Rendiconti
del
Seminario Matematico
della
Università di Padova

H. M. Srivastava

Sums of certain double q-hypergeometric series

Rendiconti del Seminario Matematico della Università di Padova,
tome 72 (1984), p. 1-8
<http://www.numdam.org/item?id=RSMUP_1984__72__1_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 72 (1984)

Sums of Certain Double q-Hypergeometric Series.

H. M. SRIVASTAVA (*)

Dedicated to Professor Leonard Carlitz
on the occasion of his seventy-fifth birthday

SUMMARY - Some special summation theorems for two classes of double hypergeometric series are observed here to follow fairly easily as special cases of certain expansion formulas given in the literature. The underlying techniques are then shown to yield analogous sums of various double q-hypergeometric series. q-Extensions of the aforementioned expansion formulas are also presented.

1. Introduction

Put

$$(1.1)\quad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1\ , & \text{if } n=0\ , \\ \lambda(\lambda+1)\ldots(\lambda+n-1)\ , & \forall n \in \{1, 2, 3, \ldots\}\ , \end{cases}$$

and let ${}_rF_s$ denote a generalized hypergeometric series with r numerator and s denominator parameters. Also let $F^{p:r;u}_{k:s;v}$ denote a generalized

(*) Indirizzo dell'A.: H. M. SRIVASTAVA: Department of Mathematics, University of Victoria, Victoria, British Columbia V8W 2Y2, Canada.
Supported, in part, by NSERC (Canada) Grant A-7353.

Kampé de Fériet double hypergeometric series defined by

$$(1.2)\qquad F^{p:r;u}_{k:s;v}\left[\begin{matrix}(\lambda_p): & (\alpha_r); & (\gamma_u); \\ & & & x,y \\ (\mu_k): & (\beta_s); & (\delta_v); \end{matrix}\right] =$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{m+n} \prod_{j=1}^{r}(\alpha_j)_m \prod_{j=1}^{u}(\gamma_j)_n}{\prod_{j=1}^{k}(\mu_j)_{m+n} \prod_{j=1}^{s}(\beta_j)_m \prod_{j=1}^{v}(\delta_j)_n} \frac{x^m}{m!}\frac{y^n}{n!},$$

where, for convenience, (α_r) abbreviates the array of r parameters $\alpha_1, \ldots, \alpha_r$, with similar interpretations for (β_s), *et cetera.*

Inspired by two earlier papers of Carlitz ([2], [3]), Jain [5] derived the following summation formula for a double hypergeometric series of type (1.2) with $p = 0$, $r = u = 3$ and $k = s = v = x = y = 1$:

$$(1.3)\qquad F^{0:3;3}_{1:1;1}\left[\begin{matrix}-: & c-a,\, c-b,\, -M & ; & a,\, b,\, -N & ; \\ & & & & & 1,1 \\ c\ : & 1-a-b+c-M; & & 1+a+b-c-N; & \end{matrix}\right] =$$

$$= \frac{(a)_M (c-a)_N (b)_M (c-b)_N}{(c)_{M+N}(a+b-c)_M(c-a-b)_N}, \qquad M, N = 0, 1, 2, \ldots,$$

provided that $a + b - c$ is not an integer.

Subsequently, Carlitz [4] showed that

$$(1.4)\qquad F^{1:2;2}_{0:2;2}\left[\begin{matrix}c\ : & a,\, -M; & a',\, -N; \\ & & & 1,1 \\ -: & b,\quad c; & b',\quad c; \end{matrix}\right] =$$

$$= \frac{(a')_M (a)_N (c)_{M+N}}{(a'-a)_M (a-a')_N (c)_M (c)_N}, \qquad M, N = 0, 1, 2, \ldots,$$

where the parameters are constrained by

$$(1.5)\qquad a = 1 + a' - b' - N\,, \quad a' = 1 + a - b - M\,.$$

In Section 2 we shall show how readily the summation formulas (1.3) and (1.4) can be derived by specializing the following expansion formulas (*cf.* [9], p. 52, Equations (13)′ and (11)′):

$$(1.6)\quad F^{0:r;u}_{1:s;v}\left[\begin{matrix} -: & (\alpha_r); & (\gamma_u); \\ c\ : & (\beta_s); & (\delta_v); \end{matrix}\ x, y\right] =$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r} (\alpha_j)_n \prod_{j=1}^{u} (\gamma_j)_n}{(c+n-1)_n (c)_{2n} \prod_{j=1}^{s} (\beta_j)_n \prod_{j=1}^{v} (\delta_j)_n} \frac{(-xy)^n}{n!} \cdot$$

$$\cdot {}_rF_{s+1}\left[\begin{matrix} (\alpha_r)+n & ; \\ (\beta_s)+n,\ c+2n; \end{matrix}\ x\right] {}_uF_{v+1}\left[\begin{matrix} (\gamma_u)+n & ; \\ (\delta_v)+n,\ c+2n; \end{matrix}\ y\right],$$

$$(1.7)\quad F^{1:r;u}_{0:s;v}\left[\begin{matrix} a\ : & (\alpha_r); & (\gamma_u); \\ -: & (\beta_s); & (\delta_v); \end{matrix}\ x, y\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n \prod_{j=1}^{r} (\alpha_j)_n \prod_{j=1}^{u} (\gamma_j)_n}{\prod_{j=1}^{s} (\beta_j)_n \prod_{j=1}^{v} (\delta_j)_n} \frac{(xy)^n}{n!} \cdot$$

$$\cdot {}_{r+1}F_s\left[\begin{matrix} (\alpha_r)+n,\ a+n; \\ (\beta_s)+n \quad ; \end{matrix}\ x\right] {}_{u+1}F_v\left[\begin{matrix} (\gamma_u)+n,\ a+n; \\ (\delta_v)+n \quad ; \end{matrix}\ y\right],$$

which hold true whenever both sides exist.

We remark in passing that the expansion formulas (1.6) and (1.7), together with their union (*cf.* [9], p. 52, Equation (8)′)

$$(1.8)\quad F^{1:r;u}_{1:s;v}\left[\begin{matrix} a: & (\alpha_r); & (\gamma_u); \\ c: & (\beta_s); & (\delta_v); \end{matrix}\ x, y\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (c-a)_n \prod_{j=1}^{r} (\alpha_j)_n \prod_{j=1}^{u} (\gamma_j)_n}{(c+n-1)_n (c)_{2n} \prod_{j=1}^{s} (\beta_j)_n \prod_{j=1}^{v} (\delta_j)_n} \frac{(xy)^n}{n!} \cdot$$

$$\cdot {}_{r+1}F_{s+1}\left[\begin{matrix} (\alpha_r)+n,\ a+n\ ; \\ (\beta_s)+n,\ c+2n; \end{matrix}\ x\right] {}_{u+1}F_{v+1}\left[\begin{matrix} (\gamma_u)+n,\ a+n\ ; \\ (\delta_v)+n,\ c+2n; \end{matrix}\ y\right],$$

were deduced as very special cases of three classes of hypergeometric transformations (or inverse series relations) studied in our earlier paper [9].

2. Derivations of special sums

For convenience, let S_1 and S_2 denote the first members of the summation formulas (1.3) and (1.4), respectively. Making use of the expansion formula (1.6) with $x = y = 1$, and

$$(2.1)\quad \begin{cases} r = u = 3\,, \quad s = v = 1\,, \quad \alpha_1 = c - a\,, \quad \alpha_2 = c - b\,, \quad \alpha_3 = -M\,, \\ \beta_1 = 1 - a - b + c - M\,, \quad \gamma_1 = a\,, \quad \gamma_2 = b\,, \quad \gamma_3 = -N\,, \\ \qquad\qquad\qquad\qquad \delta_1 = 1 + a + b - c - N\,, \end{cases}$$

and summing each of the resulting hypergeometric ${}_3F_2$ series on the right-hand side by Saalschütz's theorem [8, p. 49, equation (2.3.1.3)], we find that

$$(2.2)\qquad S_1 = \frac{(a)_M(c-a)_N(b)_M(c-b)_N}{(c)_M(c)_N(a+b-c)_M(c-a-b)_N}\,\cdot$$

$$\cdot\,{}_4F_3\left[\begin{matrix} c-1,\ \tfrac{1}{2}c+\tfrac{1}{2}, & -M\ , & -N\ ; \\ \tfrac{1}{2}c-\tfrac{1}{2}, & c+M, & c+N; \end{matrix}\ -1\right],$$

provided that $a + b - c$ is not an integer.

The terminating hypergeometric ${}_4F_3$ series occurring in (2.2) can be summed by appealing to the known result [8, p. 57, Equation (2.3.4.8)], and thus we are led immediately to the second member of the summation formula (1.3).

In order to derive Carlitz's sum (1.4) we make the following substitutions in the expansion formula (1.7):

$$(2.3)\quad \begin{cases} r = s = u = v = 2\,, \quad a \to c\,, \quad \alpha_1 = a\,, \quad \alpha_2 = -M\,, \\ \qquad\qquad\qquad\qquad \beta_1 = b\,, \quad \beta_2 = c\,, \\ \gamma_1 = a'\,, \quad \gamma_2 = -N\,, \quad \delta_1 = b'\,, \quad \delta_2 = c\,, \quad \text{and } x = y = 1\,. \end{cases}$$

Upon summing each of the resulting hypergeometric ${}_2F_1$ series by Vandermonde's theorem [8, p. 28, Equation (1.7.7)], we readily obtain

$$(2.4)\qquad S_2 = \frac{(b-a)_M(b'-a')_N}{(b)_M(b)_N}\cdot$$

$$\cdot {}_4F_3\left[\begin{matrix} a, a', -M, -N & ; \\ c, 1+a-b-M, 1+a'-b'-N; & 1\end{matrix}\right].$$

The terminating hypergeometric ${}_4F_3$ series occurring in (2.4) immediately reduces, under the constraints (1.5), to a ${}_2F_1$ series which is also summable by Vandermonde's theorem [*loc. cit.*], and Carlitz's formula (1.4) follows at once.

By employing one or more of several known summation theorems for generalized hypergeometric series (*cf.*, *e.g.*, [8], pp. 243-245) one can similarly deduce, from each of the expansion formulas (1.6), (1.7) and (1.8), a large number of double sums including, for example, the sums of the double hypergeometric $F^{0:3;3}_{1:1;1}$ series given recently by Singal ([6], [7]).

3. q-Extensions

For real or complex q, $|q|<1$, let

$$(3.1)\qquad (\lambda; q)_\mu = \prod_{j=0}^{\infty}\left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right)$$

for arbitrary λ and μ, so that

$$(3.2)\qquad (\lambda; q)_n = \begin{cases} 1\,, & \text{if } n=0\,,\\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1})\,, & \forall n\in\{1,2,3,\ldots\}\,,\end{cases}$$

and

$$(3.3)\qquad (\lambda; q)_\infty = \prod_{j=0}^{\infty}(1-\lambda q^j)\,.$$

Comparing the definitions (1.1) and (3.2), we easily have

$$\lim_{q\to 1}\left\{\frac{(q^{\lambda};\,q)_n}{(q^{\mu};\,q)_n}\right\}=\frac{(\lambda)_n}{(\mu)_n},\qquad n=0,1,2,\ldots \tag{3.4}$$

for arbitrary λ and μ, $\mu\neq 0,-1,-2,\ldots$.

Making use of familiar notations (analogous to those for ${}_rF_s$ and $F^{p:r;u}_{k:s;v}$ used in the preceding sections) for q-hypergeometric series in one and two variables, we now state the following q-extensions of the expansion formulas (1.6), (1.7) and (1.8):

$$\Phi^{0:r;u}_{1:s;v}\left[\begin{matrix}-: & (\alpha_r); & (\gamma_u); \\ c\ : & (\beta_s); & (\delta_v);\end{matrix}\quad q;\quad x,y\right]= \tag{3.5}$$

$$=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{r}(\alpha_j;\,q)_n\prod_{j=1}^{u}(\gamma_j;\,q)_n}{(cq^{n-1};\,q)_n(c;\,q)_{2n}\prod_{j=1}^{s}(\beta_j;\,q)_n\prod_{j=1}^{v}(\delta_j;\,q)_n}\,q^{\frac{1}{2}n(n-1)}\,\frac{(-cxy)^n}{(q;\,q)_n}\cdot$$

$$\cdot\,{}_r\Phi_{s+1}\left[\begin{matrix}(\alpha_r)q^n & ; \\ (\beta_s)q^n,\ cq^{2n}; & \end{matrix}\quad q,x\right]{}_u\Phi_{v+1}\left[\begin{matrix}(\gamma_u)q^n & ; \\ (\delta_v)q^n,\ cq^{2n}; & \end{matrix}\quad q,y\right],$$

$$\Phi^{1:r;u}_{0:s;v}\left[\begin{matrix}a\ : & (\alpha_r); & (\gamma_u); \\ -: & (\beta_s); & (\delta_v);\end{matrix}\quad q;\quad x,y\right]= \tag{3.6}$$

$$=\sum_{n=0}^{\infty}\frac{(a;\,q)_n\prod_{j=1}^{r}(\alpha_j;\,q)_n\prod_{j=1}^{u}(\gamma_j;\,q)_n}{\prod_{j=1}^{s}(\beta_j;\,q)_n\prod_{j=1}^{v}(\delta_j;\,q)_n}\,q^{n(n-1)}\,\frac{(axy)^n}{(q;\,q)_n}\cdot$$

$$\cdot\,{}_r\Phi_{s+1}\left[\begin{matrix}(\alpha_r)q^n,\ aq^n; \\ (\beta_s)q^n\quad ;\end{matrix}\quad q,x\right]{}_u\Phi_{v+1}\left[\begin{matrix}(\gamma_u)q^n,\ aq^n; \\ (\delta_v)q^n\quad ;\end{matrix}\quad q,y\right],$$

$$\Phi^{1:r;u}_{1:s;v}\left[\begin{matrix}a: & (\alpha_r); & (\gamma_u); \\ c: & (\beta_s); & (\delta_v);\end{matrix}\quad q;\quad x,y\right]= \tag{3.7}$$

$$= \sum_{n=0}^{\infty} \frac{(a;q)_n (c/a;q)_n \prod_{j=1}^{r} (\alpha_j;q)_n \prod_{j=1}^{u} (\gamma_j;q)_n}{(cq^{n-1};q)_n (c;q)_{2n} \prod_{j=1}^{s} (\beta_j;q)_n \prod_{j=1}^{v} (\delta_j;q)_n} q^{n(n-1)} \frac{(axy)^n}{(q;q)_n} \cdot$$

$$\cdot {}_{r+1}\Phi_{s+1} \left[\begin{matrix} (\alpha_r)q^n, aq^n; \\ (\beta_s)q^n, cq^{2n}; \end{matrix} \; q, x \right] {}_{u+1}\Phi_{v+1} \left[\begin{matrix} (\gamma_u)q^n, aq^n; \\ (\delta_v)q^n, cq^{2n}; \end{matrix} \; q, y \right],$$

which hold true whenever each side exists.

To prove (3.5), (3.6) and (3.7), we first expand each of the q-hypergeometric series occurring on their right-hand sides, collecs the coefficients of $x^l y^m$ ($l, m = 0, 1, 2, \ldots$), sum the innermost q-seriet by appealing to known results for terminating q-hypergeometric ${}_5\Phi_4$, ${}_2\Phi_1$ and ${}_6\Phi_5$ series, respectively [8, p. 247], and then interpret the resulting double series as a q-hypergeometric function of two variables by means of a definition essentially analogous to (1.2).

Each of the q-expansion formulas (3.5), (3.6) and (3.7) can be applied in conjunction with known summation theorems for special q-hypergeometric series (*cf.*, *e.g.*, [8], p. 247) to deduce sums of various double q-series. For example, (3.7) and q-Saalschütz's theorem [8, p. 247, Equation (IV.4)] would, together, yield Al-Salam's q-extensions [1] of the double sums evaluated earlier by Carlitz ([2], [3]). On the other hand, by using the technique (detailed in the preceding section) *mutatis mutandis*, we can deduce from (3.5) and (3.6) the following q-extensions of the double sums (1.3) and (1.4):

$$\Phi^{0:3;3}_{1:1;1} \left[\begin{matrix} -: & c/a, c/b, q^{-M}; & a, b, q^{-N}; \\ c\;: & cq^{1-M}/ab\;; & abq^{1-N}/c; \end{matrix} \quad q; \quad q, q \right] = \tag{3.8}$$
$$= \frac{(a;q)_M (c/a;q)_N (b;q)_M (c/b;q)_N}{(c;q)_{M+N} (ab/c;q)_M (c/ab;q)_N}, \qquad M, N = 0, 1, 2, \ldots;$$

$$\Phi^{1:2;2}_{0:2;2} \left[\begin{matrix} c\;: & a, q^{-M}; & a', q^{-N}; \\ -: & b, c\;; & b', c\;; \end{matrix} \quad q; \quad q, q \right] = \tag{3.9}$$
$$= \frac{(a';q)_M (a;q)_N (c;q)_{M+N}}{(a'/a;q)_M (a/a';q)_N (c;q)_M (c;q)_N}, \quad M, N = 0, 1, 2, \ldots,$$
$$a = a' q^{1-N}/b', \qquad a' = aq^{1-M}/b.$$

The q-summation formulas (3.8) and (3.9) are believed to be new.

REFERENCES

[1] W. A. AL-SALAM, *Saalschützian theorems for basic double series*, J. London Math. Soc. **40** (1965), pp. 455-458.

[2] L. CARLITZ, *A Saalschützian theorem for double series*, J. London Math. Soc. **38** (1963), pp. 415-418.

[3] L. CARLITZ, *Another Saalschützian theorem for double series*, Rend. Sem. Mat. Univ. Padova **34** (1964), pp. 200-203.

[4] L. CARLITZ, *Summation of a double hypergeometric series*, Matematiche (Catania) **22** (1967), pp. 138-142.

[5] R. N. JAIN, *Sum of a double hypergeometric series*, Matematiche (Catania) **21** (1966), pp. 300-301.

[6] R. P. SINGAL, *A Watson sum for non-terminating double hypergeometric series*, Jñānābha **6** (1976), pp. 45-47.

[7] R. P. SINGAL, *A Watson sum for the double hypergeometric series*, Ganita **30** (1979), pp. 68-72.

[8] L. J. SLATER, *Generalized Hypergeometric Series*, Cambridge Univ. Press, Cambridge, 1966.

[9] H. M. SRIVASTAVA, *Certain pairs of inverse series relations*, J. Reine Angew. Math. **245** (1970), pp. 47-54.

Manoscritto pervenuto in redazione il 22 Settembre 1982.