rank and with finite index $|\mathbf{O}_p(A) : \mathbf{J}(\mathbf{O}_p(A))|$. As a characteristic subgroup of the characteristic subgroup of $\mathbf{O}_p(A)$ of A also $\mathbf{J}(\mathbf{O}_p(A))$ is a characteristic subgroup A .—Analogously $\mathbf{J}(\mathbf{O}_p(B))$ is a characteristic radicable abelian p -subgroup of B with finite rank. Every finite subset of $\mathbf{J}(\mathbf{O}_p(A))$ is contained in a finite normal subgroup A_1 of A with $A_1 \subseteq \mathbf{J}(\mathbf{O}_p(A))$. Also, every finite subset of $\mathbf{J}(\mathbf{O}_p(B))$ is contained in a finite normal subgroup B_1 of B with $B_1 \subseteq \mathbf{J}(\mathbf{O}_p(B))$.

Let $S_1 = \langle A_1, B_1 \rangle$. Since A and B are torsion groups, by Kegel (5), Lemma 1.2, p. 536, the normalizer $N(S_1)$ is factorized. Since G is locally soluble, by Amberg (1), Proposition 5.3, p. 117, S_1 is finite and $\pi S_1 = \pi A_1 \cup \pi B_1 = \{p\}$.

It follows that $\langle \mathbf{J}(\mathbf{O}_p(A)), \mathbf{J}(\mathbf{O}_p(B)) \rangle$ is a p -group. Since G is locally soluble, p -subgroups of G are locally finite.

The proof of (b) is similar.

The following criterion for a factorized almost locally-nilpotent group to satisfy min- p is needed.

LEMMA 1.4. *If the almost locally-nilpotent group $G = AB$ is the product of two π -subgroups A and B with min- p for the prime p in the set of primes π , then G is a π -group with min- p .*

PROOF. The Hirsch-Plotkin radical $R = \mathbf{R}(G)$ has finite index $|G : R|$ in G . Then also the indices

$$|A : (A \cap R)| \leq |G : R| \quad \text{and} \quad |B : (B \cap R)| \leq |G : R|$$

are finite. By Amberg (1), Lemma 5.1, p. 116, also $V = \langle A \cap R, B \cap R \rangle$ has finite index in G and is contained in the π -component of R . Hence there exists a normal π -subgroup N of G contained in V with finite factor group G/N . As a finite product of two π -subgroups $G/N = (AN/N)(BN/N)$ is a π -group. Hence also G is a π -group.

As a locally finite-nilpotent group, $R = R_p \otimes R_{p'}$ is the direct product of its p -component R_p and its p' -component $R_{p'}$. The factor

group $G/R_{p'} = (AR_{p'}/R_{p'})(BR_{p'}/R_{p'})$ contains a normal p -subgroup $R/R_{p'}$ of finite index and is therefore an almost p -group.

The groups $AR_{p'}/R_{p'}$ and $BR_{p'}/R_{p'}$ are isomorphic to almost p -factors of A resp. B and therefore satisfy the minimum condition on subgroups. By Amberg (1), Corollary 3.3, p. 112, the normal subgroups of $G/R_{p'}$ satisfy the minimum condition. Since $R/R_{p'}$ has finite index in $G/R_{p'}$, also the normal subgroups of $R/R_{p'}$ satisfy the minimum condition; see Robinson (7), Theorem 5.21, p. 147.

As a locally nilpotent group with minimum condition on normal subgroups $R/R_{p'}$ is a Černikov group; see Kegel and Wehrfritz (6), Theorem 1.H.4, p. 44. Thus also $G/R_{p'}$ is a Černikov group. This implies that G satisfies min- p .

The basis of the subsequent considerations is the following proposition.

PROPOSITION 1.5. *If the locally soluble group $G = AB = AK = BK$ is the product of two π -subgroups A and B with min- p for the prime p in the set of primes π and an abelian normal subgroup K of G with $A \cap K = B \cap K = 1$, then the following holds:*

- (a) $K_p \mathbf{O}_p(A) = K_p \mathbf{O}_p(B)$, and this group is a normal p -subgroup of G ,
- (b) If $A \simeq B$ satisfies min- p for every p , then

$$H = \prod_{p \in \pi} K_p \mathbf{O}_p(A) = K_\pi \mathbf{R}(A)$$

is a normal π -subgroup of G contained in $\mathbf{R}(G)$; and if, in addition, $A \simeq B$ is almost locally-nilpotent, then G is an almost locally-nilpotent π -group with min- p for every p .

PROOF. By Lemma 1.2 (c) $K\mathbf{O}_p(A) = K\mathbf{O}_p(B)$, and this group is a normal subgroup of $AB = G$. By hypothesis $\mathbf{O}_p(A)$ and $\mathbf{O}_p(B)$ are Černikov groups, so that $J_1 = \mathbf{J}(\mathbf{O}_p(A))$ and $J_2 = \mathbf{J}(\mathbf{O}_p(B))$ are radicable abelian p -subgroups of finite rank and with finite index in $\mathbf{O}_p(A)$ resp. $\mathbf{O}_p(B)$. By Lemma 1.2 (c) $KJ_1 = KJ_2$, and this group is a normal subgroup of $AB = G$.

By Lemma 1.3 (a) $\langle J_1, J_2 \rangle$ is a p -group. If $a \in J_1$, then $a = sb$ with $b \in J_2$ and $s \in K$. Then $s = s_1 s_2$ where s_1 is a p' -element and s_2 is a p -element. Since the p -component K_p of the abelian normal subgroup K of G is a normal subgroup of G , $\langle J_1, J_2 \rangle K_p$ is a p -subgroup of G . Thus $s_1 = ab^{-1}s_2^{-1}$ is a p' -element and a p -element, and there-

fore equals 1. This shows $J_1 \subseteq K_p J_2$. Similarly $J_2 \subseteq K_p J_1$. If $P = K_p J_1 = K_p J_2$, then by Lemma 1.2 (c) P is a normal subgroup of G .

The group $G/P = (AP/P)(BP/P)$ is the product of two π -subgroups $AP/P \simeq A/(A \cap P)$ and $BP/P \simeq B/(B \cap P)$, whose maximal normal p -subgroups are finite, since by the modular law

$$J_1 \subseteq J_1(A \cap K_p) = A \cap K_p J_1 = (A \cap P) \subseteq \mathbf{O}_p(A),$$

and the index $|\mathbf{O}_p(A):J_1|$ is finite.

Therefore $\mathbf{O}_p(AP/P)$ and $\mathbf{O}_p(BP/P)$ are finite p -subgroups of G/P . By Lemma 1.3 (b) the group

$$\begin{aligned} \langle \mathbf{O}_p(AP/P), \mathbf{O}_p(BP/P) \rangle &\supseteq \langle \mathbf{O}_p(A)P/P, \mathbf{O}_p(B)P/P \rangle = \\ &= \langle \mathbf{O}_p(A), \mathbf{O}_p(B) \rangle P/P \end{aligned}$$

is a finite p -group. Then also the group isomorphic to the last group

$$\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \rangle / (P \cap \langle \mathbf{O}_p(A), \mathbf{O}_p(B) \rangle)$$

is a p -group. Hence also $\langle \mathbf{O}_p(A), \mathbf{O}_p(B) \rangle$ is a p -group.

As above it is shown that $K_p \mathbf{O}_p(A) = K_p \mathbf{O}_p(B)$, and that this group is a normal subgroup of G . This proves (a).

If $A \simeq B$ is a π -group with min- p for every prime p , then by (a) every group $K_p \mathbf{O}_p(A) = K_p \mathbf{O}_p(B)$ is a normal p -subgroup of G , which is locally nilpotent and therefore contained in the Hirsch-Plotkin radical $\mathbf{R}(G)$. Since A is locally finite, $\mathbf{R}(A)$ is the direct product of the maximal normal p -subgroups $\mathbf{O}_p(A)$. Also, $\mathbf{R}(B)$ is the direct product of the maximal normal p -subgroups $\mathbf{O}_p(B)$. Then

$$\begin{aligned} H &= \prod_{p \in \pi} K_p \mathbf{O}_p(A) = \prod_{p \in \pi} K_p \prod_{p \in \pi} \mathbf{O}_p(A) = KR(A) \\ &= \prod_{p \in \pi} K_p \mathbf{O}_p(B) = \prod_{p \in \pi} K_p \prod_{p \in \pi} \mathbf{O}_p(B) = KR(B) \end{aligned}$$

is a normal π -subgroup of G which is contained in $\mathbf{R}(G)$. In particular $\langle \mathbf{R}(A), \mathbf{R}(B) \rangle \subseteq H$.

If $A \simeq B$ is almost locally-nilpotent, then $\mathbf{R}(A)$ has finite index in A and $\mathbf{R}(B)$ has finite index in B . By Amberg (1), Lemma 5.1,

p. 116, also $\langle \mathbf{R}(A), \mathbf{R}(B) \rangle$ and therefore H has finite index in $G = AB$. Thus G is almost locally-nilpotent. By Lemma 1.4 G is a π -group with $\text{min-}p$ for every prime p . This proves (b).

2. Results.

The first theorem is formulated for a rather general class of groups \mathbf{X} .

Theorem 2.1. *Let \mathbf{X} be a class of groups closed under the forming of subgroups, epimorphic images and extensions such that for a set of primes π every soluble π -group is an \mathbf{X} -group. If the soluble group $G = AB$ is the product of an \mathbf{X} -subgroup A and a π -subgroup B one of which satisfies $\text{min-}p$ for every prime p , then G is an \mathbf{X} -group.*

PROOF. Assume that the theorem is false. Then there is a soluble group $G = AB$, which is the product of an \mathbf{X} -subgroup A and a π -subgroup B , one of which satisfies $\text{min-}p$ for every p , but itself is not an \mathbf{X} -group. By Lemma 1.1 it may be assumed that G contains an abelian normal subgroup K such that $G = AB = AK = BK$ and $A \cap K = B \cap K = 1$. Also, it may be assumed that every soluble group $T = T_1 T_2$, which is the product of an \mathbf{X} -subgroup A and a π -subgroup B , one of which satisfies $\text{min-}p$ for every p , is an \mathbf{X} -group, if the sum of the derived lengths of T , T_1 and T_2 is smaller than the sum of the derived lengths of G , A and B .

Then $A \simeq B$ is a π -group with $\text{min-}p$ for every p . By Proposition 1.5 $H = \prod_{p \in \pi} K_p \mathbf{O}_p(A)$ is a normal π -subgroup of G , which contains $\mathbf{R}(A)$ and $\mathbf{R}(B)$. The group $G/H = (AH/H)(BH/H)$ is the product of two π -subgroups $AH/H \simeq A/(A \cap H)$ and $BH/H \simeq B/(B \cap H)$ with $\text{min-}p$ for every p . By induction G/H is an \mathbf{X} -group. As a soluble π -group also H is an \mathbf{X} -group. Hence G is an \mathbf{X} -group.

If \mathbf{X} is the class of all soluble π -groups, the following corollary mentioned in the introduction, is obtained.

COROLLARY 2.2. *If the soluble group $G = AB$ is the product of two π -subgroups A and B for the set of primes π , one of which satisfies $\text{min-}p$ for every prime p , then G is a π -group; in particular G is locally finite and $\pi G = \pi A \cup \pi B$.*

Does in the situation of Corollary 2.2 G satisfy $\text{min-}p$ for every p , if A and B do? In the following this is proved under an additional

requirement for A or B . The following theorem generalizes Theorem B of Amberg (2), p. 2.

THEOREM 2.3. *Let \mathbf{X} be a class of groups closed under the forming of subgroups, epimorphic images and extensions such that for a set of primes π every almost locally-nilpotent π -group with $\min-p$ for every prime p is an \mathbf{X} -group. If the soluble group $G = AB$ is the product of an \mathbf{X} -subgroup A and an almost locally-nilpotent π -subgroup B with $\min-p$ for every p , then G is an \mathbf{X} -group.*

PROOF. Assume that the theorem is false. Then there exists a soluble group $G = AB$, which is the product of an \mathbf{X} -subgroup A and an almost locally-nilpotent π -subgroup B with $\min-p$ for every p , but which is not an \mathbf{X} -group. By Lemma 1.1 it may be assumed that G contains an abelian normal subgroup K such that $G = AB = AK = BK$ and $A \cap K = B \cap K = 1$.

Then $A \simeq B$ is an almost locally-nilpotent π -group with $\min-p$ for every p . By Proposition 1.5 (b) G is an almost locally-nilpotent π -group with $\min-p$ for every p . By hypothesis G is therefore an \mathbf{X} -group. This contradiction proves the theorem.

A soluble group G has *finite abelian section rank*, if every abelian factor (section) of G has finite p -rank, where $p = 0$ or a prime. The class of soluble groups with finite abelian section rank is closed under the forming of subgroups, epimorphic images and extensions, and a soluble torsion group has finite abelian section rank, if and only if it satisfies $\min-p$ for every prime p ; see Robinson (8), chapter 9. Thus Theorem 2.3 has the following consequence.

COROLLARY 2.4. *If the soluble group $G = AB$ is the product of two subgroups A and B with finite abelian section rank, one of which is almost locally finite-nilpotent, then G has finite abelian section rank.*

If in Theorem 2.3 \mathbf{X} is the class of soluble π -groups with $\min-p$ for every prime p , the following is obtained.

COROLLARY 2.5. *If the soluble group $G = AB$ is the product of two π -subgroups A and B with $\min-p$ for every prime p in the set of primes π , and if A or B is almost locally-nilpotent, then G is a π -group with $\min-p$ for every p .*

Corollary 2.5 may be further specialized. First the minimum condition on subgroups of finite index is considered.

LEMMA 2.6. *The following conditions of the group G are equivalent:*

- (a) *The subgroups of finite index in G satisfy the minimum condition,*
- (b) *The normal subgroups of finite index in G satisfy the minimum condition,*
- (c) *$G/\mathbf{J}(G)$ is finite,*
- (d) *There is only a finite number of subgroups of finite index in G .*

PROOF. Clearly (a) implies (b). It follows for instance from Kegel and Wehrfritz (6), Proposition 1.E.4, p. 31, that (c) is a consequence of (b). If (c) holds, G has only finitely many subgroups of finite index, so that also (d) holds. Finally (a) is an obvious consequence of (d).

The next lemma shows that the minimum condition on subgroups of finite index is inherited by products of groups.

LEMMA 2.7. *If the group $G = AB$ is the product of two subgroups A and B with finite $A/\mathbf{J}(A)$ and $B/\mathbf{J}(B)$, then $G/\mathbf{J}(G)$ is finite, and*

$$\mathbf{J}(G) = \langle \mathbf{J}(A), \mathbf{J}(B) \rangle .$$

PROOF. Since $A/\mathbf{J}(A)$ and $B/\mathbf{J}(B)$ are finite, by Amberg (1), Lemma 5.1, p. 116, also $\langle \mathbf{J}(A), \mathbf{J}(B) \rangle$ has finite index in G . Therefore $\mathbf{J}(G) \subseteq \langle \mathbf{J}(A), \mathbf{J}(B) \rangle$.

If S is any subgroup of finite index in G , then $|A:(A \cap S)| \leq |G:S|$ and $|B:(B \cap S)| \leq |G:S|$. Therefore $\mathbf{J}(A) \subseteq (A \cap S)$ and $\mathbf{J}(B) \subseteq (B \cap S)$. Hence $\langle \mathbf{J}(A), \mathbf{J}(B) \rangle \subseteq \langle A \cap S, B \cap S \rangle \subseteq S$, so that $\langle \mathbf{J}(A), \mathbf{J}(B) \rangle \subseteq \mathbf{J}(G)$. This proves the lemma.

Now Corollary 2.5 is specialized in the following way.

COROLLARY 2.8. *If the soluble group $G = AB$ is the product of two almost locally-nilpotent π -subgroups A and B with min- p for every prime p in the set of primes π and if $A/\mathbf{J}(A)$ and $B/\mathbf{J}(B)$ are finite, then G is an almost abelian π -group with min- p for every prime p , $G/\mathbf{J}(G)$ is finite and*

$$\mathbf{J}(G) = \mathbf{J}(A)\mathbf{J}(B) .$$

PROOF. By Corollary 2.5 G is π -group with min- p for every prime p . By Lemma 2.7 $G/\mathbf{J}(G)$ is finite, and $\mathbf{J}(G) = \langle \mathbf{J}(A), \mathbf{J}(B) \rangle$. Since G is a soluble torsion group with min- p for every p , by Kegel and Wehr-

fritz (6), Corollary 3.18, p. 95, $\mathbf{J}(G)$ is a radicable abelian group. Thus also $\mathbf{J}(G) = \mathbf{J}(A)\mathbf{J}(B)$.

Finally our considerations give a new proof of the following known result; see (1) and (4).

COROLLARY 2.9. *If the soluble group $G = AB$ is the product of two Černikov π -subgroups A and B for the set of primes π , then G is a Černikov π -group, and*

$$\mathbf{J}(G) = \mathbf{J}(A)\mathbf{J}(B) .$$

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