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A Stefan Problem for the Heat Equation Subject to an Integral Condition (*).

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SUMMARY - We prove existence, uniqueness and continuous dependence and we study the behaviour of the free boundary of the solution of a Stefan problem for the heat equation when the integral condition $E(t)=\int_0^{s(t)} u(x,t)\,dx$ is assigned.

1. Introduction.

In [6] the heat conduction in a slab of variable thickness $0 < x < s(t)$ is studied in the case in which no boundary conditions are assigned on the face $x = 0$, but the integral of the temperature $E(t) = \int_0^{s(t)} u(x,t)\,dx$ is prescribed as a function of time.

Obviously $E(t)$ represents the thermal energy at time t if we assume that the heat capacity of the material is constant and equal to 1.

In [3] the same problem is considered assuming that the slab is made of a material undergoing a change of phase at a fixed temperature (say $u = 0$). In this case $x = s(t)$ represents the interphase and it is assumed that $u \equiv 0$ for $x > s(t)$.

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In this problem the total thermal energy at time t consists of two terms: one is the «latent energy», $Ls(t)$ (L is the latent heat), while the «diffusing energy» is still given by the integral $E(t)$.

In [3] the well posedness of the problem is proved when $E \geqslant 0$, $\dot{E} \geqslant 0$, the initial temperature $\varphi(x)$ satisfies $0 \leqslant \varphi(x) \leqslant N(b-x)$ for all $0 \leqslant x \leqslant b$ (N constant > 0), and $\varphi'(x) \leqslant 0$.

Here we consider the case of general data (without sign specification) and we prove that a $T > 0$ exists such that the problem is well posed in the time interval $(0, T)$.

The possible non existence of a global solution (i.e. for arbitrary T) is outlined in sec. 5, where we show that if $E \leqslant 0$ $\dot{E} \leqslant 0$, $u(x, 0) \leqslant 0$, then a finite time T_0 exists such that $\lim\limits_{t \to T_0} \dot{s}(t) = -\infty$.

2. Formulation and results.

Let us consider the following problem: find a triple (T, s, u) such that

i) $T > 0$;

ii) $s(t)$ is a positive function, continuously differentiable in $[0, T)$;

iii) $u(x, t) \in c_1(\overline{D}_T)$, u_{xx}, u_t are continuous in D_T, where $D_T = \{(x, t)\colon 0 < x < s(t), 0 < t < T\}$ and $\overline{D}_T$ is its closure;

iv) the following conditions are satisfied:

$$u_{xx} - u_t = 0 \quad \text{in } D_T \tag{2.1}$$

$$s(0) = b\,, \quad b > 0 \tag{2.2}$$

$$u(x, 0) = \varphi(x)\,, \qquad 0 < x < b \tag{2.3}$$

$$u\big(s(t), t\big) = 0 \qquad 0 < t < T \tag{2.4}$$

$$\int_0^{s(t)} u(x, t)\, dx = E(t)\,, \qquad 0 < t < T \tag{2.5}$$

$$u_x\big(s(t), t\big) = -\dot{s}(t) \qquad 0 < t < T\,. \tag{2.6}$$

Here we assume

$$\text{(A)} \qquad \varphi(x) \in C_1[0, b] \quad \text{and} \quad \varphi(b) = 0$$

$$\text{(B)} \qquad E(t) \in C_1[0, T] \quad \text{and} \quad E(0) = \int_0^b \varphi(x)\, dx\,.$$

First, we state our existence and uniqueness theorem.

THEOREM 1. Under assumptions (A), (B) there exists a solution (T, s, u) of problem (i)-(iv), which is unique in $(0, T)$.
Moreover we have

THEOREM 2. Let (T, s, u) be the solution of problem (i)-(iv), then $s \in C_1[0, T) \cap C_\infty(0, T)$.
We need some more regularity on the data to state the continuous dependence theorem.
Let us consider two solutions (T_1, s_1, u_1), (T_2, s_2, u_2) of problem (i)-(iv) corresponding to data φ_1, E_1 and φ_2, E_2 respectively.
If we replace assumptions (A), (B) with

$$\text{(A)}' \qquad \varphi(x) \in C_2[0, b]\,, \quad \varphi(b) = 0$$

$$\text{(B}'\text{)} \qquad E(t) \in C_2[0, T]\,, \quad E(0) = \int_0^b \varphi(x)\, dx$$

then we prove

THEOREM 3. If assumption (A)', (B)' are satisfied then two constants $k, \hat{T}$ can be found a priori such that:

$$\|s_1 - s_2\|_{C_1(0,\hat{T})} \leqslant k\{\|\varphi_1 - \varphi_2\|_{C_1(0,b)} + \|E_1 - E_2\|_{C_1(0,\hat{T})}\hat{T}^{\frac{1}{2}}\}\,. \tag{2.7}$$

The notation of spaces and norms used here and in the following are the same as in [12]. For sake of simplicity we often use the symbol $\|\cdot\|_N$, $N \geqslant 0$ integer instead of $\|\cdot\|_{C_N}$, $\|\cdot\|_\alpha$, $\alpha \in (0, 1)$ instead of $\|\cdot\|_{H_\alpha}$, to denote the Hölder norme of order α; and $\|\cdot\|_{N+\alpha}$, instead of $\|\cdot\|_{H_{N+\alpha}}$ to denote the Hölder norm of the N-th derivative.

3. Proof of Theorem 1.

I. *An equivalent formulation.*

We begin with stating the following

LEMMA 3.1. Let (T, s, u) be a solution of (2.1)-(2.6) then

$$u_x(0,t) = -\dot{E}(t) - \dot{s}(t)\,, \qquad 0 < t \leqslant T\,. \tag{3.1}$$

PROOF. From Green's identity

$$\iint_{D_t} (vLu - uL^*v)\,dx\,d\tau = \oint_{\partial D_t} [(u_x v - u v_x)\,d\tau + uv\,dx]$$

where L is the heat operator and L^* its adjoint, with $u = u(x,t)$ and $v = 1$, we obtain

$$s(t) = b - \int_0^t u_x(0,\tau)\,d\tau - E(t) + E(0) \tag{3.2}$$

from which (3.1) follows.

LEMMA 3.2. Let $v(t) = u_x(s(t), t)$, where u, s solves (2.1)-(2.6), then

$$\begin{aligned} v(t) = {} & 2\int_0^t N_x\big(s(t), t;\, s(\tau), \tau\big)\,v(\tau)\,d\tau - \\ & - 2\int_0^t N_x\big(s(t), t;\, 0, \tau\big)\,v(\tau)\,d\tau - 2E(0)\,N_x\big(s(t), t;\, 0, 0\big) - \\ & - 2\int_0^t N_{x\tau}\big(s(t), t;\, 0, \tau\big)\,E(\tau)\,d\tau + 2\int_0^b G\big(s(t), t;\, \xi, 0\big)\,\varphi'(\xi)\,d\xi \end{aligned} \tag{3.3}$$

with

$$s(t) = b - \int_0^t v(\tau)\,d\tau\,. \tag{3.4}$$

PROOF. (3.3)-(3.4) are proved following the methods of [4]. Here $G(x, t; \xi, \tau)$ and $N(x, t; \xi, \tau)$ are Green's and Neuman's functions for the heat operator.

Thus we have reduced (2.1)-(2.6) to a system of integral equations such that if $u(x, t)$, $s(t)$ satisfy (2.1)-(2.6) then $v(t)$, $s(t)$ satisfy (3.3)-(3.4).

Conversely, if $v(t)$ is a continuous solution of (3.3) and if $s(t)$, given by (3.4), is positive, we can define $u(x, t)$ replacing $u_\xi(s(\tau), \tau)$ with $v(\tau)$ and $u_\xi(0, \tau)$ with $-\dot{E}(\tau) + v(\tau)$ in the formal rappresentation for the solution of the problem. Now it is easy to show (see [5]) that $u(x, t)$ so defined satisfies (2.1)-(2.6).

Moreover, it is known that the initial boundary problem (2.1)-(2.5), for given Lipschitz continuous and positive $s(t)$, admits a unique solution [6].

It is so proved that problem (2.1)-(2.6) is equivalent to the problem of finding a continuous solution of the integral equations (3.3), (3.4).

II. *Existence and uniqueness.*

Now we prove that the system (3.3), (3.4) has a unique solution for $0 \leqslant t \leqslant T$ where T is sufficiently small.

We consider

$$X_{T,M} = \left\{v(t) \in C[0, T] : \|v\|_0 = \max_{0 \leqslant t \leqslant T} |v(t)| \leqslant M\right\}.$$

On the set $X_{T,M}$ we define a transformation

$$\tilde{v} = \mathcal{T}(v)$$

as follows

$$\tilde{v}(t) = 2\int_0^t N_x(s(t), t; s(\tau), \tau)\, v(\tau)\, d\tau - 2\int_0^t N_x(s(t), t; 0, \tau)\, v(\tau)\, d\tau - \tag{3.5}$$

$$- 2E(0) N_x(s(t), t; 0, 0) - 2\int_0^t N_{x\tau}(s(t), t; 0, \tau)\, E(\tau)\, d\tau +$$

$$+ 2\int_0^b G(s(t), t; \xi, 0)\, \varphi'(\xi)\, d\xi$$

where

(3.6) $$s(t) = b - \int_0^t v(\tau)\, d\tau \, .$$

We shall prove that there exists a fixed point of $\mathcal{T}$.
Chosen a T such that

(3.7) $$\frac{b}{2} \leqslant s(t) \leqslant \frac{3}{2} b \, , \qquad 0 \leqslant t \leqslant T$$

we have that $\|v\|_0 \leqslant M$ implies immediately

(3.8) $$|s(t) - s(\tau)| \leqslant M(t - \tau) \, , \qquad 0 \leqslant \tau \leqslant t$$

and, from (3.5),

(3.9) $$\|\tilde{v}\|_0 \leqslant k + cM^2 T^{\frac{1}{2}}$$

having posed $T \leqslant 1$, $M \geqslant 1$. k is a constant depending on $\|E\|$, $\|\varphi\|$, b, and c is a constant depending on b only.

Thus chosen a set $X_{T,M}$ with e.g. $M = 2k$ and T such that

$$T^{\frac{1}{2}} \leqslant \frac{1}{c\,4k}$$

then $\mathcal{T}$ maps $X_{T,M}$ into itself.

Now we prove that $\mathcal{T}$ is a contraction.

For any $v_1, v_2 \in X_{T,M}$ let us consider the difference $v_1 - v_2$.

Denote

(3.10) $$\|v_1 - v_2\|_0 = \varepsilon \, , \qquad \varepsilon \leqslant 2M \, .$$

From (3.6) we have,

(3.11) $$|s_1(t) - s_2(t)| \leqslant \varepsilon t \, , \qquad \|\dot{s}_1 - \dot{s}_2\|_0 \leqslant \varepsilon \, , \qquad 0 \leqslant t \leqslant T \, .$$

From (3.5):

$$(3.12)\quad \tilde{v}_1-\tilde{v}_2=2\int_0^t[N_x(s_1(t),t;s_1(\tau),\tau)\,v_1(\tau)-N_x(s_2(t),t;s_2(\tau),\tau)\,v_2(\tau)]\,d\tau$$

$$-2\int_0^t[N_x(s_1(t),t;\,0,\,\tau)\,v_1(\tau)-N_x(s_2(t),t;\,0,\,\tau)\,v_2(\tau)]\,d\tau-$$

$$-2E(0)[N_x(s_1(t),t;\,0,0)-N_x(s_2(t),t;\,0,0)]-$$

$$-2\int_0^t[N_{x\tau}(s_1(t)t;\,0,\,\tau)-N_{x\tau}(s_2(t),t;\,0,\,\tau)]E(\tau)\,d\tau+$$

$$+2\int_0^b[G(s_1(t),t;\,\xi,0)-G(s_2(t),t;\,\xi,0)]\varphi'(\xi)\,d\xi\,.$$

To estimate the first integral on the right-hand side of (3.12), say I_1, we consider that:

$$(3.13)\qquad I_1=-\int\frac{s_1(t)-s_1(\tau)}{t-\tau}\,\Gamma(s_1(t),t;s_1(\tau),\tau)\,v_1(\tau)\,d\tau+$$

$$+\int_0^t\frac{s_2(t)-s_2(\tau)}{t-\tau}\,\Gamma(s_2(t),t;\;s_2(\tau),\tau)\,v_2(\tau)\,d\tau+I_1'$$

where I_1' denotes the sum of the first two terms on the right-hand side of (3.13) but with $s_1(\tau)$ and $s_2(\tau)$ replaced by $-s_1(\tau)$, $-s_2(\tau)$ respectively.

We can write

$$(3.14)\quad I_1=-\int_0^t\left[\frac{s_1(t)-s_1(\tau)}{t-\tau}-\frac{s_2(t)-s_2(\tau)}{t-\tau}\right]\Gamma(s_2(t),t;\,s_2(\tau),\tau)\,v_2(\tau)\,d\tau+$$

$$+\int_0^t\frac{s_1(t)-s_1(\tau)}{t-\tau}\,[\Gamma(s_2(t),t;\;s_2(\tau),\tau)-\Gamma(s_1(t),t;\;s_1(\tau),\tau)]\,v_2(\tau)\,d\tau-$$

$$-\int_0^t\frac{s_1(t)-s_1(\tau)}{t-\tau}\,\Gamma(s_1(t),t;\;s_1(\tau),\tau)\,[v_1(\tau)-v_2(\tau)]\,d\tau+I_1'\,.$$

From (3.14)

$$|I_1| \leqslant c\left\{\int_0^t \frac{\int_\tau^t |v_1(\eta) - v_2(\eta)|d\eta}{(t-\tau)^{\frac{3}{2}}} M\,d\tau + \right. \tag{3.15}$$

$$+\int_0^t \frac{M^2}{(t-\tau)^{\frac{1}{2}}}\left[1 - \exp\left[\frac{(s_1(t)-s_1(\tau))^2}{4(t-\tau)} - \frac{(s_2(t)-s_2(\tau))^2}{4(t-\tau)}\right]\right] d\tau +$$

$$\left. +\int_0^t \frac{M}{(t-\tau)^{\frac{1}{2}}}\,\varepsilon\,d\tau\right\} + |I_1'| \leqslant c\{M\varepsilon t^{\frac{1}{2}} + M^3\varepsilon t^{\frac{3}{2}} + M\varepsilon t^{\frac{1}{2}}\} + |I_1'|\,.$$

Here and in the following c will denote a constant depending on b and possibly on M. The estimate for $|I_1'|$ is obtained by means of the mean value theorem

$$|I_1'| \leqslant cM\varepsilon t^{\frac{1}{2}}\,. \tag{3.16}$$

Finally

$$|I_1| \leqslant cM\varepsilon T^{\frac{1}{2}}\,. \tag{3.17}$$

The second integral in (3.12), call it I_2, is easily estimated as follows

$$|I_2| \leqslant 2\int_0^t |N_x(s_1(t), t;\, 0, \tau) - N_x(s_2(t), t;\, 0, \tau)\|v_1(\tau)|\,d\tau +$$

$$+\int_0^t |N_x(s_2(t), t;\, 0, \tau)\|v_1(\tau) - v_2(\tau)|\,d\tau \leqslant$$

$$\leqslant 2M\|s_1 - s_2\|_0\int_0^t |N_{xx}(\bar{s}(t), t;\, 0, \tau)|\,d\tau + 2\varepsilon\int_0^t |N_x(s_2(t), t;\, 0, \tau)|\,d\tau$$

with

$$\min(s_1, s_2) \leqslant \bar{s} \leqslant \max(s_1, s_2)$$

that is

$$|I_2| \leqslant c\varepsilon T\,. \tag{3.19}$$

Estimates like these hold for the third and the fourth terms on the right-hand side of (3.12), from which we obtain

$$|I_3 + I_4| \leqslant c\|E\|_0 \varepsilon T . \tag{3.20}$$

For the last integral, say I_5, we have (see [4])

$$|I_5| \leqslant c\varepsilon\|\varphi\|_1 T^{\frac{1}{2}} . \tag{3.21}$$

From (3.17), (3.19), (3.20) and (3.21) we obtain

$$\|\tilde{v}_1 - \tilde{v}_2\|_0 \leqslant cT^{\frac{1}{2}}\|v_1 - v_2\|_0 . \tag{3.22}$$

Thus, there exists a time $\bar{T} \leqslant T$ such that $c\bar{T}^{\frac{1}{2}} < 1$, then (3.22) implies that $\mathcal{T}$ is a contractive mapping in the norm of $C[0, \bar{T}]$.

Therefore we proved that there exists a unique fixed point $v(t)$ of $\mathcal{T}$ in $X_{\bar{T},M}$, and then $v(t)$ is the unique solution of the integral equation (3.3), with $s(t)$ defined by (3.4).

REMARK 3.1. Note that in the case in which $s(t)$ is monotone non-decreasing (this happens for example for $\varphi(x) \geqslant 0$, $E(t) \geqslant 0$, $0 \leqslant \dot{E}(t) \leqslant A$ (see [3])) we can apply the proof of § 3 step by step, to obtain a solution $u(x, t)$, $s(t)$ (or $v(t)$) for all times [4].

Now we prove

THEOREM 2. If (T, s, u) solves (i)-(iv) then $s \in C_1[0, T) \cap C_\infty(0, T)$.

PROOF. Recalling [7] we can assert that $s \in C^\infty(\varepsilon, T)$ for any $\varepsilon > 0$. Moreover, performing the limit for $t \to 0$ in (3.3), we can easily prove that

$$v(0) = \varphi'(b) \tag{3.23}$$

that is $\dot{s}(t)$ is continuous at $t = 0$.

REMARK 3.2. Let us define

$$w(x, t) = \int_{s(t)}^{x} u(\xi, t)\, d\xi . \tag{3.24}$$

By strightforward computation one verifies that if (T, s, u) solves (2.1)-(2.6) then (T, s, w) solves

$$(3.25) \qquad w_{xx} - w_t = -\dot{s} \qquad \text{in } D_T$$

$$(3.26) \qquad w(x, 0) = \int_b^x \varphi(\xi)\, d\xi\,, \qquad 0 < x < b$$

$$(3.27) \qquad w(0, t) = -E(t)\,, \qquad 0 < t < T$$

$$(3.28) \qquad w(s(t), t) = 0\,, \qquad 0 < t < T$$

$$(3.29) \qquad w_x(s(t), t) = 0\,, \qquad 0 < t < T\,.$$

Thus we proved existence and uniqueness of the solution of problem (3.25)-(3.29), which is a problem with Cauchy data assigned on $x = s(t)$, which differs from those studied in [8] where the right-hand member of the parabolic equation was not allowed to depend on $\dot{s}$.

4. Proof of Theorem 3.

LEMMA 4.1. Assume (A)′, (B)′ and let (T, s, u) be the solution of problem (i)-(iv), then

$$(4.1) \qquad |t^{\frac{1}{2}}\ddot{s}(t)| \leqslant c\,, \qquad 0 \leqslant t \leqslant \tilde{T} < T\,.$$

PROOF. Replacing assumptions (A), (B) with (A)′, (B)′, we can repeat the arguments of sec. 3 to prove a contraction on the set

$$\bar{X}_{\tilde{T},H} = \{v(t) \in H_{\frac{1}{2}}[0, \tilde{T}] : \|v\|_{\frac{1}{2}} \leqslant H,\ \tilde{T} \leqslant T\}$$

where $v(t)$ is defined by (3.3), (3.4).

This implies $s \in H_{1+1/2}[0, \tilde{T}]$.

The estimates of $\|v\|_{\frac{1}{2}}$ and $\|v_1 - v_2\|_{\frac{1}{2}}$ are obtained following the methods of [9].

LEMMA 4.2. Under the hypotesis of Lemma 4.1 we have

$$(4.2) \qquad \max_{0 \leqslant x \leqslant s(t)} |u_{xx}(x, t)| \leqslant c t^{-\frac{1}{2}} \qquad 0 < t \leqslant \tilde{T}\,.$$

PROOF. In [10] it is proved that the solution $z(x, t)$ of the first boundary problem in D_T for the heat equation has a bounded second order derivative z_{xx}, when $z(s(t), t) = 0$, $z(0, t)$ is Lipschitz continuous, $z(x, 0)$ satisfies assumptions like (A)$'$.

It can be shown that in our case $u(0, t) \in H_{\frac{1}{2}}$ and that it is possible to modify the estimates of [10] to prove inequality (4.2). Details are omitted for sake of brevity.

Let (T_1, s_1, u_1), (T_2, s_2, u_2) be two solutions of problem (i)-(iv), with assumptions (A)$'$, (B)$'$, corresponding to the data φ_1, E_1, b and φ_2, E_2, b respectively. We perform the transformation (for $i = 1, 2$)

$$y = x/s_i\,, \qquad w_i(y, t) = u_i(s_i y, t)\,, \qquad \bar{\varphi}_i(y) = \varphi_i(by) \tag{4.3}$$

leading to

$$w_{it} = -s_i^{-2} w_{iyy} + y \dot{s}_i s_i^{-1} w_{iy} \qquad \text{on} \qquad D_{\hat{T}},\ \hat{T} = \min(T_1, T_2) \leqslant \tilde{T} \tag{4.4}$$

$$w_i(y, 0) = \bar{\varphi}_i(y)\,, \qquad 0 < y < 1 \tag{4.5}$$

$$w_{iy}(0, t) = -s_i(t)[\dot{s}_i(t) + \dot{E}_i(t)]\,, \qquad 0 < t < \hat{T} \tag{4.6}$$

$$w_i(1, t) = 0\,, \qquad 0 < t < \hat{T} \tag{4.7}$$

$$s_i(t) = -s_i^{-1}(t) w_{iy}(1, t)\,, \qquad 0 < t < \hat{T}\,. \tag{4.8}$$

Obviously problem (4.4)-(4.8) is equivalent to (2.1)-(2.6).

Let us introduce the following notation

$$\begin{aligned} \delta(t) &= s_2(t) - s_1(t)\,, \qquad \dot{\delta}(t) = \dot{s}_2(t) - \dot{s}_1(t) \\ W(y, t) &= w_2(y, t) - w_1(y, t) \\ \Delta\varphi(y) &= \bar{\varphi}_2(y) - \bar{\varphi}_1(y) \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\Delta E(t) = E_2(t) - E_1(t)\,, \qquad \Delta\dot{E}(t) = \dot{E}_2(t) - \dot{E}_1(t)\,, \\ &\|\delta\|_t = \max_{0 \leqslant \tau \leqslant t} |\delta(\tau)|\,, \qquad \|\dot{\delta}\|_t \max_{0 \leqslant \tau \leqslant t} |\dot{\delta}(\tau)| \end{aligned} \tag{4.10}$$

$W(y, t)$ defined by (4.9) solves:

$$\begin{aligned} &W_t = A(t) W_{yy} + B(y, t) W_y + F(y, t) \qquad \text{in } D_{\hat{T}} \\ &W(y, 0) = \Delta\varphi(y)\,, \qquad 0 < y < 1 \end{aligned} \tag{4.11}$$

$$(4.12)\qquad \begin{aligned} W_y(0,t) &= -s_2(\Delta\dot{E}+\dot{\delta})-\delta(\dot{E}_1+\dot{s}_1)=G(t)\,, && 0<t<\hat{T}\\ W(1,t) &= 0\,, && 0<t<\hat{T}\end{aligned}$$

where

$$A(t) = s_1^{-2}$$

$$B(y,t) = y\dot{s}_1 s_1^{-1}$$

$$F(y,t) = -\delta\{(s_1+s_2)s_1^{-2}s_2^{-2}w_{2yy}+ys_1^{-1}s_2^{-1}\dot{s}_2 w_{2y}\}+\dot{\delta}ys_1^{-1}w_{2y}\,.$$

We are going to study the difference

$$(4.14)\qquad \dot{\delta}=\dot{s}_2-\dot{s}_1=-s_2^{-1}W_y(1,t)-\delta s_1^{-1}s_2^{-1}w_{1y}(1,t)$$

for which we need an estimate of $W_y(1,t)$.

We split W into the sum

$$W=W_1+W_2$$

where W_1 solves problem:

$$(4.15)\qquad W_{1t}=A(t)W_{1yy}\quad \text{in } D_{\hat{T}}$$

with conditions (4.12), and W_2 solves:

$$(4.16)\qquad W_{2t}=A(t)W_{2yy}+B(y,t)W_{2y}+F_0(y,t)\,,\quad \text{in } D_{\hat{T}}$$

with zero initial and boundary conditions.

In (4.16)

$$(4.17)\qquad F_0(y,t)=F(y,t)+B(y,t)W_{1y}\,.$$

As to W_1, we split it again into the sum $W_1=z_1+z_2$, where z_1 is the solution in the half plane $x>0$ of

$$(4.18)\qquad z_{1t}=A(t)z_{1yy}$$

with

$$z_1(y,0)=0\,,\quad z_{1y}(0,t)=G(t)\,,$$

while z_2 solves the same equation (4.18) with

$$(4.19)\qquad z_2(y,0)=\Delta\varphi\,,\qquad z_{2y}(0,t)=0\,,\qquad z_2(1,t)=-z_1(1,t)\,.$$

Introducing the fundamental solution for the operator $\partial/\partial t - A(t)(\partial^2/\partial y^2)$, say $\Gamma_A(y,t;\xi,\tau)$, by means of the parametrix method of E. E. Levi, we have for z_1:

$$(4.20)\qquad z_1(y,t)=-2\int_0^t \Gamma_A(y,t;0,\tau)G(\tau)\,d\tau\,.$$

From (4.20) we have immediately the estimate

$$(4.21)\qquad |z_{1y}(1,t)|\leqslant ct(\|\dot\delta\|_t+\|\Delta\dot E\|_t)\,.$$

An estimate like (4.21) holds for $z_{1t}(1,t)$.

Now we consider z_2 as the restriction to $[0,1]\times(0,\hat T)$ of the solution of

$$(4.22)\qquad z_{2t}=A(t)z_{2yy}\qquad\qquad \text{in } (-1,1)\times(0,\hat T)$$

$$(4.23)\qquad z_2(y,0)=\overline{\Delta\varphi}\,,\qquad\qquad -1<y<1$$

$$(4.24)\qquad z_2(-1,t)=z_2(1,t)=-z_1(1,t)\,,\qquad 0<t<\hat T$$

with

$$\overline{\Delta\varphi}=\begin{cases}\Delta\varphi(y)\,, & y\geqslant 0\\ \Delta\varphi(-y)\,, & y<0\,.\end{cases}$$

We can estimate $z_{2y}(1,t)$ knowing $z_2(y,t)$ on $\partial D_{\hat T}$, by means of Lemma 3.1 p. 535 of [11].

Making use of the estimate on z_1, we obtain

$$(4.25)\qquad |z_{2y}(1,t)|\leqslant c\{\|\Delta\varphi\|_1+t(\|\dot\delta\|_t+\|\Delta\dot E\|_t)\}\,.$$

From (4.21) and (4.25) we get the estimate

$$(4.26)\qquad |W_{1y}(1,t)|\leqslant c[\|\Delta\varphi\|_1+t\|\dot\delta\|_t+t\|\Delta\dot E\|_t]\,.$$

Moreover, applying the maximum principle in $D_{\hat{T}}$, we get also the estimate

$$(4.27) \qquad |W_{1y}(y,t)| \leqslant c[\|\Delta\varphi\|_1 + \|\dot{\delta}\|_t + \|\Delta E\|_1] .$$

Finally, let us consider W_2 as the restriction in $[0,1]\times(0,\hat{T})$ of the solution of

$$(4.28) \qquad W_{2t} = A(t)\,W_{2yy} + \bar{F}(y,t) \quad \text{in } (-1,1)\times(0,\hat{T})$$

$$(4.29) \qquad W_2(y,0) = W_2(-1,t) = W_2(1,t) = 0$$

with

$$(4.30) \qquad \bar{F}(y,t) = \begin{cases} B(y,t)\,W_{2y}(y,t) + F_0(y,t)\,, & y \geqslant 0 \\ B(-y,t)\,W_{2y}(y,t) + F_0(-y,t)\,, & y < 0\,. \end{cases}$$

Using the methods of [12], sec. 4 we obtain the estimate

$$(4.31) \qquad \max_{y\in[-1,1]} |W_{2y}(y,t)| \leqslant c\left\{ \int_0^t (t-\tau)^{-\frac{1}{2}} \max_y |W_{2y}(y,t)|\,d\tau + \right.$$
$$\left. + t^{\frac{1}{2}}(\|\dot{\delta}\|_t + \|\Delta\dot{E}\|_t + \|\Delta\varphi\|_1)\right\}$$

which gives, with (4.26),

$$(4.32) \qquad |W_y(1,t)| \leqslant c\{t^{\frac{1}{2}}\|\dot{\delta}\|_t + \|\Delta\varphi\|_1 + t^{\frac{1}{2}}\|\Delta\dot{E}\|_t\} .$$

From (4.14)

$$(4.33) \qquad \|\dot{\delta}\|_t \leqslant c\{\|\Delta\varphi\|_1 + t^{\frac{1}{2}}\|\Delta E\|_1 + t^{\frac{1}{2}}\|\delta\|_t\}$$

which proves (2.7).

5. Behaviour of the free boundary.

It has been proved in [3] that if one assumes positive data $\varphi(x)$, $E(t)$ with $\varphi'(x) \leqslant 0$ and $0 \leqslant \dot{E}(t) \leqslant A$, then $s(t)$ is monotone non decreasing in t and $\dot{s}(t)$ is bounded so that a solution can exist with arbitrarily large T.

In this section we want to study the problem of the continuation of the solution and to analyze the behaviour of the free boundary when the sign restriction imposed in [3] are no longer valid.

We will assume besides of (A)′, (B)′ that

$$\varphi(x) \leqslant 0 \tag{5.1}$$

$$E(t) \leqslant 0\,, \qquad \dot{E}(t) \leqslant 0\,. \tag{5.2}$$

We first prove

LEMMA 5.1. Let (T, s, u) be the solution of (i)-(iv) with assumption (A)′, (B)′. Let the data satisfy (5.1), (5.2), then

$$u(x, t) \leqslant 0\,, \qquad \dot{s}(t) \leqslant 0\,. \tag{5.3}$$

PROOF. From (5.1) it is $\varphi'(b) \geqslant 0$, that is since $\dot{s}(t)$ is continuous,

$$\dot{s}(0) = -u_x(b, 0) \leqslant 0\,. \tag{5.4}$$

Consider the solution corresponding to

$$\varphi^{(n)}(x) = \varphi(x) - \frac{1}{n}(b - x) \tag{5.5}$$

for which $\dot{s}_n(0) < 0$.

Assume that there exists a first time $\bar{t}_n$ such that $\dot{s}_n(\bar{t}_n) = 0$, and then

$$u_{n_x}\big(s_n(\bar{t}_n), \bar{t}_n\big) = 0\,. \tag{5.6}$$

From the maximum principle in $D_{\bar{t}_n}^-$, it is $u_n(x, t) < 0$, that is, as $u_n\big(s_n(\bar{t}_n), \bar{t}_n\big) = 0$, $\big(s_n(\bar{t}_n), \bar{t}_n\big)$ is an isolated maximum for u_n, and the parabolic Hopf's Lemma [13] ensures $u_{n_x}\big(s_n(\bar{t}_n), \bar{t}_n\big) > 0$, contradicting (5.6).

Then

$$u_n(x, t) < 0\,, \qquad \dot{s}_n(t) < 0 \tag{5.7}$$

and performing the limit for $n \to \infty$ we obtain (5.3).

REMARK 5.1. If one excludes the trivial case in which $\dot{s} \equiv 0$, corresponding to data $E = \varphi \equiv 0$, then immediately

$$\dot{s}(t) < 0 , \quad 0 < t < T . \tag{5.8}$$

THEOREM 4. Let us consider two sets of data (E_1, φ_1, b_1), (E_2, φ_2, b_2) for problem (i)-(iv), and assume that both of them satisfy assumptions (A)′, (B)′ and (5.1), (5.2).

Let (T_1, s_1, u_1), (T_2, s_2, u_2) be the correspondent solutions and assume:

$$E_2 \geqslant E_1 , \quad \varphi_2 \geqslant \varphi_1 , \quad b_2 > b_1 \tag{5.9}$$

$$\int_x^{b_2} [\varphi_2(y) + 1]\, dy \geqslant 0 , \quad 0 < x < b_2 \tag{5.10}$$

$$\int_0^{b_2} [\varphi_2(y) + 1]\, dy > 0 .$$

Then

$$s_1(t) < s_2(t) , \quad t < T_0 \tag{5.11}$$

where $T_0 = \min \{T_1, T_2, \sup \bar{t} \colon s_2(\bar{t}) > -E_2(\bar{t})\}$.

PROOF. Lemma 3.1 ensures

$$u_{ix}(0, t) = -\dot{E}_i(t) - \dot{s}_i(t) , \quad i = 1, 2 . \tag{5.12}$$

Thus we can consider (T_1, s_1, u_1), (T_2, s_2, u_2) as solutions of two boundary problems with assigned flux, and then (see [14], Lemma 2.10) (5.11) holds.

We conclude this section with the following

THEOREM 5. Let the hypothesis of Lemma 5.1 hold.

Then there exists a finite time T_0 such that:

$$\lim_{t \to T_0 -} \dot{s}(t) = -\infty . \tag{5.13}$$

PROOF. The existence of a solution of problem (i)-(iv) with arbitrarily large T implies (see [14], Cor. 2.12) that

$$\iint_{D_t} |u(x,t)|\, dx\, dt = \int_0^t |E(\tau)|\, d\tau < +\infty \tag{5.14}$$

for all $t > 0$, which is contradictory with (5.2).

Now if we suppose that a time $\bar{t}$ exists such that $\lim_{t\to\bar{t}} s(t) = 0$ then

$$\lim_{t\to\bar{t}} \dot{s}(t) = -\infty . \tag{5.15}$$

Indeed if (5.15) is not true then

$$\lim_{t\to\bar{t}} u_x(s(t), t) \qquad \text{and} \qquad \lim_{t\to\bar{t}} u_x(0, t)$$

exist and are bounded, and then u is continuous in $(s(\bar{t}), \bar{t})$ (and equal to 0).

That implies

$$E(\bar{t}) = \lim_{t\to\bar{t}} \int_0^{s(t)} u(x,t)\, dx = 0 \tag{5.16}$$

which cannot hold because of (5.2).

Of course we excluded the trivial case $E \equiv 0$.

The theorem is proved recalling [15], Theorem 8.

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