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**A remark on a Harnack inequality for degenerate parabolic equations**

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# A Remark on a Harnack Inequality for Degenerate Parabolic Equations.

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## 0. Introduction.

We study in a cylinder $Q = \Omega \times ]0, T[$ ($\Omega$ bounded open set in $\boldsymbol{R}^n$, $n \geqslant 3$) the degenerate parabolic equation

$$(0.1) \qquad \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_i}\right) = \frac{\partial}{\partial t}(w(x)u)$$

with the assumption

$$(0.2) \qquad \lambda^{-1} w(x)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i \xi_j \leqslant \lambda w(x)|\xi|^2, \qquad \forall \xi \in \boldsymbol{R}^n, \text{ a.e. in } Q$$

where $w(x) \geqslant 0$ is an $A_2$ weight in $\boldsymbol{R}^n$ (see sec. 1 for the definition).

We prove a Harnack principle for positive solutions of (0.1) on the usual parabolic cylinders of size $(\varrho, \varrho^2)$ (see sect. 2 for precises statements).

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Our interest for this problem arose while studying the equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_i}\right)=\frac{\partial u}{\partial t} \tag{0.3}$$

with hypotheses (0.2).

In fact in [C.S$_1$] we gave examples showing that a Harnack principle for positive solutions of (0.3) is in general false, at least on the standard parabolic cylinders of size $(\varrho, \varrho^2)$.

Moreover in [C.S$_2$] we proved that a more general Harnack principle holds true for positive solutions of (0.3) but only under integrability hypotheses on $[w(x)]^{-1}$ stronger that the ones necessary for degenerate elliptic equations (and depending on the space dimension $n$).

This Harnack principle holds on cylinders of size $(\varrho, \varphi(\varrho))$, where $\varphi(\varrho)$ (a continuous, strictly increasing function) is, in general, different from $\varrho^2$ and varies from point to point depending on the degeneracy of the weight $w(x)$.

In conclusion the results in [C.S$_1$] and [C.S$_2$] show the different behaviour of parabolic degenerate operators like (0.3) if compared both to the corresponding degenerate elliptic operators (as studied in [F.K.S.]) and to the usual non degenerate parabolic operators.

Given this we considered somewhat interesting to point that equation (0.1) on the contrary presents a « perfectly normal » behaviour; in fact the main result of this paper is that a Harnack inequality holds for solutions of (0.1)

(i) on the standard parabolic cylinders;

(ii) with only the $A_2$ condition on $w(x)$.

Moreover it is interesting to remark that parabolic degenerate equations with also a non negative coefficient in front of the time derivative are more natural from the physical point of view than equations like (0.3). There are in fact two relevant physical quantities in heat diffusion processes: the conductivity coefficient and the specific heat; precisely this last one appears in the equation in front of the time derivative.

Finally we point out that a number of papers have been devoted to the study of equations like (0.1) but in non divergence form (see e.g. [F.W.], [W$_1$], [W$_2$]) and that a preliminary study concerning the local boundedness of the solutions of (0.1) can be found in [Ch.F].

Since the pattern of the proof follows closely Moser's proof in

[$M_2$] we have just sketched the proofs, stressing only some parts in which the presence of the weights originates some technical difficulties.

## 1. Notations. Some preliminary results.

We will say that a real, non negative, measurable function defined in $\boldsymbol{R}^n$ is an $A_2$ weight, if:

$$\sup_C \left(\frac{1}{|C|}\int_C w(x)\,dx\right)\left(\frac{1}{|C|}\int_C \frac{1}{w(x)}\,dx\right) = c_0 < +\infty$$

where $C$ is any $n$-dimensional cube and $|C|$ is the Lebesgue measure of $C$. $c_0$ will be indicated as the $A_2$ constant of $w(x)$.

Let $\Omega$ be an open bounded set in $\boldsymbol{R}^m (m \geqslant 3)$, $T > 0$ and $Q = \Omega \times ]0, T[$.

$L^p(\Omega, w(x))$ $(1 \leqslant p \leqslant +\infty)$ is the Banach space of the (classes of) measurable functions $u(x)$ s.t. the norm

$$|u|_{p,w;\Omega} = \left(\int_\Omega |u(x)|^p w(x)\,dx\right)^{1/p} < +\infty$$

$H^1(\Omega, w(x))$ [resp. $H_0^1(\Omega, w(x))$] is the completion of $C^\infty(\bar{\Omega})$ [resp. $C_0^\infty(\Omega)$] under the norm

$$\|u\|_{1,w;\Omega} = \left(\int_\Omega (u^2(x) + |Du|^2)\,w(x)\,dx\right)^{\frac{1}{2}}$$

$\left[\text{resp. } \|u\|_{1,w;\Omega} = \left(\int_Q |Du|^2 w\,dx\right)^{\frac{1}{2}}\right]$; here $Du$ is the gradient of $u$.

$$W = \left\{u \in L^2\left(0, \mathrm{T}; H_0^1(\Omega, w(x))\right) \colon\ u_t \in L^2\left(0, T; L^2(\Omega, w(x))\right)\right\}.$$

Let us now state some lemmas we will use in the following. Here we suppose $w(x)$ to be an $A_2$ weight with $A_2$ constant $c_0$.

LEMMA 1.1 (see [F.K.S.], Th. 1.2). *There are two positive constant $c_1$ and $\delta$ such that*

$$(1.2)\qquad \left(\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}|u|^{2k}w(x)\,dx\right)^{1/2k}\leqslant$$

$$\leqslant c_1R\left(\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}|Du|^2w(x)\,dx\right)^{\frac12}\ (^1)\,,$$

*for all $u\in H_0^1(B_{x_0}(R),\ w(x))$ and $k\in[1,\chi]$ $(\chi=n/(n-1)+\delta)$; here $c_1$ depends only on $n$ and $c_0$.*

LEMMA 1.2. *Let $u\in L^\infty\big(a,b;L^2(B_{x_0}(R),w(x))\big)\cap L^2\big(a,b;H_0^1(B_{x_0}(R), w(x))\big)$. Then*

$$(1.3)\qquad \left(\frac{1}{b-a}\int_a^b\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}|u|^{2\bar\chi}w(x)\,dx\,dt\right)^{1/2\bar\chi}\leqslant$$

$$\leqslant(c_1R)^{1/\bar\chi}\left[\operatorname*{ess\,sup}_{(a,b)}\left(\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}u^2(x,t)\,w(x)\,dx\right)^{\frac12}\right]^{1-1/\bar\chi}\cdot$$

$$\cdot\left(\frac{1}{b-a}\int_a^b\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}|Du|^2w(x)\,dx\,dt\right)^{1/\bar\chi}$$

*where $\bar\chi=(2\chi-1)/\chi>1$ ($\chi$ is the number of Lemma 1.1), the constant $c_1$ depend only on $n$ and $c_0$.*

The proof is a straightforward consequence of Lemma 1.1.

LEMMA 1.3. *Let $\varphi(x)$ be a continuous function compactly supported in the ball $B_{x_0}(R)$ of $\mathbf{R}^n$. Assume that $\varphi$ has convex level sets, that $\varphi$ is not identically zero and satisfies $0\leqslant\varphi\leqslant1$. Then for any $u\in H^1(B_{x_0}(R)$;*

(1) Here and in the following

$$B_{x_0}(R)=\{x\in\mathbf{R}^n\colon |x-x_0|<R\}\,,\qquad w(B_{x_0}(R))=\int_{B_{x_0}}w(x)\,dx\,.$$

*$w(x)$)*

$$\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}\left(u(x)-A_{B_{x_0}(R)}\right)^2\varphi(x)\,w(x)\,dx\right)^{\frac12}\leqslant \tag{1.4}$$

$$\leqslant c_2\frac{|B_{x_0}(R)|}{\varphi(B_{x_0}(R))}\,R\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}|Du|^2\varphi(x)\,w(x)\,dx\right)^{\frac12};$$

*here*

$$A_{B_{x_0}(R)}=\frac{1}{\varphi w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}u(x)\,\varphi(x)\,w(x)\,dx$$

*and $c_2$ depends only on $n$ and $c_0$* [2].

PROOF. The following inequality holds:

$$|u(x)-\tilde{A}_{B_{x_0}(R)}|\sqrt{\varphi(x)}\leqslant c\,\frac{|B_{x_0}(R)|}{\varphi(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}\frac{|Du(z)|}{|z-x|^{n-1}}\sqrt{\varphi(z)}\,dz\,.$$

Here,

$$\tilde{A}_{B_{x_0}(R)}=\frac{1}{\varphi(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}u(x)\,\varphi(x)\,dx\,.$$

The proof can be found in [C.S$_2$] ((2.7) of Lemma 2.4). Furthermore

$$\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}\left(\int\limits_{B_{x_0}(R)}\frac{|Du(z)|}{|x-z|^{n-1}}\sqrt{\varphi(z)}\,dz\right)^{2\chi}w(x)\,dx\right)^{1/2\chi}$$

$$\leqslant cR\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}|Du|^2\varphi(x)\,w(x)\,dx\right)^{\frac12}$$

(see [F.K.S.] the proof of Theorem 1.2).

From these it follows

$$\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}|u(x)-\tilde{A}_{B_{x_0}(R)}|^{2\chi}\sqrt{\varphi(x)}^{\,2\chi}w(x)\,dx\right)^{1/2\chi}\leqslant$$

$$\leqslant c\,\frac{|B_{x_0}(R)|}{\varphi(B_{x_0}(R))}\,R\left(\frac{1}{w(B_{x_0}(R))}\int\limits_{B_{x_0}(R)}|Du|^2\varphi(x)\,w(x)\,dx\right)^{\frac12},$$

[2] For the meaning of $\varphi(B(R))$ and $\varphi w(B(R))$ see [1].

so that:

$$\left(\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}(u(x)-\tilde{A}_{B_{x_0}(R)})^2\varphi(x)\,w(x)\,dx\right)^{\frac{1}{2}}\leqslant$$

$$\leqslant c\,\frac{|B_{x_0}(R)|}{\varphi(B_{x_0}(R))}\,R\left(\frac{1}{w(B_{x_0}(R))}\int_{B_{x_0}(R)}|Du|^2\varphi(x)\,w(x)\,dx\right)^{\frac{1}{2}}$$

Finally one can replace $\tilde{A}_{B_{x_0}(R)}$ with $A_{B_{x_0}(R)}$ as in the conclusion of Th. 1.5 of [F.K.S.].

REMARK 1.1. Another property of the $A_2$ weights we will use frequently is the « doubling property » ([Co.F.]). That is

$$\exists d = d(c_0, n) > 0: \quad \forall x_0 \in \boldsymbol{R}^n, \quad \forall r > 0 \quad w(B_{x_0}(2R)) \leqslant d w(B_{x_0}(R)) .$$

## 2. The Harnack inequality.

Let us consider in $Q$ (see n. 1), the degenerate parabolic equation:

$$\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}a_{ij}(x)\,\frac{\partial u}{\partial x_i}\right)-\frac{\partial}{\partial t}\,(w(x)u)=0\,. \tag{2.1}$$

We assume that the coefficients $a_{ij}(x)$ are measurable functions a.e. defined in $Q$ such that

$$\begin{cases} a_{ij}(x) = a_{ji}(x) \qquad i, j = 1, \ldots, n \\ \exists\, \lambda > 0: \dfrac{1}{\lambda}\, w(x)|\xi|^2 \leqslant \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \leqslant \lambda w(x)|\xi|^2, \\ \hfill \text{a.e. in } \Omega,\ \forall \xi \in \boldsymbol{R}^n \end{cases} \tag{2.2}$$

where $w(x)$ is an $A_2$ weight in $\boldsymbol{R}^n$ with $A_2$ constant $c_0$.

DEFINITION 2.1. *We say that* $u(x,t) \in L^2(0, T; H^1(\Omega, w(x)))$ *is a solution of* (2.1) *in* $Q$ *if*:

$$\int_Q\left\{\sum_{i,j=1}^{n}a_{ij}(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial\varphi}{\partial x_j}-w(x)\,u\,\frac{\partial\varphi}{\partial t}\right\}dx\,dt=0 \tag{2.3}$$

$$\forall \varphi \in W, \qquad \varphi(0) = \varphi(T) = 0\,.$$

Before stating our main result we introduce some further symbols.

Let $(x_0, t_0) \in Q$, $\varrho > 0$. We put:

$$D_{x_0,t_0}(\varrho) = \{(x,t) \in Q\colon |t - t_0| < \varrho^2, |x - x_0| < 2\varrho\}$$

$$D^+_{x_0,t_0}(\varrho) = \{(x,t) \in Q\colon t_0 + \tfrac{3}{4}\varrho^2 < t < t_0 + \varrho^2, |x - x_0| < \tfrac{1}{2}\varrho\}$$

$$D^-_{x_0,t_0}(\varrho) = \{(x,t) \in Q\colon t_0 - \tfrac{3}{4}\varrho^2 < t < t_0 - \tfrac{1}{4}\varrho^2, |x - x_0| < \tfrac{1}{2}\varrho\}$$

$$R_{x_0,t_0}(\varrho) = B_{x_0}(\varrho) \times ]t_0 - \varrho^2, t_0 + \varrho^2[;$$

$$R^+_{x_0,t_0}(\varrho) = B_{x_0}(\varrho) \times ]t_0, t_0 + \varrho^2[;$$

$$R^-_{x_0,t_0}(\varrho) = B_{x_0}(\varrho) \times ]t_0 - \varrho^2, t_0[$$

Finally:

$$R(\varrho) = R_{0,0}(\varrho), \qquad R^+(\varrho) = R^+_{0,0}(\varrho), \qquad R^-(\varrho) = R^-_{0,0}(\varrho).$$

THEOREM 2.1 (Harnack inequality). *Let $u(x,t)$ be a positive solution in $D_{x_0,t_0}(\varrho)$ of (2.1). Assume (2.2) holds. Then it exists $\gamma = \gamma(c_0, \lambda, n) > 0$ such that*

$$\sup_{D^-_{x_0,t_0}(\varrho)} u(x,t) \leqslant \gamma \inf_{D^+_{x_0,t_0}(\varrho)} u(x,t) \qquad \forall \varrho\colon D_{x_0,t_0}(\varrho) \subseteq Q. \tag{2.4}$$

REMARK 2.1. Theorem 2.1, and the following lemmas, will be proved only for a particular choise of $D, D^+, D^-$ $(\varrho = 1)$.

To get (2.4) in its full generality it will be enough to perform a change of coordinates of the kind:

$$T\colon y = \varrho x + x_0, \qquad \tau = \varrho^2 t + t_0.$$

Such a transformation takes $D(1)$, $D^+(1)$, $D^-(1)$ into $D(\varrho)$, $D^+(\varrho)$, $D^-(\varrho)$ while the equationn (2.1) is changed in a similar equation for the unknown function $v = u \circ T$, and $w(x)$ is changed in $w \circ T$. Obviously $w \circ T \in A_2$, with the same constant $c_0$. This, recalling that the constant $\gamma$ in the statement of Theorem 2.1 (like the ones in the following Lemmas 2.1, 2.2, 2.3) depends only on $c_0$, $n$ and $\lambda$, implies the validity of Theorem 2.1 in the general situation.

Let us remark in particular that the « doubling property » (see Remark 1.1) holds with the same constant $d$ for all the weights $w \circ T$.

LEMMA 2.1. *Let* $\frac{1}{2}\leqslant\varrho<r\leqslant 1$ *and* $u$ *be a positive solution of* (2.1). *Then it exists a constant* $k_1=k_1(c_0,\lambda,n)>0$ *such that*

$$\operatorname*{ess\,sup}_{R(\varrho)} u(x,t)\leqslant k_1\left(\frac{1}{(r-\varrho)^{n+2}w(B_0(1))}\int_{R(r)}(u(x,t))^p w(x)\,dx\,dt\right)^{1/p} \tag{2.5}$$

$\forall p\in\,]0,2\lambda[$, *and*:

$$\operatorname*{ess\,sup}_{R^-(\varrho)} u(x,t)\leqslant k_1\left(\frac{1}{(r-\varrho)^{n+2}w(B_0(1))}\int_{R^-(r)}(u(x,t))^p w(x)\,dx\,dt\right)^{1/p} \tag{2.5$'$}$$

$\forall p\in\,]-2\lambda,0[$.

LEMMA 2.2. *Let* $u$ *be a positive solution of* (2.1) *in* $D_{0,0}(1)$. *Then there are constants* $k_2=k_2(c_0,n)>0$ *and* $a=a(c_0,n,u)>0$ *such that*

$$\begin{aligned} w\{(x,t)\in R^+(1)\colon \log u(x,t)<-s+a\}&+\\ +\,w\{(x,t)\in R^-(1)\colon \log u(x,t)>s+a\}&\leqslant k_2\frac{\lambda}{s}\,w(B_0(1))\,. \end{aligned} \tag{2.6}$$

LEMMA 2.3. *Let* $\mu$, $k_3$ *and* $\theta\in[\frac{1}{2},1[$ *be some positive constants. Let* $v$ *be a positive function defined in a neighbourhood of* $Q(1)$ (³) *such that*

$$\operatorname*{ess\,sup}_{Q(\varrho)} v\leqslant k_3\left[\frac{1}{(r-\varrho)^{\mu}w(Q(1))}\int_{Q(r)}v^p w(x)\,dx\,dt\right]^{1/p} \tag{2.7}$$

*for all* $\varrho$, $r$ *and* $p$ *such that* $\frac{1}{2}\leqslant\theta\leqslant\varrho<r\leqslant 1$, $0<p<2\lambda$.

*Moreover assume that*

$$w\{(x,t)\in Q(1)\colon \log v>s\}\leqslant k_3\frac{\lambda}{s}\,w(Q(1))\,,\qquad \forall s>0\,. \tag{2.8}$$

*Then it exists a constant* $\gamma=\gamma(\theta,\mu,k_3,\lambda)$ *such that*

$$\sup_{Q(\theta)} v<\gamma\,.$$

Proof of LEMMA 2.1. We will get the conclusion for $\varrho=\frac{1}{2}$, $r=1$ only.

(³) Here $Q(\varrho)$ can be indifferently $R(\varrho)$, $R^+(\varrho)$, $R^-(\varrho)$.

Using standard techniques (see e.g. [L.S.U.], p. 189 having in mind that one can assume $u$ to be locally bounded for the results in [Ch.F.]) we get the inequality:

$$\sup_{]-\varrho^2,\varrho^2[}\int_{B_0(\varrho)} v^2 w\,dx + \int_{R(\varrho)} |Dv|^2 w(x)\,dx\,dt \leqslant \frac{k_4}{r-\varrho}\int_{R(r)} v^2 w(x)\,dx\,dt\,. \tag{2.9}$$

Here: $\frac{1}{2} \leqslant \varrho < r \leqslant 1$, $v = u^{p/2}$, $p \in ]0, +\infty[\setminus\{1\}$, $k_4$ depends on $\lambda$ and $p$. An application of LEMMA 1.2 to (2.9) yields:

$$\left(\frac{1}{\varrho^2 w(R_0(\varrho))}\int_{R(\varrho)} v^{2\overline{\chi}} w(x)\,dx\,dt\right)^{1/2\overline{\chi}} \leqslant k_5\left(\frac{1}{r^2 w(B_0(r))}\int_{R(r)} v^2 w(x)\,dx\,dt\right)^{\frac{1}{2}}$$

where $k_5$ depends on $\lambda$, $p$, $n$, $c_0$, $r$ and $\varrho$.

The conclusion can be achieved as in $[M_2]$, p. 735. Let us only point out that it is possible to control the ratio $w(B_0(1))/w(B_0(2))$ in terms of the constant $d$ in the doubling property, hence in terms of $c_0$ only.

Proof of LEMMA 2.2.

Let:

$$S_h \equiv u_h \equiv \frac{1}{h}\int_t^{t+h} u(x, s)\,ds \quad (h > 0)\,.$$

Then, with standard calculations, it is possible to derive the following inequality valid for all positive solutions in $D(1)$

$$\int_{t_1}^{t_2}\int_{B_0(2)}\left\{\sum_{i,j=1}^{n} S_h\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right)\frac{\partial[\psi^2(x)(u_h)^{-1}]}{\partial x_j} + \frac{\partial S_h u}{\partial t}\frac{1}{S_h u}\psi^2(x)w(x)\right\}dx\,dt = 0\,.$$

Here $\psi(x)$ is a Lipschitz continuous function compactly supported in $B_0(2)$ and: $-1 < t_1 < t_2 < 1$.

We have:

$$\int_{t_1}^{t_2}\int_{B_0(2)}\left\{\sum_{i,j=1}^{n} a_{ij}(x)\,S_h u_{x_i}\,2\psi\psi_{x_j}\frac{1}{S_h u} - \right.$$
$$\left. - \sum_{i,j=1}^{n} a_{ij}(x)\,S_h u_{x_i}\psi^2\frac{1}{(S_h u)^2}\,S_h u_{x_j} + \frac{\partial S_h u}{\partial t}\frac{1}{S_h u}\psi^2(x)w(x)\right\}dx\,dt = 0\,.$$

Let: $v_{(h)} = -\log(S_h u)$. Then:

$$\int_{t_1}^{t_2}\int_{B_0(2)}\Big\{\sum_{i,j=1}^{n} a_{ij}(x)(v_{(h)})_{x_i} 2\psi\psi_{x_j} + \\ + \sum_{i,j=1}^{n} a_{ij}(x)(v_{(h)})_{x_i}(v_{(h)})_{v_j}\psi^2 + \frac{\partial v_{(h)}}{\partial t}\psi^2 w\Big\} dx\, dt = 0 .$$

From this it follows:

$$\frac{1}{\lambda}\int_{t_1}^{t_2}\int_{B_0(2)}\sum_{i=1}^{n}(v_{(h)})^2_{x_i}\psi^2(x)w(x)\,dx\,dt + \Big[\int_{B_0(2)} v_{(h)}\psi^2 w\,dx\Big]_{t=t_1}^{t=t_2} \leqslant \\ \leqslant \varepsilon\int_{t_1}^{t_2}\int_{B_0(2)}\psi^2\sum_{i,j=1}^{n} a_{ij}(v_{(h)})_{x_i}(v_{(h)})_{x_j} dx\,dt + \frac{1}{\varepsilon}\int_{t_1}^{t_2}\int_{B_0(2)}\sum_{i,j=1}^{n} a_{ij}(x)\psi_{x_i}\psi_{x_j} dx\,dt .$$

After a convenient choice of $\varepsilon$, and after taking $\psi$ with convex level sets and such that $0 \leqslant \psi \leqslant 1$, $\psi(1) = 1$ in $B_0(1)$ the last inequality together with LEMMA 1.3 yields

$$\Big[\int_{B_0(2)} v_{(h)}(x, t)\,\psi^2(x)\,w(x)\,dx\Big]_{t=t_1}^{t=t_2} + k_6\int_{D(1)}\big(v_{(h)}(x, t) - V_{(h)}(t)\big)^2\psi^2(x)\,w(x)\,dx\,dt \leqslant \\ \leqslant k_7\Big(\int_{B_0(2)} w(x)\,dx\Big)(t_2 - t_1) .$$

Here $k_6$ and $k_7$ depend on $\lambda$ ($k_6$ on $c_0$ too), and

$$V_{(h)}(t) = \frac{\int_{B_0(2)} v_{(h)}(x, t)\,\psi^2(x)\,w(x)\,dx}{\int_{B_0(2)}\psi^2(x)\,w(x)\,dx} .$$

The last estimate implies

$$\frac{V_{(h)}(t_2) - V_{(h)}(t_1)}{t_2 - t_1} + \frac{k_6}{d\int_{B_0(1)} w(x)\,dx}\cdot\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\int_{B_0(1)}\big(v_{(h)}(x,t) - V_{(h)}(t)\big)^2 w(x)\,dx\,dt \leqslant k_7 d$$

from which, going to the limit for $t_2$ approaching $t_1$, it is possible to get a relation similar to (4.7) in [M$_1$].

From this, one gets (2.6) (with $v_{(h)}$ in the place of $\log u$) following the same argument of [M$_1$].

Finally it is possible to get the conclusion letting $h$ go to zero.

Proof of LEMMA 2.3. It is possible to suppose $\lambda = 1$. We let:

$$\varphi(\varrho) = \sup_{Q(\varrho)} \log v \qquad \text{for } \varrho \in [\theta, 1[$$

We now split $Q(r)$ $(\theta \leqslant \varrho < r \leqslant 1)$ in two subsets:

$$Q_1 = \{(x, t) \in Q(r)\colon \log v > \tfrac{1}{2}\varphi(r)\}$$

and

$$Q_2 = \{(x, t) \in Q(r)\colon \log v \leqslant \tfrac{1}{2}\varphi(r)\}\,.$$

So that

$$\int_{Q(r)} v^p w(x)\, dx\, dt = \int_{Q_1} + \int_{Q_2} \leqslant w(Q(1))\left[\exp[p\,\varphi(r)/2] + \exp[p\varphi(r)]\frac{2k_3}{\varphi(r)}\right]$$

Now a convenient choice of $p$ gives:

$$\int_{Q(r)} v^p w(x)\, dx\, dt \leqslant 2 \exp\left[\frac{p\varphi}{2}\right] w(Q(1))$$

so that:

$$\varphi(\varrho) < \frac{1}{p}\log\left(k_3\,\frac{2\exp[p\varphi/2]}{(r-\varrho)^{\mu}}\right).$$

From now on the proof is the same as the one of LEMMA 3 in [M$_1$].

Proof of THEOREM 2.1. The theorem follows immediately from the LEMMAS 2.1, 2.2, 2.3 (see [M$_1$]).

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