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$\mathcal{F}$-Constraint of the Automorphism Group of a Finite Group.

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If $\mathcal{F}$ is a homomorph and we denote $\mathcal{F}' = \{G | S^{\mathcal{F}} = S \ \forall S \leqslant G\}$, we say that a group G is $\mathcal{F}$-constrained when there is a maximal normal $\mathcal{F}$-subgroup $\bar{M}$ of $\bar{G} = G/G_{\mathcal{F}'}$, such that $C_{\bar{G}}(\bar{M}) \leqslant \bar{M}$.

In ([7]), *R.* Laue proves that if G has no direct abelian factors and $C_G(F(G)) \leqslant F(G)$ (i.e. G is Nilpotent-constrained) then $C_{\mathrm{Aut}\, G}$ $(F(\mathrm{Aut}\, G)) \leqslant F(\mathrm{Aut}\, G)$.

In ([10]) it is proved for a saturated Fitting formation and a group G verifying that:

i) $G_{\mathcal{F}'} \leqslant \Phi(G)$,

ii) $G/G_{\mathcal{F}'}$ has no direct abelian factors,

iii) G is $\mathcal{F}$-constrained,

that $\mathrm{Aut}\, G$ is $\mathcal{F}$-constrained.

The purpose of this paper is mainly to prove the above result when $\mathcal{F}$ is a homomorph closed for direct products (D_0-closed) and normal subgroups that is: i) saturated or ii) closed for central extensions.

All group considered are finite.

If $\mathcal{F}$ is a homomorph then the class $\mathcal{F}'$ is a s-closed (i.e. closed for subgroups)extensible Fitting formation and a group G is said $\mathcal{F}$-sepa-

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rable if it possesses a normal series:

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \dots \trianglerighteq G_r = 1$$

whose factor groups G_i/G_{i+1} are either $\mathcal{F}$-groups or $\mathcal{F}'$-groups.

If we do not state the contrary here on we shall suppose that $\mathcal{F}$ is a n-closed (i.e. closed for normal subgroups) homomorph.

The class of $\mathcal{F}$-separable groups is an extensible Fitting formation that contains the solvable groups.

The product of all normal semisimple subgroups of a group G is again a semisimple normal subgroup of G, and it is denoted by $L(G)$. It is called the semisimple radical of G([6]).

We use the concepts of semisimple and perfect-quasisimple groups given by Gorenstein and Walter [6].

LEMMA 1 (s. [5] p. 127 or [9] p. III-32). *For every group G, we have*:

$$C_G(F(G)L(G)) \leqslant F(G) .$$

From the definition and properties of class $\mathcal{F}'$ it follows that G is $\mathcal{F}$-constrained if and only if G/N is $\mathcal{F}$-constrained, when $N \leqslant G_{\mathcal{F}'}$.

LEMMA 2. *If $\mathcal{F}$ is a saturated Fitting formation, the following are equivalent.* 1) *G is $\mathcal{F}$-constrained,* 2) $L(\bar{G}) \in \mathcal{F}$([8]).

Our remainder notation is standard and it is based on Huppert's book ([4]).

In the following we shall say that $\mathcal{F}$ verifies:

A) If $\{Q, E_\Phi, D_0, S_n\}\mathcal{F} = \mathcal{F}$;

B) If $\{Q, E_z, D_0, S_n\}\mathcal{F} = \mathcal{F}$, where $E_z\mathcal{F} = \{G|N;\ N \leqslant Z(G),\ G/N \in \mathcal{F}\}$.

As a consequence of the following proposition we obtain that the $\mathcal{F}$-constraint is equivalent to the constraint with respect to a suitable saturated Fitting formation.

PROPOSITION 3. *If $\bar{G} = G/G_{\mathcal{F}'}$ and if $\mathcal{F}$ verifies A or B, the following are equivalent*:

1) *G is $\mathcal{F}$-constrained,*

2) *$\bar{G}$ is ($\mathcal{F}$-separable)-constrained,*

3) $L(\bar{G}) \varepsilon \mathcal{F}$.

PROOF. 1) $\Rightarrow$ 2) Obvious.

2) $\Rightarrow$ 3). By lemma 2, $L(\bar{G})$ is $\mathcal{F}$-separable, hence $L(\bar{G})/Z(L(\bar{G}))$ is $\mathcal{F}$-separable and direct product of non-abelian simple groups ([6]). Moreover, since $\mathcal{F}'$ is saturated $\big(L(\bar{G})/Z(L(\bar{G}))\big)_{\mathcal{F}'} = 1$, thus $L(\bar{G})/Z(L(\bar{G})) \in \mathcal{F}$, but since $Z(L(\bar{G})) = \Phi(L(\bar{G}))$, $L(\bar{G}) \in \mathcal{F}$. 3) $\Rightarrow$ 1). Since $\bar{G}_{\mathcal{F}'} = 1$, both $L(\bar{G})$ and $F(\bar{G})$ are normal $\mathcal{F}$-subgroups of $\bar{G}$ and since $[L(\bar{G}), F(\bar{G})] = 1$, then $L(\bar{G})F(\bar{G})$ is a normal $\mathcal{F}$-subgroup of $\bar{G}$. Let $\bar{M}$ be a maximal normal $\mathcal{F}$-subgroup of $\bar{G}$ containing $L(\bar{G})F(\bar{G})$, by lemma 1 it follows:

$$C_{\bar{G}}(\bar{M}) \leqslant C_{\bar{G}}(L(\bar{G})F(\bar{G})) \leqslant F(\bar{G}) \leqslant \bar{M}$$

COROLLARY. *If G is $\mathcal{F}$-constrained group then $C_{\bar{G}}(\bar{M}) \leqslant \bar{M}$ for all maximal normal $\mathcal{F}$-subgroups of $\bar{G}$.*

PROOF. By proposition 3, $L(\bar{G}) \in \mathcal{F}$. Since $L(C_{\bar{G}}(\bar{M})) \trianglelefteq L(\bar{G})$ then $L(C_{\bar{G}}(\bar{M})) \in \mathcal{F}$ hence $L(C_{\bar{G}}(\bar{M})) \leqslant \bar{M}$ by the maximality of $\bar{M}$ and so $L(C_{\bar{G}}(\bar{M})) = 1$, thus $C_{\bar{G}}(\bar{M})$ is a $\mathcal{N}$-constrained group. On the other hand $F(C_{\bar{G}}(\bar{M})) \in \mathcal{F}$ hence $F(C_{\bar{G}}(\bar{M})) \leqslant \bar{M}$.

Clearly:

$$C_{\bar{G}}(\bar{M}) \leqslant C_{\bar{G}}\big(F(C_{\bar{G}}(\bar{M}))\big) \cap C_{\bar{G}}(\bar{M}) = C_{C_{\bar{G}}(\bar{M})}\big(F(C_{\bar{G}}(\bar{M}))\big) \leqslant F(C_{\bar{G}}(\bar{M})) \leqslant \bar{M}$$

REMARKS. The following conditions, are not equivalent: i) $\bar{G}$ is ($\mathcal{F}$-separable)-constrained and ii) G is ($\mathcal{F}$-separable)-constrained. In fact, in [6] it is proved that $G = C_X(\tau)$, where $X = SL(4, 2^a)$, $a > 1$, and τ is the central involution $I_4 + xE_{14}$, $x \neq 0$, is 2-constrained, hence G is (2-separable)-constrained or equivalently (2′-separable)-constrained, however G is not 2′-constrained. Thus G is (2′-separable)-constrained but $G/O_2(G)$ is not (2′-separable)-constrained.

If $\mathcal{F}$ is a saturated Fitting formation it is known that:

i) The class of the $\mathcal{F}$-constrained groups is a Fitting class that contains the solvable groups ([8])

ii) G is $\mathcal{F}$-constrained if and only if Inn G is $\mathcal{F}$-constrained ([10]),

iii) The class of the $\mathcal{F}$-constrained groups is extensible ([10]).
As a consequence of the above proposition, the properties i), ii), iii) are still valid when $\mathcal{F}$ verifies A or B.

LEMMA 4. *Let $\mathfrak{F}$ be a homomorph that verifies A or B, then if G is a group without direct abelian factors and $G_{\mathfrak{F}'} = 1$, it follows* $(\mathrm{Aut}\, G)_{\mathfrak{F}'} = 1$.

PROOF. Let H be a normal $\mathfrak{F}'$-subgroup of $\mathrm{Aut}(G)$, then $H \cap \mathrm{Inn}(G)$ is a normal $\mathfrak{F}'$-subgroup of $\mathrm{Inn}(G)$; as $(G/Z(G))_{\mathfrak{F}'} = 1$, consequently $H \cap \mathrm{Inn}\, G = 1$, thus $[H, \mathrm{Inn}\, G] = 1$ and so $H \leqslant C_{\mathrm{Aut}\, G}(\mathrm{Inn}\, G) \leqslant F(\mathrm{Aut}\, G)$ because G has no direct abelian factors ([7]). Set $\pi = \mathrm{car}\, \mathfrak{F} = \{p | C_p \in \mathfrak{F}\}$, since $F(G)$ is nilpotent and $G_{\mathfrak{F}'} = 1$ it follows that $F(G)$ is a π-group and thus $F(\mathrm{Aut}\, G)$ is a π-group ([7]). Since $H \in \mathfrak{F}'$, H is a π'-group and so $H = 1$.

COROLLARY 1. *Let $\mathfrak{F}$ be a homomorph that verifies A or B, and G a group such that $G/G_{\mathfrak{F}'}$ has no direct abelian factors, then if G is $\mathfrak{F}$-constrained,* $\mathrm{Aut}\,(G/G_{\mathfrak{F}'})$ *is $\mathfrak{F}$-constrained.*

PROOF. As G is $\mathfrak{F}$-constrained group, so $G/G_{\mathfrak{F}'}$ is $\mathfrak{F}$-constrained. By proposition 3, $G/G_{\mathfrak{F}'}$ is ($\mathfrak{F}$-separable)-constrained. We can apply (2.6) of [10] to obtain that $\mathrm{Aut}\,(G/G_{\mathfrak{F}'})$ is ($\mathfrak{F}$-separable)-constrained. By lemma 4 $(\mathrm{Aut}\,(G/G_{\mathfrak{F}'}))_{\mathfrak{F}'} = 1$, hence by proposition 3, $\mathrm{Aut}\,(G/G_{\mathfrak{F}'})$ is $\mathfrak{F}$-constrained.

COROLLARY 2. *Let $\mathfrak{F}$ be a homomorph verifying B, and G a group whithout direct abelian factors. If G is $\mathfrak{F}$-constrained, then* $\mathrm{Aut}\, G$ *is $\mathfrak{F}$-constrained.*

PROOF. If $\mathfrak{F}$ verifies B), then $\mathfrak{F}' = 1$ now we can apply corollary 1, to obtain the result.

However, in the case of saturated homomorph, it does not appear that the constraint of $\mathrm{Aut}\, G$ can be obtained as an easy consequence of the constraint of $\mathrm{Aut}\,(G/G_{\mathfrak{F}'})$.

THEOREM 5. *Let $\mathfrak{F}$ be a homomorph verifying A, and G a group that verifies:*

i) $G_{\mathfrak{F}'} \leqslant \Phi(G)$

ii) *$G/G_{\mathfrak{F}'}$ has no direct abelian factors*

iii) *G is $\mathfrak{F}$-constrained.*

Then $\mathrm{Aut}\, G$ *is $\mathfrak{F}$-constrained.*

PROOF. We denote $\overline{\text{Aut}\, G} = \text{Aut}\, G/(\text{Aut}\, G)_{\mathcal{F}'}$. We have:

$$\begin{aligned}\overline{\text{Inn}\, G} &= \text{Inn}\, G(\text{Aut}\, G)_{\mathcal{F}'}/(\text{Aut}\, G)_{\mathcal{F}'}\\ &\simeq \text{Inn}\, G/\text{Inn}\, G \cap (\text{Aut}\, G)_{\mathcal{F}'} = \text{Inn}\, G/(\text{Inn}\, G)_{\mathcal{F}'} \simeq\\ &\simeq (G/Z(G))/(G/Z(G))_{\mathcal{F}'} = (G/Z(G))/(G_{\mathcal{F}'}Z(G))/Z(G) \simeq\\ &\simeq G/G_{\mathcal{F}'}Z(G)\,.\end{aligned}$$

Since G is $\mathcal{F}$-constrained we know that $\text{Inn}(G)$ is $\mathcal{F}$-constrained and so $\overline{\text{Inn}\, G}$ is $\mathcal{F}$-constrained too. By proposition 3, $L(\overline{\text{Inn}\, G})$ is a characteristic $\mathcal{F}$-subgroup of $\overline{\text{Inn}\, G}$. Since $(\overline{\text{Inn}\, G})_{\mathcal{F}'} = 1$, consequently $F(\overline{\text{Inn}\, G})$ is also a characteristic $\mathcal{F}$-subgroup of $\overline{\text{Inn}\, G}$. Since $[L(\overline{\text{Inn}\, G}), F(\overline{\text{Inn}\, G})] = 1$ and $\mathcal{F}$ is a D_0-closed homomorph, hence $L(\overline{\text{Inn}\, G})$ $F(\overline{\text{Inn}\, G})$ is a normal $\mathcal{F}$-subgroup of $\overline{\text{Aut}\, G}$. Let $\bar{M}$ be a maximal normal $\mathcal{F}$-subgroup of $\overline{\text{Aut}\, G}$ that contains $L(\overline{\text{Inn}\, G})F(\overline{\text{Inn}\, G})$. Set $\bar{L} = C_{\overline{\text{Aut}\, G}}(\bar{M})$, since $\bar{L}$ centralizes $\bar{M}$, $\bar{L}$ will centralize $L(\overline{\text{Inn}\, G})$ $F(\overline{\text{Inn}\, G})$. Let L be the subgroup of $\text{Aut}\, G$ such that $\bar{L} = L/(\text{Aut}\, G)_{\mathcal{F}'}$, and let H be the normal subgroup of G such that $L(\overline{\text{Inn}\, G})F(\overline{\text{Inn}\, G}) \simeq$ $\simeq H/G_{\mathcal{F}'}Z(G)$. Then L centralizes $H/G_{\mathcal{F}'}Z(G)$ and hence L centralizes $G/C_G(H/G_{\mathcal{F}'}\, Z(G))$, too. Now since $H/G_{\mathcal{F}'}\, Z(G) = L(G/G_{\mathcal{F}'}\, Z(G))$ $F(G/G_{\mathcal{F}'}Z(G))$, by lemma 1 consequently:

$$C_{G/G_{\mathcal{F}'}Z(G)}(H/G_{\mathcal{F}'}Z(G)) \leqslant H/G_{\mathcal{F}'}Z(G)\,,$$

hence $C_G(H/G_{\mathcal{F}'}Z(G)) \leqslant H$, and so L centralizes G/H, therefore L stabilizes the series:

$$G/G_{\mathcal{F}'}Z(G) \geqslant H/G_{\mathcal{F}'}Z(G) \geqslant 1\,,$$

thus $L/C_L(G/G_{\mathcal{F}'}Z(G))$ is nilpotent by a well-known P. Hall's result. We consider now $C_{\text{Aut}\, G}(G/G_{\mathcal{F}'}Z(G))$. This group induces an automorphism group of $G/G_{\mathcal{F}'}$ which is isomorphic to:

$$C_{\text{Aut}\, G}(G/G_{\mathcal{F}'}Z(G))/C_{\text{Aut}\, G}(G/G_{\mathcal{F}'}) \tag{1}$$

By ii) $G/G_{\mathcal{F}'}$ has no direct abelian factors, hence we know ([7]) that:

$$C_{\text{Aut}\,(G/G_{\mathcal{F}'})}(G/G_{\mathcal{F}'}/Z(G/G_{\mathcal{F}'}))$$

is nilpotent and so:

$$C_{\mathrm{Aut}\,(G/G_{\mathcal{F}'})}(G/G_{\mathcal{F}'}/Z(G)\,G_{\mathcal{F}'}/G_{\mathcal{F}'}) \tag{2}$$

is nilpotent, too.

Since the subgroup of $\mathrm{Aut}(G/G_{\mathcal{F}'})$ isomorphic to (1) is contained in (2), we deduce that (1) is nilpotent. Now, since $C_{\mathrm{Aut}\,G}(G/\Phi(G))$ is nilpotent ([11]) and by i) $C_{\mathrm{Aut}\,G}(G/G_{\mathcal{F}'})$ is nilpotent too. Consequently $C_{\mathrm{Aut}\,G}(G/G_{\mathcal{F}'}Z(G))$ is solvable and so L and hence $\bar{L}$ are solvable. Now $F(\bar{L}) \in \mathcal{F}$, hence $F(\bar{L})\bar{M} \in \mathcal{F}$, as $\bar{M}$ is a maximal normal $\mathcal{F}$-subgroup of $\overline{\mathrm{Aut}\,G}$, then $F(\bar{L}) \leqslant \bar{M}$, therefore:

$$\bar{L} = C_{\overline{\mathrm{Aut}\,G}}(\bar{M}) \leqslant C_{\overline{\mathrm{Aut}\,G}}(F(\bar{L})) \cap \bar{L} = C_{\bar{L}}(F(\bar{L})) \leqslant F(\bar{L}) \leqslant \bar{M}\,.$$

Next, we give some counterexamples which prove that the conditions imposed in theorem 5, are not superfluous.

1) The assumption $G_{\mathcal{F}'} \leqslant \Phi(G)$ is necessary.

It is enough to take $G = C_3 \times C_3 \times C_3 \times Q$ where Q is the quaternion group of order 8. The $\mathrm{Aut}\,G = GL(3,3) \times S_4$. If $\mathcal{F}$ is the class of 2-groups, then $G_{\mathcal{F}'} = C_3 \times C_3 \times C_3$ is not a subgroup of $\Phi(G) = Z(Q)$. Make note that G is 2-constrained because it is solvable, now, if $\mathrm{Aut}\,G$ is 2-constrained, $GL(3,3)$ would be also 2-constrained and hence his normal subgroup $SL(3,3)$ would be also 2-constrained, but $SL(3,3)$ is a simple group that is neither a 2-group nor a $2'$-group and therefore it is not 2-constrained group.

2) The condition that $G/G_{\mathcal{F}'}$ has no direct abelian factors is equally necessary. We can consider $G = C_3 \times C_3 \times C_3$ and as $\mathcal{F}$ the class of nilpotent groups, then G is $\mathcal{N}$-constrained but $\mathrm{Aut}\,G$ is not $\mathcal{N}$-constrained, because $GL(3,3)$ is not $\mathcal{N}$-constrained by a similar argument, to the above one.

Finally, we give some examples of classes verifying A or B that are not saturated Fitting formations

a) The class of the supersolvable groups is a formation that verifies A and B but it is not a Fitting class.

b) Let $\mathcal{F} = \mathcal{N}\mathcal{A} \cap \mathfrak{S}_\pi$ be, where $\mathcal{A}$ is the class of abelian groups and $\mathfrak{S}_\pi$ is the class of π-groups, π is a set of prime numbers. This class is a n-closed saturated formation but it is neither a Fitting class nor

E_z-closed. Of course if we take $\pi = \{2\}$ and $G = C_5 \times Q$ where Q is the quaternion group of order 8, then $G/Z(G) \simeq C_2 \times C_2 \in \mathcal{F}$ but $G \notin \mathcal{F}$. On the other hand if we consider:

$$\pi = \{2, 5\}, \qquad G = [C_5 \times C_5]\cdot Q_8,$$

$$C = [C_5 \times C_5]\langle a\rangle \quad \text{and} \quad D = [C_5 \times C_5]\langle b\rangle$$

where $Q = \langle a, b | a^4 = 1 \;\; a^2 = b^2,\; b^{-1}ab = a^{-1}\rangle$ then C, $D \in \mathcal{F}$ but $G \notin \mathcal{F}$ because G' is not nilpotent.

c) The class of solvable groups with absolute arithmetic 3-rank a $3'$-number is a not saturated Fitting formation [3]; but it is an E_z-closed class, because the 3-chief factors under $Z(G)$ have order 3.

d) Following Cossey, we consider inside solvable groups, the class $\mathfrak{X} = \mathcal{S}_p \mathcal{S}_{p'} \mathcal{S}_p$ where $\mathcal{S}_{p'}$, $\mathcal{S}_p$ are the class of p'-groups and p-groups respectively. We denote by $c_p(G)$ the least common multiple of the absolute degrees of the complemented p-chief factors. Thus the class:

$$\mathfrak{Y} = \{G \in \mathfrak{X} | c_p(G) \text{ is coprime to } p\}$$

is both a Fitting class and a Schunck class but it is not a formation [1]. This class verifies A and also satisfies B by a similar argument to that of *c*).

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