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Positive Functionals and the Axiom of Choice.

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In this note we prove, that a proposition which is useful in integration theory, is equivalent to a weak axiom of choice.

Theorem. The axiom MC^{ω} of countable multiple choice is equivalent to the assertion, that every positive linear functional on a B-lattice is continuous.

Our set theory is ZF^0, the Zermelo-Fraenkel system minus the axioms of choice and foundation. MC^{ω} says, that a countable sequence of nonempty sets $(S_n)_{n\in\omega}$ contains a sequence of nonempty finite subsets $F_n \subseteq S_n$. In ZF^0 MC^{ω} is not provable, and MC^{ω} does not imply the countable axiom of choice AC^{ω}. It is unknown, whether in ZF ($= ZF^0 +$ foundation) $MC^{\omega} \Leftrightarrow AC^{\omega}$.

A B-lattice (AB-lattice in [4]) is a vector lattice together with a Frechet-complete (Cauchy sequences are convergent, c.f. [3] for a discussion of diverse completeness properties) Riesz-norm $\|\cdot\|$ (c.f. [7], p. 61 and p. 101), i.e. (i) $-y \leqq x \leqq y$ implies $\|x\| \leqq \|y\|$ and (ii) if $\|x\| < 1$ there is a $y \geqq 0$ such that $-y \leqq x \leqq y$ and $\|y\| < 1$. A positive linear functional for X is a linear mapping $f\colon X \to \mathbb{R}$ such that $fx \geqq 0$ whenever $x \geqq 0$.

Proof of the Theorem. «$\Rightarrow$»: Assume MC^{ω}. Let $f\colon X \to \mathbb{R}$ be positive and linear. If f is not continuous, the standard argument proves, that in ZF^0 f is not bounded. We observe, that $S_n = \{x \in X$:

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$\|x\| \leqq 1, fx \geqq n, x \geqq 0\}$ is nonempty. Since f is not bounded, there is a $x \in X$, $\|x\| \leqq 1$, such that $|fx| \geqq 2n$. According to the Riesz decomposition theorem, $x = x^+ - x^-$, $|x| = x^+ + x^-$, where $x^+ = \sup \{x, 0\}$, $x^- = (-x)^+$, $|x| = \sup \{x, -x\}$. Since $\|\cdot\|$ is a Riesz norm, lemma 9.16 of [7] applies, which shows: $\|x^+\| \leqq 1$, $\|x^-\| \leqq 1$. Since f is positive $fx^+ \geqq 0$. If $fx^+ < n$ and $fx^- < n$, then $|fx| = |fx^+ - fx^-| < 2n$, a contradiction. So $x^+ \in S_n$ or $x^- \in S_n$. MC^ω provides us with a sequence $F_n \subseteq S_n$ of finite sets $F_n \neq \emptyset$; x_n is the arithmetic mean of F_n: $x_n \geqq 0$, $\|x_n\| \leqq 1$ and $fx_n \geqq n$. We set $z = \sum_{n=1}^{\infty} (1/n^2)x_n$. As this series converges absolutely and X is complete, the limit z exists; $z = \lim y_n$, $y_n = \sum_{i=1}^{n} (1/i^2)x_i$. Since $y_n \leqq z$, $\sum_{i=1}^{n} 1/i \leqq fy_n \leqq fz$. This is a contradiction: The harmonic series is unbounded.

The above proof is an adaption of [6], p. 273, where this result was proved for a class of bornological ordered vector spaces. Examples of B-lattices are the spaces L_p, $p \geqq 1$, and $C(X)$, X compact T_2, with the usual order. If the cone of a B-lattice has a base, there is —in $ZF^0 + MC^\omega$—a positive-, continuous-linear functional f, such that $B = C \cap f^{-1}(1)$ (B the base, $C = \{x \in X: x \geqq 0\}$ the cone). In [6], p. 272, it was shown without the axiom of choice, that a positive linear functional on a topological ordered vector space is continuous, if the cone has interior points. Since this result applies to $C(X)$, it was of interest, whether its extension to B-lattices depends on the axiom of choice. This is shown next. A modification of the argument in [5]. p 24, combined with «$\Leftarrow$» proves, that the following assertion is equivalent to MC^ω. In a B-lattice a convex, balanced set which absorbs all order bounded sets absorbs all bounded sets, too. This property also implies the continuity of positive linear functionals (in ZF_0).

«$\Leftarrow$»: We first observe, that MC^ω is equivalent to the following weaker principle PMC^ω: If $(S_n)_{n\in\omega}$ is a sequence of nonempty sets, there exists a subsequence $(S_{n(k)})_{k\in\omega}$ and a sequence $(E_k)_{k\in\omega}$ of nonempty finite $E_k \subseteq S_{n(k)}$. For if PMC^ω is applied to the sequence $(T_n)_{n\in\omega}$, T_n the set of all choice functions on $(S_i)_{i\in n}$, we define a finite $\emptyset \neq F_i \subseteq S_i$ through $F_i = \{f(i): f \in E_{k(i)}\}$, where $E_k \subseteq T_{n(k)}$ is finite and $k(i) = \min \{j: i < n(j)\}$.

We next assume, that MC^ω—and hence PMC^ω—is false and let $(S_n)_{n\in\omega}$ be a counterexample of PMC^ω, consisting of infinite and pairwise disjoint sets. We define a B-lattice X with a positive, unbounded functional φ. $S = \bigcup_{n\in\omega} S_n$ and $X = \{f \in S^\omega: \|f\| < \infty\}$, where $\|f\|^2 = \sum_{i=1}^{\infty}\left(\sum_{s\in S_i}|f(s)|\right)^2$. X is the l_2-sum of the spaces $l_1(S_i)$. As a normed space, X is complete, as can be proved in ZF^0 (c.f. [4]). The ordering $f \leq g$, iff $f(s) \leq g(s)$ for all $s \in S$, defines a vector lattice on X and $\|\cdot\|$ is a Riesz-norm with respect to this structure. We want define φ as $\varphi(f) = \sum_{s\in S} f(s)$.

This definition is correct, i.e.: $\sum f(s)$ is convergent. For set $F_n = \{s \in S: |f(s)| \geqq 1/n\}$. Since $\|f\| < \infty$, F_n is finite. If $F = \bigcup_{n\in\omega} F_n$, $M = \{n \in \omega: S_n \cap F \neq \emptyset\} = \{n \in \omega: \exists x \in S_n: f(x) \neq \emptyset\}$ is finite, for otherwise we can define a partial multiple choice function $(E_n)_{n\in M}$: $E_n = S_n \cap F_{n(k)}$, $n(k) = \min\{m: S_n \cap F_m \neq \emptyset\}$. Hence $\varphi(f) = \sum_{n\in M}\sum_{s\in S_n} f(s)$ is absolutely convergent by the definition of X and hence the series which defines $\varphi(f)$ is (unconditionally) convergent, by completeness.

That φ is linear and positive is obvious. But φ is not bounded. For otherwise we choose N so that $\sum_{n=1}^{N} 1/n > \pi/\sqrt{6}\cdot\|\varphi\|$. Choose $s_i \in S_i$ for $i \in N$ and set

$$f(s_i) = \frac{1}{i+1}, \quad i \in N, \quad f(s) = 0 \text{ otherwise}.$$

Since

$$\|f\| = \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}} \leqq \frac{\pi}{\sqrt{6}}$$

we obtain

$$\frac{\pi}{\sqrt{6}}\cdot\|\varphi\| < \sum_{n=1}^{N}\frac{1}{n} = |\varphi(f)| \leqq \|\varphi\|\cdot\|f\| \leqq \|\varphi\|\cdot\frac{\pi}{\sqrt{6}},$$

a contradiction. This proves the theorem.

Axioms similar to PMC^ω were studied in [2]. Another application of l_p-sums of l_ϱ-spaces to the axiom of choice appeared in [1].

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