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Products of several Jacobi or Laguerre polynomials

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Products of Several Jacobi or Laguerre Polynomials.

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SUMMARY - New product formulas of several Laguerre or Jacobi polynomials are established.

SUNTO - Vengono determinate due formule coinvolgenti il prodotto di più polinomi di Laguerre o di Jacobi.

1. The main result.

Product formulas of classical polynomials have been considered in several papers, [2], [3], [4]. A sample about the main results is given in [1]. The following product of Jacobi polynomials is believed to be new

$$(1)\qquad P_{m_1}^{(\alpha_1-m_1,\beta_1-m_1)}\left(1+\frac{2a_1x}{c_1}\right)\dots P_{m_n}^{(\alpha_n-m_n,\beta_n-m_n)}\left(1+\frac{2a_nx}{c_n}\right)=$$

$$=c_1^{-m_1}c_2^{-m_2}\dots c_n^{-m_n}\binom{\alpha_1+\beta_1}{m_1}\binom{\alpha_2+\beta_2}{m_2}\dots\binom{\alpha_n+\beta_n}{m_n}\cdot$$

$$\cdot\sum_{j=0}^{m_1+\dots+m_n}A_j^{(m_1,\dots,m_n)}c_0^j\left[\binom{\alpha_0+\beta_0}{j}\right]^{-1}P_j^{(\alpha_0-j,\beta_0-j)}\left(1+\frac{2a_0x}{c_0}\right)$$

where $A_j^{(m_1,\dots,m_n)}$ satisfy

$$(2)\qquad \sum_{m_1,\dots,m_n=0}^{\infty}A_j^{(m_1,\dots,m_n)}\frac{u_1^{m_1}\dots u_n^{m_n}}{m_1!\dots m_n!}=$$

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$$= \frac{{}_1F_1(-\alpha_1; -\alpha_1-\beta_1, c_1u_1)\dots {}_1F_1(-\alpha_1; -\alpha_n-\beta_n, c_nu_n)}{j!\,{}_1F_1[-\alpha_0; -\alpha_0-\beta_0, (a_1u_1+\dots+a_nu_n)c_0]}\cdot (a_1u_1+\dots+a_nu_n)^j .$$

Here ${}_1F_1$ is the Kummer's function [6, p. 64].

The explicit espression of the $A_j^{(m_1,\dots,m_n)}$'s coefficients is given by

$$A_j^{(m_1,\dots,m_n)} = \sum_{j_1=0}^{m_1}\dots\sum_{j_n=0}^{m_n}\binom{m_1}{j_1}\dots\binom{m_n}{j_n}\binom{j_1+\dots+j_n}{j} a_1^{j_1}\dots a_n^{j_n}\cdot \frac{f_{j_1}+\dots+j_n-j}{2^{m_1+\dots+m_n-j_1-\dots-j_n}}\cdot\frac{(-\alpha_1)_{m_1-j_1}\dots(-\alpha_n)_{m_n-j_n}}{(-\alpha_1-\beta_1)_{m_1-j_1}\dots(-\alpha_n-\beta_n)_{m_n-j_n}} \tag{3}$$

with $(-\alpha_1)_{m_1-j_1} = \Gamma(m_1-j_1-\alpha_1)/\Gamma(-\alpha_1)$, etc. and where the coefficients f_k satisfy

$$\sum_{k=0}^{\infty} f_k \frac{t^k}{k!} = [{}_1F_1(-\alpha_0; -\beta_0-\alpha_0, c_0t)]^{-1} . \tag{4}$$

We can also prove

$$L_{m_1}^{(\alpha_1)}\left(\frac{y_1}{c_1x}\right)\dots L_{m_n}^{(\alpha_n)}\left(\frac{y_n}{c_nx}\right) = \frac{(m_1+1)_{\alpha_1}\dots(m_n+1)_{\alpha_n}}{(c_1x)^{m_1}\dots(c_nx)^{m_n}}\sum_{j=0}^{m_1+\dots+m_n} B_j^{(m_1,\dots,m_n)}\frac{x^j}{(j+1)_{\alpha_0}}L_j^{(\alpha_0)}\left(\frac{y_0}{c_0x}\right) \tag{5}$$

where the $B_j^{(m_1,\dots,m_n)}$'s are generated by

$$\sum_{j_1,\dots,j_n=0}^{\infty} B_j^{(j_1,\dots,j_n)}\frac{u_1^{j_1}\dots u_n^{j_n}}{j_1!\dots j_n!} = \frac{[y_0(c_1u_1+\dots+c_nu_n)]^{\alpha_0/2}}{(u_1y_1)^{\alpha_1/2}\dots(u_ny_n)^{\alpha_n/2}}\,\frac{J_{\alpha_1}(2\sqrt{u_1y_1})\dots J_{\alpha_n}(2\sqrt{u_ny_n})}{J_{\alpha_0}(2\sqrt{y_0(c_1u_1+\dots+c_nu_n)})}\cdot\frac{(c_1u_1+\dots+c_nu_n)^j}{j!} .$$

Here $J_{\alpha_i}(2\sqrt{u_iy_i})$ are Bessel functions of order α_i.

Other products of Laguerre polynomials can be found in [3].

2. Proofs.

We consider the product of two polynomials. The extension to the cases $n > 2$ is straightforward; we omit it.

The proof of (1) is based on a recent result about the Jacobi polynomials, [5], and uses arguments of [4]. From theorem 2 of [5] it follows that the polynomials

$$Q_m(x) = Q_m^{(\alpha,\beta)}(x, c) = \frac{m!}{(\alpha+1)_m} x^m P_m^{(\alpha,\beta-m)}\left(1 + \frac{2c}{x}\right) \tag{6}$$

form an Appell set with respect to x. We recall that a set of polynomials $\{Q_m(x)\}$ is called Appell set if $Q'_m(x) = mQ_{m-1}(x)$, $m = 0, 1, \ldots$.

By using the explicit espression of Jacobi polynomials, [6, p. 94], equation (6) can be written

$$Q_m(x) = \sum_{k=0}^{m} \binom{m}{k} Q_k(0)\, x^{m-k}. \tag{7}$$

Thus

$$\sum_{m=0}^{\infty} Q_m(x) \frac{t^m}{m!} = e^{xt} F(t), \tag{8}$$

where

$$F(t) = {}_1F_1(\alpha+\beta+1;\ \alpha+1,\ ct). \tag{9}$$

Now, if we set

$$f(t) = [F(t)]^{-1} = \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \tag{10}$$

then from (8) we have

$$x^n = \sum_{k=0}^{n} \binom{n}{k} f_{n-k} Q_k(x).$$

This and equation (7) give

$$Q_m^{(\alpha_1,\beta_1)}(a_1 x, c_1) Q_n^{(\alpha_2,\beta_2)}(a_2 x, c_2) = \sum_{j=0}^{m+n} A_j^{(m,n)} Q_j^{(\alpha_0,\beta_0)}(a_0 x, c_0) \tag{11}$$

with

$$(12)\quad A_j^{(m,n)} = \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \binom{m}{\mu}\binom{n}{\nu}\binom{\mu+\nu}{j} a_1^{\mu} a_2^{\nu} f^{(0)}_{\mu+\nu-j} Q_{m-\mu}^{(\alpha_1,\beta_1)}(0, c_1) Q_{m-\nu}^{(\alpha_2,\beta_2)}(0, c_2)\,,$$

where the coefficients $f^{(0)}_{\mu+\nu-j}$ satisfy (10) with $F(t) = {}_1F_1(\alpha_0+\beta_0+1;\ \alpha_0+1,\ c_0 t)$.

We now observe that

$$P_m^{(\alpha_i,\beta_i-m)}\left(1+\frac{2c_i}{x}\right) = \left(-\frac{c_i}{x}\right)^m P_m^{(-\alpha_i-\beta_i-1-m,\,\beta_i-m)}\left(1+\frac{2x}{c_i}\right), \quad i = 0, 1, 2\,.$$

By using this into (6) and by substituing the resulting equation with $\alpha_i+\beta_i+1$ replaced by $-\alpha_i$ into (11), we see that the product formula (1), with $n = 2$, is readily established.

Moreover we have

$$\sum_{m,n=0}^{\infty} A_j^{(m,n)} \frac{u^m v^n}{m!\,n!} =$$

$$= \sum_{\mu,\nu,h,k=0}^{\infty} \frac{u^{k+\mu} v^{k+\nu} a_1^{\mu} a_2^{\nu}}{h!\,k!\,\mu!\,\nu!} \binom{\mu+\nu}{j} f^{(0)}_{\mu+\nu-j} Q_h^{(\alpha_1,\beta_1)}(0, c_1) Q_k^{(\alpha_2,\beta_2)}(0, c_2) =$$

$$= (a_1 u + a_2 v)^j \frac{{}_1F_1(\alpha_1+\beta_1+1;\alpha_1+1, c_1 u)\, {}_1F_1(\alpha_2+\beta_2+1;\alpha_2+1, c_2 v)}{j!\,{}_1F_1(\alpha_0+\beta_0+1;\alpha_0+1, (a_1 u + a_2 v) c_0)}$$

which proves equation (2) with $n = 2$, apart from an irrilevant replacement of $\alpha_i+\beta_i+1$ with $-\alpha_i$.

The proof of (5) is also based on theorem 2 of [5] and is similar to the above one. We omit it.

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