Rendiconti
del
Seminario Matematico
della
Università di Padova

Mimma De Acutis

On the quadratic optimal control problem for Volterra integro-differential equations

Rendiconti del Seminario Matematico della Università di Padova, tome 73 (1985), p. 231-247
<http://www.numdam.org/item?id=RSMUP_1985__73__231_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 73 (1985)

On the Quadratic Optimal Control Problem for Volterra Integro-Differential Equations.

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SUMMARY - This paper is concerned with the optimal control problem for an integro-differential Volterra equation having a quadratic cost functional. In the first part the problem is studied for special kernels and we get an optimal control of feedback type involving also the past history of the solutions. In the second part, using the semigroup theory approach, a similar result is proved in the general scalar case.

0. Introduction.

In this paper we consider the synthesis of the following optimal control problem

(P_1) Minimize the « cost functional »

$$J(u) = \frac{1}{2}\int_0^T \{|u(t)|^2 + |x(t)|^2\}\, dt$$

over all $u \in L^2(0, T)$ and $x \in C(0, T)$ which solves

$$(P_2) \qquad \begin{cases} x'(t) = \int_0^t K(t-s)x(s)\, ds + u(t) \\ x(0) = x_0 \end{cases}$$

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Work partially supported by CNR-GNAFA.

We suppose that K is a scalar Kernel subject to the additional hypotheses which will be stated in the following.

It is well known that in the optimal theory for ordinary differential equations (see for instance Curtain and Pritchard [5]) we get a feedback control of the form

$$u(t) = -H(t)x(t) \tag{1}$$

where H is the solution of a related Riccati equation. In this case we shall give a similar result writing (P_2), in some way, as an ordinary differential equation on suitable functional spaces. It is important to point out that in our case we shall give an expression of type (1) involving also the past hystory of the solution up to the time t.

In the first part of this paper we shall obtain a constructive result by means of some particular kernels. This result could be useful, we hope, in the future to study numerical approximations of the optimal control function.

A more theoretical result will be given in the second section using the techniques of Chen-Grimmer [4]. For general references on the integral equation theory it is possible to see Miller [11], Grimmer-Miller [9]. Da Prato-Iannelli [6]. For the optimal control theory we refer to Curtain-Pritchard [5], Barbu-Da Prato [3], Banks-Manitius [1], Delfour-Mitter [8].

1. Theory with special kernels.

In this section we shall be concerned with kernels of the following particular type

$$a) \qquad k(t) = \sum_{i=0}^{N} a_i t^i \exp[-\alpha_i t]\,; \qquad \alpha_i,\, a_i \in \mathbb{R}$$

$$b) \qquad k(t) = \int_0^1 \exp[-st]\varphi(s)\,ds$$

where φ is a prescribed function.

In order to study the case $a)$ we set [1]

$$z_{ik} = \left(t^{i-k}\exp[-\alpha_i t]\right) * x \quad (k = 0, 1, \ldots, i)\,, \qquad M = (N^2 + 3N + 4)/2\,.$$

Therefore we obtain the following $M \times M$ system

$$\begin{cases} x' = \sum\limits_{i=0}^{N} a_i z_{i,0} \\ z'_{0.0} = x - \alpha_0 z_{0,0} \\ z'_{1,0} = z_{11} - \alpha_1 z_{1,0} \\ z'_{i,k} = (i-k) z_{i,k+1} - \alpha_i z_{i,k} \qquad k < i,\ i = 0, \ldots, N \\ z'_{i,i} = x - \alpha_i \cdot z_{i,i} \end{cases}$$

Hence the problem (P_2) can be studied by means of the finite dimensional problem

$$(P_3) \qquad X'(t) + AX(t) = Bu(t)\,, \qquad X(0) = X_0$$

where A is the following $M \times M$ matrix

$$\begin{bmatrix} 0 & -a_0 & -a_1 & 0 & -a_2 & 0 & 0 & -a_3 & 0 & 0 & 0 & -a_4\cdots \\ -1 & \alpha_0 & & & & & & & & & & \\ 0 & & \alpha_1 & -1 & & & & & & & & \\ -1 & & & \alpha_1 & & & & & & & & \\ 0 & & & & \alpha_2 & -2 & & & & & & \\ 0 & & & & & \alpha_2 & -1 & & & & & \\ -1 & & & & & & \alpha_2 & & & & & \\ 0 & & & & & & & \alpha_3 & -3 & & & \\ 0 & & & & & & & & \alpha_3 & -2 & & \\ 0 & & & & & & & & & \alpha_3 & -1 & \\ -1 & & & & & & & & & & \alpha_3 & \\ \vdots & & & & & & & & & & & \alpha_4 \end{bmatrix}$$

[1] Here $*$ denotes the usual convolution product. That is

$$(f * g)(t) = \int_0^t f(t-s)g(s)\,ds\,.$$

Moreover we set $B\colon \mathbb{R} \to \mathbb{R}^M$

$$By = \begin{bmatrix} y \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \text{for all } y \in \mathbb{R}$$

and

$$X = \begin{bmatrix} x \\ z_{0,0} \\ z_{1,0} \\ z_{1,1} \\ \vdots \\ z_{i,0} \\ z_{i,1} \\ \vdots \\ z_{i,i} \\ z_{i+1,0} \\ \vdots \end{bmatrix}, \quad X_0 = \begin{bmatrix} x_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Bx_0 .$$

Let us consider, now, the new cost functional

$$(P_4) \qquad J_1(u) = \tfrac{1}{2}\int_0^T \{|u(t)|^2 + \langle CX(t), X(t)\rangle\}\, dt$$

where C is an $M \times M$ matrix such that

$$CX = \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{for all } X \in \mathbb{R}^M.$$

We shall study the optimal control problem (P_3), (P_4).

The problem (P_1), (P_2) is completely equivalent to (P_3), (P_4) which is well-known in the literature (see for instance Curtain-Pritchard [5]). We know, there exists a unique selfadjoint matrix H, which is a solution to the following Riccati equation

$$(R_1) \qquad H' - A^*H - HA + C - HBB^*H = 0\,, \quad H(T) = 0$$

such that there exists a feedback optimal control of the form

$$u(t) = -B^* H(t) X(t) .$$

In this way the following result holds.

PROPOSITION (1.1). *Assume the kernel $k(t)$ of the above type a), therefore there exists a unique optimal control of « feedback » type*

$$u(t) = -h_{11}(t)x(t) - \int_0^t h(t, s)x(t-s)\,ds$$

where

$$h(t, s) = \sum_{i,j=0}^{M} h_{1,i(i+1)/2+j+2}(t)\cdot s^{i-j}\cdot \exp[-\alpha_i s]$$

and $\{h_{1,s}\}_{s=1,\dots,M}$ is the first row of matrix $H(t)$ solution to (R_1). That is

$$u(t) = -h_{11}(t)x(t) - \sum_{i,j=0}^{M} h_{1,i(i+1)/2+j+2}(t)\int_0^t k_{i,j}(s)x(t-s)\,ds$$

where

$$k_{i,j}(s) = s^{i-j}\cdot \exp[-\alpha_i s] .$$

REMARK (1.2). This proposition shows that is not possible, in our case, to have an optimal control depending only on $x(t)$, indeed if this is the case we can obtain, via the equivalence between (P_1), (P_2) with (P_3), (P_4), a new matrix $H_1(t)$ which solves (R_1). This fact is impossible by virtue of the uniqueness of the solution to the Riccati equations (see for instance Curtain-Pritchard [5]).

REMARK (1.3). We wish to point out that also in this case the calculation of the optimal control is reduced to solving a Riccati equation.

Let us consider now, a kernel $k(t)$ of the type b) mentioned above, that is

$$k(t) = \int_0^1 \exp[-\tau t]\varphi(\tau)\,d\tau .$$

This kind of kernel have been considered by Da Prato-Iannelli [6].

In order to solve our problem we need the following lemma

LEMMA (1.4). *Let* $\alpha > 0$ *and* $\varphi \in L^{2+\alpha}(0,1)$ *then*

$$\sum_{n=0}^{\infty} (k^{(n)}(0))^2 < +\infty$$

where

$$k^{(n)}(0) = \left.\frac{d^n k}{dt^n}\right|_{t=0}.$$

PROOF. Let $r \in \mathbf{N}$ then

$$\sum_{n=0}^{r} (k^n(0))^2 = \sum_{n=0}^{r} \left(\int_0^1 \tau^n \varphi(\tau)\, d\tau \right)^2$$

therefore

$$\left(\int_0^1 \tau^n \varphi(\tau)\, d\tau \right)^2 \leqslant \left(\int_0^1 \tau^n\, d\tau \right) \cdot \int_0^1 \tau^n \varphi^2(\tau)\, d\tau \leqslant \frac{1}{n+1} \cdot \int_0^1 \tau^n \varphi^2(\tau)\, d\tau$$

since $\varphi^2 \in L^{1+\alpha/2}$ one has

$$\left(\int_0^1 \tau^n \varphi(\tau)\, d\tau \right)^2 \leqslant \frac{1}{n+1} \left(\int_0^1 \tau^{n(\alpha+2)/2}\, d\tau \right)^{\alpha/(\alpha+2)} \cdot \left(\int_0^1 \varphi^{2+\alpha}(\tau)\, d\tau \right)^{2/(\alpha+2)}.$$

Hence

$$\left(\int_0^1 \tau^n \varphi(\tau)\, d\tau \right)^2 \leqslant \frac{1}{1+n} \left(\frac{1}{1+n} \right)^{\alpha/(\alpha+2)} \cdot \|\varphi\|^2_{L^{2+\alpha}} = \frac{1}{(1+n)^{\beta+1}} \cdot \|\varphi\|^2_{L^{2+\alpha}}$$

where

$$\beta = \frac{\alpha}{\alpha+2} > 0 .$$

Hence the lemma is proved. #

For our purposes we set, for all $n \in \mathbb{N}$

$$z_n(t) = \big(k^{(n)} * x\big)(t)$$

then we get

$$z_n'(t) = z_{n+1} + k^{(n)}(0) \cdot x(t) .$$

Substituting into the integral equation we have the following infinite system

$$\begin{cases} x' - z_0 = u \\ z_0' - z_1 - k_{0,0} \cdot x = 0 \\ \vdots \\ z_n' - z_{n+1} - k_{n,0} \cdot x = 0 \\ \vdots \end{cases}$$

where

$$k_{n,0} = k^{(n)}(0)$$

In order to give sense to the above formal calculations we shall use the Hilbert space $l^2(\mathbb{R})$ with the scalar product

$$\langle \{x_n\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} x_n \cdot y_n .$$

Let us define the linear operator $A : l^2(\mathbb{R}) \to l(\mathbb{R})$ in the following way

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & \cdots \\ -k_{0,0} & 0 & -1 & 0 & \cdots & \cdots \\ -k_{1,0} & 0 & 0 & -1 & \cdots & \cdots \\ \vdots & & & & -1 & \\ \vdots & & & & & \end{bmatrix}$$

that is

$$(Ax)_1 = -x_2 , \qquad (Ax)_{n+1} = -k_{n-1,0} \cdot x_1 - x_{n+2} , \qquad n \geqslant 1$$

we get the following lemma.

LEMMA (1.5). *Let $\varphi \in L^{2+\alpha}(0,1)$, $\alpha > 0$, then A is a bounded linear operator on $l^2(\mathbb{R})$.*

PROOF. For all $\{x_n\} \in l^2(\mathbb{R})$ one has

$$\|Ax\|^2_{l^2(\mathbb{R})} = x_2^2 + \sum_{n=1}^{\infty} |k_{n-1,0}x_1 + x_{n+2}|^2 \leqslant$$
$$\leqslant x_2^2 + 2\left\{\sum_{n=1}^{\infty} (k_{n-1,0}\cdot x_1)^2 + \sum_{n=1}^{\infty}(x_{n+2})^2\right\} \leqslant 2\,\|x\|^2_{l^2(\mathbb{R})}\cdot(1+\|k_0\|^2)$$

where

$$\|k_0\|^2 = \sum_{n=1}^{\infty} k_{n,0}^2 . \qquad \#$$

Let us denote by

$$X(t) = \begin{bmatrix} x(t) \\ z_0(t) \\ \vdots \\ z_1(t) \\ z_n(t) \\ \vdots \end{bmatrix}$$

therefore one has the following

COROLLARY (1.6). *Let $\varphi \in L^{2+\alpha}(0,1)$, $\alpha > 0$, then for all $t \geqslant 0$ it follows that*

$$X(t) \in l^2(\mathbb{R}) .$$

PROOF. It follows at once by the variation constant formula being A and B bounded linear operators. $\#$

Let us define the operator $B: \mathbb{R} \to l^2(\mathbb{R})$, such that

$$By = \begin{bmatrix} y \\ 0 \\ \vdots \end{bmatrix}$$

for all $y \in \mathbb{R}$. Therefore the problem (P_1), (P_2) is equivalent to search the optimal control to

$$(P_5) \qquad \begin{cases} X'(t) + AX(t) = Bu(t) \\ X(0) = X_0 = Bx_0 \end{cases}$$

with the new «cost functional»

$$(P_6) \qquad J_2(u) = \tfrac{1}{2}\int_0^T \{|u(t)|^2 + \langle X(t), CX(t)\rangle_{l^2(\mathbb{R})}\}\, dt$$

where $C: l^2(\mathbb{R}) \to l^2(\mathbb{R})$ is defined by

$$C\{x_n\} = \begin{bmatrix} x_1 \\ 0 \\ \vdots \end{bmatrix}$$

for all $\{x_n\} \in l^2(\mathbb{R})$.

Therefore by applying the standard theory (see [5]) we obtain a feed back optimal control

$$u(t) = -B^*Q(t)X(t)$$

where $Q(t)$ is a linear bounded operator for all $t \geqslant 0$ and the mapping $t \to Q(t)$ is the unique solution in the class of selfadjoint linear operator to the related Riccati equation

$$(R_2) \qquad Q' - A^*Q - QA + C - QBB^*Q = 0, \qquad Q(T) = 0\,.$$

Let us denote by $\{l_i\}_{i=1,2,\dots}$ the canonical orthonormal basis for $l^2(\mathbb{R})$, that is $l_i = \{\delta_{ij}\}_{j=1,2,\dots}$

If we set

$$q_{ij}(t) = \langle Q(t)l_j, l_i\rangle_{l^2(\mathbb{R})}$$

then one has

$$u(t) = -q_{11}(t)x(t) - \sum_{j=2}^{\infty} q_{ij}(t)z_{j-2}(t)\,.$$

Therefore we have the following result

PROPOSITION (1.7). *Assume that the integral kernel $k(t)$ is of the above type b) and $\varphi \in L^{2+\alpha}(0,1)$, $\alpha > 0$. therefore the problem (P_1), (P_2) is equivalent to (P_5), (P_6) stated above and there exists a unique optimal* *control*

$$u(t) = -q_{11}(t)x(t) - \sum_{j=2}^{\infty} q_{1j}(t)\left(k^{(j-2)} * x\right)(t)$$

where

$$q_{1j}(t) = \langle Q(t)l_j, l_1\rangle_{l^2(\mathbb{R})} \qquad .$$

$Q(t)$ is the solution to the above Riccati equation (R_2) and the series converges uniformly on $[0, T]$.

PROOF. It remains to prove only the uniform convergence of the above series. Let $r,\ p \in \mathbb{N}$.

Then one has to prove that

$$\lim_{r,p\to\infty} \sup_{t\in[0,T]} \left| \sum_{j=r}^{r+p} q_{1j}(t) z_{j-2}(t) \right| = 0$$

To do this we shall prove that

$$\sum_{j=1}^{\infty} q_{1j}^2(t)$$

converges uniformly on $[0, T]$ and

$$\lim_{r,p\to\infty} \sum_{j=r}^{r+p} \sup_{t\in[0,T]} |z_j(t)|^2 = 0 \ .$$

Hence, we have

$$\sup_t \sum_{j=r}^{r+p} q_{1j}(t) z_{j-2}(t) \leqslant \tfrac{1}{2}\left(\sup_t \sum_{j=r}^{r+p} q_{1j}^2(t) + \sup_t \sum_{j=r}^{r+p} |z_j(t)|^2\right)$$

that proves the uniform convergence.

In order to get the uniform convergence of $\sum_{j=1}^{\infty} q_{1j}^2(t)$ we set

$$D(t) = B^*Q(t)$$

then $D(t) \in [l^2(\mathbb{R})]^* \simeq [l^2(\mathbb{R})]$ for all $t \geqslant 0$. Moreover since for all

$\{y_n\} \in l^2(\mathbb{R})$ one has

$$D(t)\ \{y_n\} = \sum_{j=1}^{\infty} q_{1j}(t) y_j$$

it follows that

$$\|D(t)\|^2_{[l^2(\mathbb{R})]^*} = \|\{q_{1j}(t)\}\|^2_{l^2(\mathbb{R})} = \sum_{j=1}^{\infty} q^2_{1j}(t) .$$

Since the solution to the Riccati equation (R_2) has that property $Q(t) \in \mathfrak{L}(l^2(\mathbb{R}))$ for all $t \geqslant 0$, we get

$$\|D'(t)\|^2 = \sum_{j=1}^{\infty} q'^2_{1j}(t) .$$

We know that the solution to (R_2) has the property that $Q \in C([0, T),$ $\mathfrak{L}_s(l^2(\mathbb{R}))$ and there exists $M > 0$ such that

$$\sup_{t\in[0,T]} \|Q(t)\| \leqslant M$$

therefore

$$\sup_t \sum q^2_{1j}(t) = \sup_t \|D(t)\|^2 \leqslant M$$

From (R_3) we obtain

$$\|D'(t)\| \leqslant \|B^*\|\,\|A^*\|\,\|Q(t)\| + \|B^*\|\,\|A\|\,\|Q(t)\| + \|B^*\| \cdot \|C\| + \\ + \|Q(t)\|^2 \cdot \|B\| \cdot \|B^*\| \leqslant M(1 + M + 2\|A\|) = M_1 .$$

If we consider the sequence of mappings, defined in the following way

$$q_n(t) = \sum_{j=1}^{n} q^2_{1j}(t)$$

it is possible to prove that $\{q_n\}$ is a relatively compact subset of $C([0, T])$ by virtue of the Ascoli-Arzelà theorem.

Indeed we have

$$\sup_t |q_n(t)| \leqslant M , \qquad \sup_t |q'_n(t)| \leqslant M^2 + M_1^2 .$$

Hence we can say there exists a uniformly convergent subsequence, but if one takes into account the pointwise convergence of the complete sequence, it follow the uniform convergence of the serie $\sum_{j=1}^{\infty} q_{1j}^2(t)$ on $[0, T]$.

The other part is easily proved since, using the same arguments of the lemma (1.6), it is possible to see

$$|z_j(t)|^2 \leqslant \text{const}\left(\|\varphi\|_{L^{2+\alpha}} T\right)\cdot \frac{1}{(1+n)^{1+\beta}}$$

where $\beta > 0$. $\#$

2. Abstract theory.

Let us consider the linear Volterra integrodifferential equation

$$\text{(2.1)} \qquad \begin{cases} x'(t) = \int_0^t k(t-s)x(s)\,ds + f(t), & t \geqslant 0 \\ x(0) = x_0 \end{cases}$$

where $k \in H^1(\mathbb{R}_+)$ and $f \in L^2(\mathbb{R}_+)$. We wish to transform (2.1) into an abstract ordinary differential equation on a suitable Hilbert space. Let us denote by $H = \mathbb{R} \times L^2(\mathbb{R}_+)$ and by

$$\delta_0 f = f(0), \qquad D_s f = f'$$

for all $f \in H^1(\mathbb{R}_+)$. Denote by

$$\mathfrak{D}(G) = \mathbb{R} \times H^1(\mathbb{R}_+)$$

and by $G\colon \mathfrak{D}(G) \to \mathbb{R} \times L^2(\mathbb{R}_+)$

$$G = \begin{bmatrix} 0 & \delta_0 \\ k(\cdot) & D_s \end{bmatrix}$$

If we set

$$z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$y(t)(s) = f(t+s) + \int_0^t k(t+s-\tau)x(\tau)\,d\tau$$

the problem (2.1) is equivalent (see Chen-Grimmer [4]) to the problem

$$(2.2)\qquad \begin{cases} Z' = GZ \\ Z(0) = Z_0 = \begin{bmatrix} x_0 \\ f \end{bmatrix} \end{cases}$$

To solve (2.2) we shall use the Lumer-Phillips theorem (see A. Pazy [13]) in a way close to the proof of the theorem 5.1 of [4]

PROPOSITION (2.1). *G is the infinitesimal generator of a C_0-semigroup* $\{S(t)\}$ of bounded linear operator on H.

PROOF. Let us consider the operator

$$G_\alpha = G - \alpha I_H, \qquad \alpha > 0$$

where

$$\mathfrak{D}(G_\alpha) = \mathfrak{D}(G)\,.$$

We shall prove that for $\alpha > \frac{1}{2}(1 + \|k\|_{L^2})$, G_α is a dissipative operator. Assume $z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{D}(G)$ then

$$\langle G_\alpha z, z\rangle_H = (-\alpha x + \delta_0 y)x + \langle k(\cdot)x + (D_s - \alpha I_{L^2})y \cdot y\rangle_{L^2} =$$
$$= -\alpha x^2 - \alpha\|y\|^2_{L^2} + x\cdot\delta_0 y + \langle k(\cdot)x, y\rangle_{L^2} + \langle D_s y, y\rangle_{L^2}\,.$$

One has easily

$$\langle D_s y, y\rangle = -\tfrac{1}{2}\,|\delta_0 y|^2\,,$$

$$\langle G_\alpha z\, z\rangle_H \leqslant -\alpha\|z\|^2_H + \tfrac{1}{2}\|z\|^2_H + \tfrac{1}{2}\|k\|^2_{L^2}\cdot|x|^2 \leqslant -\left(\alpha - \tfrac{1}{2} - \tfrac{1}{2}\|k\|^2_{L^2}\right)\|z\|^2_H\,.$$

Then for $\alpha > \frac{1}{2}\cdot(1 + \|k\|^2_{L^2})$, G_α is, dissipative.

Now consider $(\lambda - G_\alpha)^{-1} = R(\lambda + \alpha, G)$, we need to show that the range of this operator is the whole space H, for some $\lambda > 0$.

Observe that the equation

$$(\lambda - G)\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

has a solution $\begin{bmatrix}\xi\\ \eta\end{bmatrix} \in D(G)$ for all $\begin{bmatrix}x\\ y\end{bmatrix} \in H$ if and only if

$$\lambda \neq \hat{k}(\lambda) .$$

Indeed it is equivalent to solving

$$\begin{cases} \lambda\eta(s) - k(s)\xi - y(s) = \eta'(s) \\ \eta(0) = \lambda\xi - x \end{cases}$$

then by evaluating the Laplace transform of the first equation in λ we obtain

$$\xi = (\lambda - \hat{k}(\lambda))^{-1}\,[x + \hat{y}(\lambda)]$$

Since one has for all $\lambda > 0$

$$|\hat{k}(\lambda)| \leqslant \frac{1}{(2\lambda)^{\frac{1}{2}}}\, \|k\|_{L^2}$$

we obtain for all $\alpha > 0$, there exists $\lambda > 0$ such that

$$\lambda + \alpha \neq \hat{k}(\lambda + \alpha) . \qquad \#$$

REMARK (2.2). We wish to point out that, when a control function is considered we have the equivalence between the problem

$$x'(t) = \int_0^t k(t-s)\, x(s)\, ds + u(t) , \qquad x(0) = x_0$$

and the non-homogeneous equation on H

$$(P_a) \qquad Z'(t) = GZ(t) + Bu(t), \qquad Z(0) = \begin{bmatrix}x_0\\ 0\end{bmatrix}$$

where $B\colon \mathbb{R} \to H$ is defined by $Bx = \begin{bmatrix}x\\ 0\end{bmatrix}$.

Therefore we have that the range of B is contained in $\mathfrak{D}(G)$.

Let us introduce the new « cost functional » (2)

$$(P_a') \qquad J_a(u) = \tfrac{1}{2}\int_0^T \{|u(t)|^2 + \langle Z(t), DZ(t)\rangle_H\}\,dt\,.$$

Therefore to solve the problem (P_1), (P_2) is equivalent to study the problem (P_a), (P_a').

As in the above case, the problem (P_a), (P_a') can be investigated by means of the standard theory on the Riccati equations in Hilbert spaces. Hence we obtained the following

THEOREM (2.2). *Under the above assumptions there exists an «optimal control» u expressed by the «feedback» type formula*

$$u(t) = -B^* H(t) Z(t)$$

where H is the solution to the Riccati equation

$$(R_a) \qquad H' + G^*H + HG + D - HBB^*H = 0, \qquad H(T) = 0\,.$$

Let us define the following linear operator

$$P\colon L^2(0, T) \to \mathbb{R}\times L^2(0, T)$$

such that

$$Py = \begin{bmatrix} 0 \\ y \end{bmatrix}$$

and by

$$H_{11}(t) = B^* H(t) P, \qquad H_{12}(t) = B^* H(t) P$$

$$H_{21}(t) = H_{12}^*(t) = P^* H(t) B, \qquad H_{22}(t) = P^* H(t) P\,.$$

Hence

$$H(t) = \begin{bmatrix} H_{11}(t) & H_{12}(t) \\ H_{21}(t) & H_{22}(t) \end{bmatrix}.$$

(2) $D\colon H \to H$ is defined by

$$DZ = \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \text{for all } z = \begin{bmatrix} x \\ y \end{bmatrix} \in H\,.$$

Observe that $H_{12}(t)$, for fixed t, is an element of $L^2(0, T)$ and the boundedness of $H(t)$ as a map from $[0, T]$ to $\mathfrak{L}(H)$ implies that

$$\sup_t \|H_{12}(t)\|_{L^2(0,T)}$$

is bounded. Moreover since $t \to H(t)$ is continuous in the strong operator topology one has for all $y \in L^2(0, T)$ the continuity of the map

$$t \to \langle H_{12}(t), y \rangle_{L^2} .$$

Hence we get

$$u(t) = -H_{11}(t)x(t) - \langle H_{12}(t), y \rangle_{L^2} .$$

From the definition of $y(t)$ one has

$$u(t) = -H_{11}(t)x(t) - \int_0^T H_{12}(t, s) \int_0^t k(t + s - \tau)x(\tau)\, d\tau\, ds .$$

Now it follows that for all t, τ the integral

$$\int_0^T H_{12}(t, s)k(t + s - \tau)\, ds = \bar{H}(t, \tau)$$

converges, since both the factors are L^2-functions and it is continuous separately in t and τ. So Fubini's theorem can be applied.

Therefore one has

THEOREM (2.3). *Under the above hypotheses, the optimal control of* (P_1) *and* (P_2) *is given by*

$$u(t) = -H_{11}(t)x(t) - \int_0^t \bar{H}(t, \tau)x(\tau)\, d\tau$$

where $\bar{H}(t, \tau)$ is given by the above formula.

REMARK (2.4). The remark (1.2) concerning the uniqueness property can be extended to this case, also.

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Manoscritto pervenuto in redazione il 10 maggio 1984.