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Solutions of Lower Semicontinuous Differential Inclusions on Closed Sets.

ALBERTO BRESSAN (*) (**)

SUNTO - Si dimostra un teorema di esistenza di soluzioni per la relazione differenziale $\dot{x} \in F(x)$, a secondo membro inferiormente semicontinuo, su di un compatto D soddisfacente opportune ipotesi di tangenza alla Nagumo.

1. Introduction.

Let F be a mapping defined on a Banach space E , taking values in the family of closed and bounded subsets of E . By a solution of the autonomous system

$$(1) \quad \dot{x} \in F(x)$$

$$(2) \quad x(0) = x_0,$$

we mean an absolutely continuous mapping $x: [0, \tau) \rightarrow E$, with $x(0) = x_0$ and $\dot{x} \in F(x)$ almost everywhere (a.e.) on some interval $[0, \tau)$. If F is Hausdorff continuous and convex valued, the evolution system (1)-(2) behaves very much like an ordinary differential equation, and existence results are comparatively easy to obtain. In the case where F is continuous but $F(x)$ is not necessarily convex, existence of solutions was first proven by A. Filippov [3]. Lojasiewicz [5] and the author [2]

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solved the lower semicontinuous case, using the selection technique developed in [1]. Apparently less attention has received the problem of finding solutions of (1)-(2) taking values inside a closed set $D \subseteq E$. In the present paper we construct a solution lying on a compact set D , provided that the following condition of Nagumo type holds:

$$(3) \quad \forall x \in D, \forall y \in F(x), \quad \lim_{h \rightarrow 0^+} d(x + hy, D) h^{-1} = 0 .$$

The proof given here relies on the careful construction of a family of piecewise linear approximate solutions, and shows that the original technique used by A. Filippov can be adapted to the lower semicontinuous case as well. We also show how our result yields a general method for deriving the existence of solutions of (1)-(2) from the properties of the convex-valued orientor field

$$(1^*) \quad \dot{x} \in \overline{\text{co}} F(x) .$$

2. Notations and statement of the main result.

To fix the ideas, we assume that D is a locally closed subset of a Banach space E and F maps D into \mathcal{K} , where \mathcal{K} is the family of nonempty, closed but not necessarily convex subsets of the ball $\{x \in E: \|x\| \leq M\}$, for some $M > 0$. We use the symbol $h(\cdot, \cdot)$ for the Hausdorff distance on \mathcal{K} . $d(x, A)$ stands for the distance $\inf \{\|x - a\| : a \in A\}$ from the point x to the set A , ΔA for the diameter of A , $B[A, \varepsilon]$ for the closed ball $\{x: d(x, A) \leq \varepsilon\}$ of radius $\varepsilon > 0$ about the set A . The closure and the convex closure of A are denoted by \bar{A} and $\overline{\text{co}} A$ respectively. The map F is lower semicontinuous iff

$$\forall x_0 \in D, \forall \varepsilon > 0, \quad \exists \delta > 0: x \in D, \quad \|x - x_0\| \leq \delta \Rightarrow F(x_0) \subseteq B[F(x), \varepsilon] .$$

We denote by \mathbb{N} the set of natural numbers and by \mathcal{L}^1 the set of Lebesgue integrable mappings from $[0, 1]$ into E . Our main result is the following

THEOREM. *Let D be compact and $F: D \rightarrow \mathcal{K}$ be a lower semicontinuous orientor field for which (3) holds. Then, for any $x_0 \in D$, the system (1)-(2) has a solution x defined on $[0, +\infty)$ with values in D .*

3. A set of integrable functions.

It clearly suffices to prove the existence of a solution on the interval $[0, 1]$. To do this, we begin by choosing a set of integrable functions that will be used in the construction of approximate solutions. For each integer $n \geq 1$, define inductively a finite set of points $A_n = \{a_{n1}, \dots, a_{nk}\}$ and open neighborhoods V_{n1}, \dots, V_{nk} with the following properties:

$$(4) \quad a_{ni} \in V_{ni},$$

$$(5) \quad \bigcup_i V_{ni} \supseteq D,$$

$$(6) \quad F(a_{ni}) \subseteq B[F(x), 2^{-n}] \quad \forall x \in D \cap V_{ni},$$

$$(7) \quad \Delta V_{ni} \leq \lambda_{n-1}/2 \quad (\text{if } n \geq 2),$$

where λ_{n-1} is a Lebesgue number for the open covering $\{V_{(n-1),i}\}$, i.e. any closed ball with center at some point $x \in D$ and radius λ_{n-1} is entirely contained in some $V_{(n-1),i}$. Note that all this can be done because of the compactness of D and of the lower semicontinuity of F . From now on, if $a \in A_n$, we shall write $V(a)$ for the neighborhood of a that has been selected in the present construction. Next define $B_n \subseteq A_1 \times A_2 \times \dots \times A_n$ to be the set of all sequences (a_1, \dots, a_n) with $a_i \in A_i$ and $V(a_i) \subseteq V(a_{i-1})$. By (5) and (7), for every $a_n \in A_n$ there are $a_i \in A_i$ ($i = 1, \dots, n-1$) such that $(a_1, \dots, a_n) \in B_n$. To each $b \in B_n$ assign, in an inductive way, a point $y(b) \in E$ satisfying

$$(8) \quad y(b) = y(a_1, \dots, a_n) \in F(a_n),$$

and, if $n > 1$,

$$(9) \quad \|y(b) - y(a_1, \dots, a_{n-1})\| \leq 2^{-n+1}.$$

A suitable $y(b)$ exists because $a_n \in V(a_{n-1})$, hence by (6)

$$y(a_1, \dots, a_{n-1}) \in B[F(a_n), 2^{-n+1}].$$

We notice that from (9) by induction follows that for any $m, j \geq 1$,

$$(a_1, \dots, a_{m+j}) \in B_{m+j},$$

$$(10) \quad \|y(a_1, \dots, a_{m+j}) - y(a_1, \dots, a_m)\| \leq 2^{-m+1}.$$

Take now a sequence of numbers $\{h_n\}$, decreasing to zero, with

$$(11) \quad 1/h_n, \quad h_n/h_{n+1} \in \mathbf{N}, \quad (M+2) \cdot h_n \leq \lambda_n/2 \quad (n = 1, 2, \dots)$$

Denote by C_n the set of all functions $c: [0, 1] \rightarrow B_n$ such that if $c(t) = (\alpha_1(t), \dots, \alpha_n(t))$ then α_i is constant on intervals of the type $[sh_i, (s+1)h_i]$ for any integer $s = 0, 1, \dots, 1/h_i - 1$. We can now define the sets

$$(12) \quad W_n = \{u: [0, 1] \rightarrow E: u(t) = y(c(t)) \text{ for some } c \in C_n\}.$$

Every W_n is then a finite set of piecewise constant functions.

4. A basic lemma.

The building blocks for the construction of approximate solutions are now provided.

LEMMA. *Let n be a fixed positive integer, $z_0 \in D$, $B(z_0, \lambda_n) \subseteq V(a)$ for some $a \in A_n$. Then for any $y \in F(a)$ there exists a continuous, piecewise linear mapping $z: [0, h_n] \rightarrow E$ such that*

$$(13) \quad z(0) = z_0, \quad z(h_n) \in D,$$

$$(14) \quad \|z(t) - y\| \leq 2^{-n+2} \quad \text{a.e. in } [0, h_n].$$

PROOF. Fix $\varepsilon > 0$. Let Γ be the set of mappings $v: [0, \tau] \rightarrow E$ for which some $t_0, t_1, \dots, t_k \in [0, 1]$ exist and satisfy

$$\text{i) } t_0 = 0, t_k = \tau, 0 < t_i - t_{i-1} \leq \varepsilon,$$

$$\text{ii) } v(0) = z_0, v(t_i) \in D, v \text{ is linear on } [t_{i-1}, t_i] \quad (i = 1, \dots, k),$$

$$\text{iii) } \|\dot{v}(t) - y\| \leq 2^{-n+1} \text{ a.e. in } [0, \tau],$$

and let $\Omega = \{\tau: \exists v \in \Gamma, v(\tau) \in D\}$.

We claim that $T = \sup \Omega > h_n$. Indeed, if $T \leq h_n$, let $\{\tau_m\}$ be a

sequence in Ω tending to T and let $v_m(\tau_m)$ be the corresponding sequence of points in D . By compactness, we have

$$\lim_{m' \rightarrow +\infty} v_{m'}(\tau_{m'}) = \bar{x}$$

for some subsequence $v_{m'}$: Choose $\bar{y} \in F(\bar{x})$ for which $\|\bar{y} - y\| \leq 2^{-n}$. Note that this is possible because, by iii), $\|\dot{v}_{m'}\| \leq M + 2$ a.e., hence $\|\bar{x} - z_0\| \leq (M + 2)h_n \leq \lambda_n/2$ and $\bar{x} \in V(a)$. From (3) now follows that there exist $\delta > 0$, $x \in D$ with $\delta < \varepsilon$ and $\|\delta^{-1}(x - \bar{x}) - \bar{y}\| \leq 2^{-n-1}$. Therefore, for the same x ,

$$\|\delta^{-1}(x - v_{\bar{m}}(\tau_{\bar{m}})) - \bar{y}\| \leq 2^{-n} \quad \text{and} \quad \sigma = \tau_{\bar{m}} + \delta > T$$

for some \bar{m} large enough. The map $v_{\bar{m}}$ can therefore be extended to $\tilde{v} \in I$ on the interval $[0, \sigma]$ defining $\tilde{v}(t) = v_{\bar{m}}(t)$ on $[0, \tau_{\bar{m}}]$, $\tilde{v}(\sigma) = x$, \tilde{v} linear on $[\tau_{\bar{m}}, \sigma]$. This contradiction shows that $\sup \Omega \geq h_n$.

Let now $v \in I$ be defined on $[0, \tau]$ with $\tau \geq h_n$. By i) and ii) $v(\bar{\tau}) \in D$ for some $\bar{\tau} \in (h_n - \varepsilon, h_n]$. If ε was chosen to be $\leq 2^{-n} \cdot h_n / (M + 2)$, one checks that the map z defined by

$$z(t) = v((\bar{\tau}/h_n) \cdot t) \quad t \in [0, h_n]$$

satisfies both (13) and (14).

5. Approximate solutions.

Fix any positive integer n . We now consider an approximate solution x_n with the following properties:

i) x_n is continuous and piecewise linear on $[0, 1]$

$$(15) \quad x_n(0) = x_0, \quad x_n(sh_n) \in D \quad (s = 1, 2, \dots, 1/h_n)$$

ii) there exists some $c_n \in C_n$, $c_n(t) = (\alpha_1(t), \dots, \alpha_n(t))$ such that

$$(16) \quad B[x_n(t), \lambda_j/2] \subseteq V(\alpha_j(t)), \quad j = 1, \dots, n, \quad t \in [0, 1],$$

$$(17) \quad \|\dot{x}_n(t) - y(c_n(t))\| \leq 2^{-n+2} \quad \text{a.e. in } [0, 1].$$

To construct x_n , we proceed by induction on s .

Assume that x_n and c_n have been defined for $t \in [0, sh_n]$ for some s , $0 \leq s < 1/h_n$. Set

$$I = \{i: 1 \leq i \leq n, sh_n = ph_i \text{ for some integer } p\}.$$

From (11) it follows that $I = \{i: k \leq i \leq n\}$ for a suitable integer k . For $t \in [sh_n, (s+1)h_n]$ we put

$$(18) \quad \begin{cases} \alpha_i(t) = \alpha_i((s-1)h_n) & \text{if } i \notin I \\ \alpha_i(t) = a_i \in A_i & \text{if } i \in I \end{cases}$$

where the new constants a_i satisfy

$$(19) \quad B[x_n(sh_n), \lambda_i] \subseteq V(a_i), \quad \forall i \in I.$$

We need to show that $c_n(t) = (\alpha_1(t), \dots, \alpha_n(t)) \in B_n$, i.e. that

$$(20) \quad V(\alpha_i(t)) \subseteq V(\alpha_{i-1}(t)) \quad (i = 2, \dots, n).$$

If $i < k$, then (20) is true by inductive hypothesis because the corresponding a_i did not change. If $i > k$, by (7)

$$V(\alpha_i(t)) \subseteq B[x_n(sh_n), \lambda_{i-1}] \subseteq V(\alpha_{i-1}(t)).$$

For $i = k$, denoting by \bar{q} the largest integer q for which $qh_{k-1} \leq sh_n$, we have $sh_n - \bar{q}h_{k-1} < h_{k-1}$. Therefore (8), (11), (17) and (19) yield

$$\|x_n(sh_n) - x_n(\bar{q}h_{k-1})\| \leq (M+2)h_{k-1} \leq \lambda_{k-1}/2,$$

$$V(a_k) \subseteq B[x_n(sh_n), \lambda_{k-1}/2] \subseteq B[x_n(\bar{q}h_{k-1}), \lambda_{k-1}] \subseteq V(a_{k-1}),$$

proving (20). Using the Lemma, we can now define x_n on $[sh_n, (s+1)h_n]$ to be any function assuming the previously assigned value $x_n(sh_n)$ at sh_n and satisfying

$$x_n((s+1)h_n) \in D,$$

$$\|\dot{x}_n(t) - y(c(t))\| \leq 2^{-n+2} \quad \text{a.e.}$$

When $x_n(t)$ has been defined for all $t \in [0, 1]$ following the above procedure, it is clear that each function $\alpha_i(t)$, $i = 1, \dots, n$, is constant on every interval of the type $[sh_i, (s + 1)h_i]$, hence $c_n = (\alpha_1, \dots, \alpha_n) \in C_n$. If $t \in [0, 1]$, $1 \leq i \leq n$, denote by \bar{q} the largest integer q for which $qh_i \leq t$. Then

$$\|x_n(t) - x_n(\bar{q}h_i)\| \leq h_i(M + 2) \leq \lambda_i/2,$$

$$B[x_n(t), \lambda_i/2] \subseteq B[x_n(\bar{q}h_i), \lambda_i] \subseteq V(\alpha_i(\bar{q}h_i)) = V(\alpha_i(t)).$$

This proves (16) and completes our construction of approximate solutions.

6. Completion of the proof.

Let $\{x_n\}$ be a sequence of approximate solutions defined according to i) -ii) in the previous section. Then the sequence of derivatives $\Sigma = \{\dot{x}_n: n \geq 1\}$ is relatively compact in \mathcal{L}^1 . Indeed, using (10) and (17), one checks that for any $m \geq 1$ the finite set of balls

$$\{B[\dot{x}_n, 2^{-m+2}], n \leq m\} \cup \{B[u, 2^{-m+2}], u \in W_m\}$$

covers Σ .

By compactness of $\bar{\Sigma}$ there exists some $\eta \in \mathcal{L}^1$ and a subsequence $\{x_{n'}\}$ such that

$$\|\dot{x}_{n'} - \eta\|_{\mathcal{L}^1} \rightarrow 0 \quad \text{and} \quad \dot{x}_{n'}(t) \rightarrow \eta(t) \quad \text{a.e.}$$

This implies that

$$x_{n'}(t) \rightarrow x_0 + \int_0^t \eta(s) ds = x(t)$$

uniformly on $[0, 1]$. We claim that x is a solution of (1)-(2) on D . Note that $x_n(sh_i) \in D$ for all integers $s = 1, \dots, 1/h_i$, $i \leq n$, therefore $x(t) \in D$ on the dense subset $\{t \in [0, 1], t = sh_i, 0 \leq s \leq 1/h_i, i \geq 1\}$. By continuity, $x(t) \in D$ for every $t \in [0, 1]$.

To show that (1) holds, we consider the sequence of functions $\{c_n\}$: $c_n(t) = (\alpha_{n1}(t), \dots, \alpha_{nn}(t))$ corresponding to the X_n in (16)-(17). For a

fixed m , the set of mappings $\{t \rightarrow (\alpha_{n1}(t), \dots, \alpha_{nm}(t)), n \geq 1\} \subseteq C_m$ is finite. Taking a subsequence of $\{x_n\}$, call it $\{x_\nu\}$, we can therefore assume that $(\alpha_{\nu 1}(\cdot), \dots, \alpha_{\nu m}(\cdot))$ is the same mapping $c_m = (\alpha_1(\cdot), \dots, \alpha_m(\cdot))$ for every ν . By (16)

$$B[x_\nu(t), \lambda_m/2] \subseteq V(\alpha_m(t)).$$

Taking the limit as $\nu \rightarrow \infty$ we get

$$(21) \quad x(t) \in V(\alpha_m(t)) \quad \forall t \in [0, 1].$$

This yields the inequality

$$(22) \quad \begin{aligned} d(\dot{x}(t), F(x(t))) &\leq \|\dot{x}(t) - \dot{x}_\nu(t)\| + \|\dot{x}_\nu(t) - y(c_\nu(t))\| + \\ &+ \|y(c_\nu(t)) - y(c_m(t))\| + d(y(c_m(t)), F(x(t))), \quad \text{a.e. in } [0, 1]. \end{aligned}$$

Letting $\nu \rightarrow \infty$, the first two terms on the right hand side of (22) tend to zero a.e.. Using (10), (21) and (16) we can now estimate the last two terms and get

$$d(x(t), F(x(t))) \leq 2^{-m+1} + 2^{-m} \quad \text{a.e. on } [0, 1]$$

Since m was arbitrary, our theorem is proved.

7. Concluding remarks.

It is worth noticing that (3) cannot be replaced by the weaker condition

$$(23) \quad \forall x \in D, \exists y \in F(x), \quad \lim_{h \rightarrow 0^+} d(x + hy, D) h^{-1} = 0.$$

To see this, consider the set $D = \Gamma \cup \{0\} \subseteq \mathbb{R}^2$, where Γ is the spiral defined in polar coordinates by

$$\Gamma = \{(\varrho, \theta): \varrho = 2\pi \cdot \theta^{-1}, \theta \geq 2\pi\}.$$

Take $x_0 = 0$, $F(x) = \{y \in \mathbb{R}^2: \|y\| = 1\}$.

Then F satisfies (23), but the system (1)-(2) has no solution because any path on D connecting the origin with any other point of D has infinite length.

We remark that from our result one can derive a technique for solving some existence problems for a general orientor field (1) simply by means of the properties of the corresponding system (1*).

PROPOSITION. *Let $D \subseteq E$ and let $F: D \rightarrow \mathcal{K}$ be lower semicontinuous. Assume that the orientor field (1*) has the following properties:*

- a) *The set A of the solutions of (1*)-(2) on $[0, 1]$ with values in D is nonempty and compact,*
- b) *For each $z_0 \in \mathcal{F}$, $y \in F(z_0)$, there exist $z \in A$, $t_0 \in [0, 1]$ for which*

$$\lim_{h \rightarrow 0^+} (z(t_0 + h) - z_0) h^{-1} = y,$$

where \mathcal{F} is the funnel $\{x \in D: \exists z \in A, \exists t \in [0, 1), z(t) = x\}$.

Then the system (1)-(2) has a solution on D .

Indeed by a) $\overline{\mathcal{F}}$ is a compact subset of D and by b) the tangential condition (3) holds for each $x \in \mathcal{F}$. The same kind of arguments used in the proof of our theorem now provide the existence of a solution of (1)-(2) taking values in $\overline{\mathcal{F}}$.

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