RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

GIORGIO BUSETTO
FEDERICO MENEGAZZO

Groups with soluble factor groups and projectivities

Rendiconti del Seminario Matematico della Università di Padova, tome 73 (1985), p. 249-260
<http://www.numdam.org/item?id=RSMUP_1985__73__249_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 73 (1985)

Groups with Soluble Factor Groups and Projectivities.

GIORGIO BUSETTO - FEDERICO MENEGAZZO

1. Introduction.

In [1] (Problem 40) the following question was asked: if G is a soluble group and π is a projectivity from G to some group $\bar{G}$ (i.e. π is a lattice isomorphism from the lattice $L(G)$ of subgroups of G to the lattice $L(\bar{G})$ of subgroups of $(\bar{G})$, is $\bar{G}$ soluble as well? In the finite case the answer was obtained by Suzuki ([7], Theorem 12) and Zappa ([11]). The general answer was given by Yakovlev ([9]), who also gave a bound for the derived length of $\bar{G}$ in terms of the one of G (namely $4n^3 + 14n^2 - 8n$ if n is the derived length of G). In the present paper we deal with the following more general question: if $N \lhd G$, G/N is soluble and $\pi: G \to \bar{G}$ is a projectivity, does some term of the derived series of $\bar{G}$ lie inside $\bar{N}$ $(= N^\pi)$? We prove that, if n is the derived length of G/N, then $\bar{G}^{(6n)}$ (the $6n$-th term of the derived series of $\bar{G}$) lies inside $\bar{N}$ (see Theorem 4.1). Our main tool is a detailed study of the structure, and particularly of the derived length, of finite p-groups of the form $G = H\langle a \rangle$, where $H \lhd G$, having a projective image $\bar{G}$ such that the image of H is core-free in $\bar{G}$ (we shall refer to this situation as to the « reduced case »). As a corollary of Theorem 4.1 we are able to improve Yakovlev's bound to $6n$-4 (see Corollary 4.2).

(*) Indirizzo degli AA.: Istituto di Algebra e Geometria, Università di Padova, Via Belzoni 7, 35100 Padova.

The authors wish to thank C.N.R. and M.P.I. for the partial financial support.

2. Preliminary results.

Our notation will be standard. For lattice theoretical definitions see [8].

LEMMA 2.1. *Let G be a finite modular p-group of exponent p^r, where p is a prime. If G is not Hamiltonian and $G/\Omega_{r-1}(G)$ is not cyclic, then G contains a characteristic abelian subgroup A such that G/A is cyclic and every automorphism of G induces the identity on G/A.*

PROOF. Assume that G is not abelian. By Iwasawa's theorem on the structure of finite modular p-groups ([8], ch. 1, Th. 18), $G = N\langle t\rangle$ where N is abelian and t induces on N some power $1+p^\lambda$, say, with $p^\lambda > 2$. By hypothesis N has exponent p^r. Set $A = C_G(N)$; A is abelian, G/A is cyclic and $C_G(A) = A$. We now distingush two cases.

(i) *$N/\Omega_{r-1}(N)$ is not cyclic.*

Let $a = xt^i \in A$, where $x \in N$ and let $\alpha \in \operatorname{Aut} G$. We shall show that $a^\alpha \in A$. Since $t^i \in Z(G) \leqslant A$, $a^\alpha \in A$ if and only if $x^\alpha \in A$. As $\mho_{r-1}(N)$, being isomorphic to $N/\Omega_{r-1}(N)$, is not cyclic, there exists $u \in N$ of order p^r such that $\langle u\rangle \cap \langle x^\alpha\rangle = 1$. $\langle u\rangle$ and $\langle x^\alpha\rangle$ are both normal subgroups of G, therefore they commute: it follows that the power automorphism induced by x^α on N is the identity. Thus $x^\alpha \in A$ and A is characteristic. Moreover, t^α induces on N^α the power $1+p^\lambda$ and it induces a power automorphism on N as well. These power automorphisms coincide on $N \cap N^\alpha$, which has exponent p^r (otherwise $N/\Omega_{r-1}(N)$ would be a quotient of the cyclic group $N/N \cap N^\alpha$), and therefore t^α induces on N the power $1+p^\lambda$. Thus $t^{-1}t^\alpha \in A$, as required.

(ii) *$N/\Omega_{r-1}(N)$ is cyclic.*

In this case t must have order p^r. Moreover, since N has exponent p^r and $\langle t\rangle \cap A = C_{\langle t\rangle}(N)$, it follows that $\langle t\rangle \cap A = \langle t^{p^{r-\lambda}}\rangle$ and $A = N\langle t^{p^{r-\lambda}}\rangle$. Then, let $xt^{ip^\mu} \in A$ of order p^r, where $x \in N$ and $\mu \geqslant r-\lambda$. Since $(xt^{ip^\mu})^{p^\lambda} = x^{p^\lambda}$ we have

$$(xt^{ip^\mu})^t = xt^{ip^\mu}x^{p^\lambda} = (xt^{ip^\mu})^{1+p^\lambda}.$$

Therefore t operates on A as the power $1+p^\lambda$. Hence, recalling that the group of power automorphisms of an abelian group lies in the centre of the whole automorphism group, in order to complete case (ii) it is sufficient to prove that A is characteristic in G. In order to show this we shall prove that A coincides with the subgroup B of G generated by the cyclic normal subgroups of G of order p^r. Clearly $A \leqslant B$. Conversely, let $\langle b\rangle$ be a cyclic normal subgroup of G of order p^r. We can write $b = t^{p^\nu}y$, where $y \in N$ and $\nu \geqslant 0$. Assume $\nu = 0$. Then $G = N\langle b\rangle$ and so, using the well-known fact that «raising to the power p^{r-1}» is an endomorphism of G, since $G/\Omega_{r-1}(G)$ is not cyclic whereas $N/\Omega_{r-1}(N)$ is cyclic, it follows that there exists $h \in N$ of order p^r such that $\langle b\rangle \cap \langle h\rangle = 1$. Thus $[b, h] = 1$ and this forces b to centralize N, contradicting our assumption that G is not abelian. Hence $\nu \geqslant 1$ and so $|y| = p^r$. Since $\langle b\rangle \lhd G$, we have

$$[b, t] = [t^{p^\nu}y, t] = [y, t] = y^{p^\lambda} \in \langle b\rangle$$

and, moreover,

$$\langle y^{p^\lambda}\rangle = \langle b^{p^\lambda}\rangle = \langle (t^{p^\nu}y)^{p^\lambda}\rangle = \langle t^{p^{\nu+\lambda}}y^k\rangle\,,$$

for some integer k. It follows that $t^{p^{\nu+\lambda}} \in \langle t\rangle \cap \langle y\rangle = 1$. Therefore $\nu \geqslant r - \lambda$ and so $b \in N\langle t^{p^{r-\lambda}}\rangle = A$, as required. □

In the following lemma we collect, for the convenience of the reader, some basic facts occurring in projectivities of some particular finite p-groups (what we call the «reduced case»). Similar results can be found in [3]. All these facts are easy consequences of results by F. Gross ([4], Lemma 3.1). Also, it seems useful to introduce a special notation. If $\pi\colon G \to \bar{G}$ is a projectivity, and H is a subgroup of G, we will set $\bar{H} = H^\pi$; and, if $H \leqslant K \leqslant G$, we often set $H^{\bar{K}} = (\bar{H}^{\bar{K}})^{\pi^{-1}}$ and $H_{\bar{K}} = (\bar{H}_{\bar{K}})^{\pi^{-1}}$.

LEMMA 2.2. *Let $G = H\langle a\rangle$ and $\bar{G}$ be finite p-groups, where $1 \neq H \lhd G$, and let $\pi\colon G \to \bar{G}$ be a projectivity such that $\bar{H}$ is core-free in $\bar{G}$. Then the following hold:*

(i) *$\langle a\rangle \cap H = 1$ and $\Omega_i(G) = \Omega_i(H)\,\Omega_i(\langle a\rangle)$ for every $i \geqslant 0$.*

(ii) *$\Omega_1(G)$ and $\Omega_1(\bar{G})$ are elementary abelian and there exist bases $\{e_0, e_1, \dots, e_m\}$, $\{f_0, f_1, \dots, f_m\}$ of $\Omega_1(G)$ and $\Omega_1(\bar{G})$ respectively, such that $\{e_1, \dots, e_m\} \subset H$, $e_0 \in \langle a\rangle$, $\overline{\langle e_i\rangle} = \langle f_i\rangle$, $e_1^a = e_1$, $e_i^a = e_i\, e_{i-1}$ for $i \geqslant 2$.*

(iii) π *induces a projectivity from* $G/\Omega_i(G)$ *to* $\bar{G}/\Omega_i(\bar{G})$ *and* $\bar{H}\Omega_i(\bar{G})/\Omega_i(\bar{G})$ *is core-free in* $\bar{G}/\Omega_i(\bar{G})$ *for* $i \geqslant 0$.

(iv) *The* (p^{i-1})*th power determines an endomorphism of* $\Omega_i(G)$ *and of* $\Omega_i(\bar{G})$, *for every* $i \geqslant 1$.

(v) $\langle e_0 \rangle \leqslant Z(G)$.

(vi) *If* $p = 2$ *and* H *has exponent* 2^r, *then* $|G/\Omega_r(G)| \geqslant 4$.

PROOF. The arguments are the same as those in [3], Lemma 1. □

THEOREM 2.3. *Let* G *and* $\bar{G}$ *be finite* p*-groups,* $\pi\colon G \to \bar{G}$ *a projectivity,* H *a normal abelian subgroup of exponent* p^r *such that* $G = H\langle a \rangle$ *and* $\bar{H}$ *is core-free in* $\bar{G}$. *Then*

(*a*) $\Omega_r(G)$ *is a modular* p*-group;*

(*b*) $\bar{G}$ *is metabelian.*

PROOF. Let $\{e_0, \ldots, e_m\}$, $\{f_0, \ldots, f_m\}$ be bases of $\Omega_1(G)$ and of $\Omega_1(\bar{G})$ respectively, chosen as in Lemma 2.2, and set $\langle a_1 \rangle = \langle a \rangle^\pi$. In order to prove (*a*) we show first that

(1) $\langle a^\beta \rangle$ induces a group of power automorphisms on H

where $\langle a^\beta \rangle = \Omega_r(\langle a \rangle)$. This is obvious if H is cyclic. Thus, suppose that H is not cyclic and write $s = \min\{i : i \in \mathbb{N} \text{ and } H/\Omega_i(H) \text{ is cyclic}\}$. $\mho_{r-1}(H)$ is a non-trivial normal subgroup of G and therefore, by Lemma 2.2 (ii), it contains e_1 and, by (iv), it coincides with the set $\{x^{p^{r-1}} : x \in H\}$. Therefore there exists $h \in H$ such that $h^{p^{r-1}} = e_1$. Then

$$\Omega_1(\overline{\langle h \rangle}^{a_1\pi^{-1}}) = \langle e_1^\gamma e_0^\delta \rangle$$

where $1 \leqslant \gamma, \delta \leqslant p-1$. Hence, using Lemma 2.2 (ii), it follows that $\overline{\langle h \rangle}^{a_1\pi^{-1}} = \langle a^\beta h' \rangle$, for some $h' \in H$. Set $Q = H \cap \bar{H}^{a_1\pi^{-1}} \cdot \bar{H}^{a_1\pi^{-1}}/Q$ is cyclic of order at most p^r and $\langle a^\beta h' \rangle \cap Q = 1$, by (2). Thus

$$\bar{H}^{a_1\pi^{-1}} = Q\langle a^\beta h' \rangle . \tag{3}$$

Moreover, since $\bar{H}^{a_1\pi^{-1}}$ is a modular p-group (as the image of the abelian p-group H via the projectivity $\pi a_1 \pi^{-1}$) we have

$$\langle q, a^\beta h' \rangle \cap Q = \langle q \rangle \lhd \langle q, a^\beta h' \rangle$$

for all $q \in Q$. In other words $a^\beta h'$, and therefore a^β, induces a power automorphism on Q. Then $a^\beta = (a^\beta)^{a^n}$ induces a power automorphism on Q^a and moreover, a^β induces the same power on Q and on Q^{a^n} for every n, i.e. a^β induces a power automorphism on Q^G. Since $e_1 \notin Q$ and H/Q is cyclic, we have $H = Q \times \langle h \rangle$ and also that Q is core-free in G. Thus, since Q has exponent p^s, we have $Q^G = Q \times \Omega_s(\langle h \rangle) = \Omega_s(H)$ and (1) is proved if $s = r$. If $s < r$, we may assume that a^β induces the power μ on $\Omega_s(H)$ and, using Lemma 2.2 (iii) and induction on $|H|$, the power λ on $H/\Omega_1(H)$. Hence $h^{a^\beta} = h^\lambda x$ where $x \in \Omega_1(H)$, $(h^{p^{r-s}})^{a^\beta} = h^{p^{r-s}\lambda} = h^{p^{r-s}\mu}$ and so $\lambda \equiv \mu$ mod. p^s. $\langle x \rangle$ is normalized by a^β (because $\Omega_1(H) \leqslant \Omega_s(H)$) and therefore a^β acts as the power λ on $H/\langle x \rangle$. If $x \in \langle h \rangle$, i.e. $x = h^{\nu p^{r-1}}$ for some integer ν, set $\lambda' = \lambda + \nu p^{r-1}$; as for λ, $\lambda' \equiv \mu$ mod. p^s. For all $y = h^i z \in H$, where $z \in Q$, we have $y^{a^\beta} = (h^i z)^{a^\beta} = h^{i\lambda'} z^\mu = (h^i z)^{\lambda'} = y^{\lambda'}$, namely a^β acts as the power λ' on H. On the other hand, if $x \notin \langle h \rangle$, then $\langle x \rangle \cap \langle x \rangle^a = 1$. Also, a^β acts as the power λ both on $H/\langle x \rangle$ and on $H/\langle x \rangle^a$. In particolar $h^{a^\beta} = h^\lambda x$ gives $h^{a^\beta}\langle x \rangle^a = (h\langle x \rangle^a)^\lambda = h^\lambda x \langle x \rangle^a$, and so $x \in \langle x \rangle^a$, a contradiction. This completes the proof of (1). Thus, by Lemma 2.2 (i), $\Omega_r(G)$ is a modular p-group if $p \neq 2$. On the other hand, if $p = 2$, $\Omega_r(G)/\Omega_{r-2}(G)$ is abelian ([2], Lemma 1), and therefore $[H, a^\beta] \leqslant \Omega_{r-2}(G) \cap H = \Omega_{r-2}(H)\, \Omega_{r-2}(\langle a \rangle) \cap H = \Omega_{r-2}(H)$. This shows that a^β induces on H a power congruent to 1 mod. 4, namely that $\Omega_r(G)$ is modular. As far as (b) is concerned, observe that $\Omega_r(\bar{G})$ is a modular non-Hamiltonian p-group (see e.g. Lemma 2.2 (iv)), and $\Omega_r(\bar{G})/\Omega_{r-1}(\bar{G})$ is not cyclic (since $a^\beta \notin \langle h, \Omega_{r-1}(G) \rangle$). Then, as a result of Lemma 2.1, $\Omega_r(\bar{G})$ contains an abelian subgroup A normal in $\bar{G}$ such that $\Omega_r(\bar{G})/A \leqslant Z(\bar{G}/A)$. Therefore, since $\bar{G}/\Omega_r(\bar{G})$ is cyclic, (b) follows. □

3. The « reduced case ».

For the convenience of the reader we state the following theorem, which was proved by the second author for p odd ([5]), and by the first one for $p = 2$ ([3]).

THEOREM 3.1. *Let G and $\bar{G}$ be finite p-groups, $\pi: G \to \bar{G}$ a projectivity, $H \lhd G$ such that $G = H\langle a \rangle$ and $\bar{H}$ is core-free in $\bar{G}$. Then*

(a) *if $p \neq 2$, then H is abelian (hence the hypothesis that H is abelian in the statement of Theorem 2.3 is unnecessary if $p \neq 2$).*

(*b*) *if* $p = 2$, *set* $r = min \{n \in \mathbb{N}: |\Omega_1(H/\Omega_n(H))| \leqslant 4\}$. *Then* $\Omega_r(H)$ *is abelian and* $H/\Omega_r(H)$ *is a metacyclic modular non-Hamiltonian group* (*).

Making a combined use of Theorems 2.3 and 3.1 we obtain

THEOREM 3.2. *Let* G *and* $\bar{G}$ *be finite* p*-groups*, $\pi: G \to \bar{G}$ *a projectivity*, $H \lhd G$ *such that* $G = H\langle a \rangle$ *and* $\bar{H}$ *is core-free in* $\bar{G}$. *Then* $\bar{G}$ *has derived length at most* 6.

PROOF. If p is odd, then $\bar{G}$ is metabelian by Theorems 2.3 and 3.1(a). Thus, suppose that $p = 2$. Let

$$r = \min \{n \in \mathbb{N}: |\Omega_1(H/\Omega_n(H))| \leqslant 4\} .$$

Suppose first that $r = 0$. Then H is a metacyclic modular non Hamiltonian group, by Theorem 3.1(b). We show that, in this case

(4) $\bar{G}$ *has derived length at most* 4.

Set $\bar{X} = (\overline{H'})^{\bar{H}}$ and suppose that $|H'| = 2^s$. By [9], Lemma 6, $(\overline{H'})^{h_1\pi^{-1}} \lhd G$ for all $h_1 \in \bar{H}$. Hence X is the join of cyclic normal subgroups of G of order 2^s. Therefore G', and consequently H', centralise X. Moreover X has exponent $|H'|$ and then, since X is metacyclic, X/H' is cyclic. It follows that X is abelian. Now consider $G/\Omega_s(G)$. π induces a projectivity from $G/\Omega_s(G)$ to $\bar{G}/\Omega_s(\bar{G})$, $\bar{H}\Omega_s(\bar{G})/\Omega_s(\bar{G})$ is core-free in $\bar{G}/\Omega_s(\bar{G})$ (Lemma 2.2 (iii)) and $H\Omega_s(G)/\Omega_s(G)$ is abelian. Then, by Theorem 2.3 (b),

(5) $\bar{G}/\Omega_s(\bar{G})$ *is metabelian*

and so (4) is proved if $s = 0$. Suppose $s > 0$ and consider the group $X\langle a \rangle = Y$, say. π induces a projectivity from Y to $\bar{Y}$, $\bar{X}$ is core-free in $\bar{Y}$ and X is abelian. Therefore, by Theorem 2.3 (a), $\Omega_s(Y)$ is a modular p-group. Also, by Lemma 2.2 (iii), applied to $\bar{Y}$, $\bar{X}\Omega_{s-1}(\bar{Y})/\Omega_{s-1}(\bar{Y})$ is non-trivial and core-free in $\bar{Y}/\Omega_{s-1}(\bar{Y})$ and therefore $\Omega_s(\bar{Y})/\Omega_{s-1}(\bar{Y})$ is not cyclic. Moreover, $\Omega_s(\bar{Y}) \lhd \Omega_s(\bar{G})$, since $\Omega_s(\bar{Y}) = \bar{X}\Omega_s(\overline{\langle a \rangle})$ (Lemma 2.2 (i)) $= \bar{X}^{\bar{Y}} = \bar{X}^{\bar{G}}$ and, since $\Omega_s(H)/X$ is cyclic, it follows also that $\Omega_s(\bar{G})/\Omega_s(\bar{Y})$ is cyclic. Then, applying Lemma 2.1 to $\Omega_s(\bar{Y})$,

(*) The proof of (b) makes use of an unpublished result due to the second author. For completeness reasons we shall give a proof here, in the Appendix.

it follows that $\Omega_s(\overline{G})$ is metabelian. Hence, recalling (5), (4) follows.

Suppose now that $r > 0$. π induces a projectivity from $G/\Omega_r(G)$ to $\overline{G}/\Omega_r(\overline{G})$, $\overline{H}\Omega_r(\overline{G})/\Omega_r(\overline{G})$ is core-free in $\overline{G}/\Omega_r(\overline{G})$ (Lemma 2.2 (iii)) and $|\Omega_1(H\Omega_r(G)/\Omega_r(G))| = |\Omega_1(H/\Omega_r(H))| \leqslant 4$. Therefore, by applying what we have just proved to $G/\Omega_r(G)$, we see that $\overline{G}/\Omega_r(\overline{G})$ has derived length at most 4. Also, by Theorem 3.1 (*b*), $\Omega_r(H)$ is abelian. Hence, by applying Theorem 2.3 (*a*) to $\Omega_r(H)\langle a\rangle$, it follows that $\Omega_r(\overline{G})$ $(= \Omega_r(\overline{H})\Omega_r(\overline{\langle a\rangle}))$ is a modular p-group. In particular it is metabelian. Consequently $\overline{G}$ has derived length at most 6, as required. □

4. On the derived series of a projective image of a group having a normal subgroup with soluble factor group.

THEOREM 4.1. *Let G and $\overline{G}$ be groups, $\pi\colon G \to \overline{G}$ a projectivity, $N \lhd G$ and suppose that G/N is soluble of derived length $\leqslant n$. Then $\overline{G}^{(6n)} \leqslant \overline{N}$.*

PROOF. Clearly we may assume that G is finitely generated. Using induction on the derived length of G/N, we may assume that $(\overline{G'N})^{(6n-6)} \leqslant \overline{N}$. G/G' is a finitely generated abelian group, therefore $G/G' = \langle c_1\rangle G'/G' \times \langle c_2\rangle G'/G' \ldots \times \langle c_t\rangle G'/G'$ for some suitable elements $c_i \in G$ such that $\langle c_i\rangle/\langle c_i\rangle \cap G'$ is either infinite cyclic or a p-group. Set $H_i = \langle G', c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_t\rangle$, for $1 \leqslant i \leqslant t$. If $\langle c_i\rangle/\langle c_i\rangle \cap G'$ is infinite cyclic, then, as a result of [10], Corollario 1, $\overline{H}_i \lhd \overline{G}$. If $|\langle c_i\rangle/\langle c_i\rangle \cap G'|$ is a prime power, then $|\overline{G}:\overline{H}_i|$ is finite ([10], Teorema *A*) and $\overline{G}/\overline{H}_i$ is a chain. Therefore, assuming that $\overline{H}_i$ is not normal in $\overline{G}$, according to [6], Lemmas 2 and 3, the following two possibilities can occur:

(*a*) *$\overline{G}/(\overline{H}_i)_{\overline{G}}$ is a non-abelian group of order pq where p and q are prime numbers. In particular $\overline{G}/(\overline{H}_i)_{\overline{G}}$ is metabelian.*

(*b*) *$\overline{G}/(\overline{H}_i)_{\overline{G}}$ is a finite non-abelian p-group. $(H_i)_{\overline{G}}$ is normal in G([2]) and π induces a projectivity from $G/(H_i)_{\overline{G}}$ to $\overline{G}/(\overline{H}_i)_{\overline{G}}$. By [7], Theorem 3, $G/(H_i)_{\overline{G}}$ is also a p-group. Theorem 3.2 implies that $\overline{G}/(\overline{H}_i)_{\overline{G}}$ has derived length $\leqslant 6$.*

Now, in all cases $\overline{G}^{(6)} \leqslant \overline{H}_i$, for all $1 \leqslant i \leqslant t$. Therefore $\overline{G}^{(6)} \leqslant \bigcap_{i=1}^{t} \overline{H}_i = \overline{G'} \leqslant \overline{G'N}$. It follows that $\overline{G}^{(6n)} \leqslant \overline{N}$, as required. □

COROLLARY 4.2. *Let G and $\bar{G}$ be groups, $\pi: G \to \bar{G}$ a projectivity, and suppose that G is soluble of derived length $\leqslant n$. Then $\bar{G}$ is soluble of derived length $\leqslant 6n-4$.*

PROOF. By Theorem 4.1, $\bar{G}^{(6(n-1))} \leqslant \overline{G^{(n-1)}}$. Moreover, since $G^{(n-1)}$ is abelian, $\overline{G^{(n-1)}}$ is metabelian ([8], ch. 1, Theorems 17 and 18). It follows that $\bar{G}^{(6(n-1)+2)} = 1$, as required. □

5. Appendix.

As it was mentioned in the statement of Theorem 3.1, we give here a proof of the following result, which was needed in the proof of Theorem 2.4 (*b*).

PROPOSITION 5.1. *In the hypotheses of Theorem* 3.1 *and with* $p = 2$, *if* 2^n *is the exponent of* H, *then*

(*a*) $\Omega_1(G) \leqslant Z(\Omega_n(G))$;

(*b*) $\Omega_1(\bar{G}) \leqslant Z(\Omega_n(\bar{G}))$.

PROOF. We first show (*a*). Suppose (*a*) false, and choose a counterexample G with $|H|$ minimal. Let $\{e_0, \ldots, e_m\}$, $\{f_0, \ldots, f_m\}$ be bases of $\Omega_1(G)$ and $\Omega_1(\bar{G})$ respectively, as in Lemma 2.2 (ii) and set $\langle a_1 \rangle = \overline{\langle a \rangle}$. Then

$$e_1 \in \bar{H}^{a_1 \pi^{-1}}. \tag{6}$$

Set $(\bar{H}^{a_1\pi^{-1}})_G = \bar{K}^{a_1\pi^{-1}}$. $\bar{K}^{a_1\pi^{-1}} \lhd G$, and it does not contain e_1. Thus $\bar{K}^{a_1\pi^{-1}} \cap H = 1$ and this implies that $\bar{K}^{a_1\pi^{-1}}$ and consequently K are cyclic. Moreover, since $e_1 e_0 \in \bar{H}^{a_1\pi^{-1}} \cap Z(G)$ (Lemma 2.2), $K \neq 1$ and K is normal in G ([2] applied to the projectivity $\pi a_1 \pi^{-1}: G \to G$). Then the minimality of $|H|$ yields $[\Omega_1(G/K), H/K] = 1$. In particular

$$[\Omega_1(G), H] \leqslant \Omega_1(K) = \langle e_1 \rangle. \tag{7}$$

Now $\mho_{n-1}(H)$ is a non-trivial normal subgroup of G contained in $\Omega_1(H)$. Thus $\mho_{n-1}(H) \geqslant \langle e_1 \rangle$. Also, by Lemma 2.2 (iv), $\mho_{n-1}(H) = \{h^{2^{n-1}} | h \in H\}$. It follows that there exists $h \in H$ of order 2^n such that $h^{2^{n-1}} = e_1$. Then $\overline{\langle h \rangle}^{a_1\pi^{-1}} \cap H = 1$ and so, since $\bar{H}^{a_1\pi^{-1}}/H \cap \bar{H}^{a_1\pi^{-1}}$

is cyclic, we get

$$(8) \qquad \bar{H}^{a_1\pi^{-1}} = (H \cap \bar{H}^{a_1\pi^{-1}})\overline{\langle h\rangle}^{a_1\pi^{-1}}.$$

$\langle h\rangle$ is normalised by $\Omega_1(G)$ (7), therefore $\overline{\langle h\rangle}^{a_1\pi^{-1}}$ is normalised by $\Omega_1(G)$ as well. Moreover, since it is quasi-normal in G, $\bar{H}^{a_1\pi^{-1}}$ is normalised by $\Omega_1(G)$. Hence, from (8)

$$(9) \qquad [\Omega_1(G), \bar{H}^{a_1\pi^{-1}}] = [\Omega_1(G), (H \cap \bar{H}^{a_1\pi^{-1}})\, \overline{\langle h\rangle}^{a_1\pi^{-1}}] \leqslant$$

$$\leqslant (\langle e_1\rangle \cap \bar{H}^{a_1\pi^{-1}})(G' \cap \overline{\langle h\rangle}^{a_1\pi^{-1}}) = 1\,.$$

Using the decomposition $\Omega_n(G) = \bar{H}^{a_1\pi^{-1}}\langle h\rangle$ (see Lemma 2.2 (i)), we get

$$(10) \qquad [\Omega_1(G), \Omega_n(G)] = \langle e_1\rangle\,.$$

Let $\langle b\rangle$ be any cyclic subgroup of G of order 2^n containing $\langle e_0\rangle$. Considering that, if $\langle b\rangle$ is normalised by $\Omega_1(G)$, then it is centralised by $\Omega_1(G)$, it follows from (9) and from the decomposition $\Omega_n(G) = \bar{H}^{a_1\pi^{-1}}\langle b\rangle$ (see Lemma 2.2 (i)) that

$$(11) \qquad \langle b\rangle \text{ is not normalised by } \Omega_1(G)\,.$$

Set $\langle h_1\rangle = \overline{\langle h\rangle}$. $\langle h_1\rangle$ is normalised by $\Omega_1(\bar{G})$. Suppose that $C_{\Omega_1(\bar{G})}(\langle h_1\rangle) \geqslant \langle f_0, f_1, \dots, f_{m-1}\rangle$. Then $f_m^{a_1} \equiv f_m$ mod. $C_{\Omega_1(\bar{G})}(\langle h_1\rangle)$ and so either f_m centralises both $\langle h_1\rangle$ and $\langle h_1^{a_1}\rangle$ or it induces on $\langle h_1\rangle$ and $\langle h_1\rangle^{a_1}$ the same power $1 + 2^{n-1}$. In both cases

$$[f_m, h_1 h_1^{a_1}] \in \langle f_0\rangle = \langle (h_1 h_1^{a_1})^{2^{n-1}}\rangle$$

by Lemma 2.2 (iv). It follows that $\langle h_1 h_1^{a_1}\rangle^{\pi^{-1}}$ is normalised by $\Omega_1(G)$ and contains e_0, contradicting (11). Therefore $\langle f_0, f_1, \dots, f_{m-1}\rangle \nleqslant C_{\Omega_1(\bar{G})}(\langle h_1\rangle)$, and so we can find $x \in \langle e_1, e_2, \dots, e_{m-1}\rangle$ such that $x \notin \langle e_1, e_2, \dots, e_{m-2}\rangle$ and $\overline{\langle x\rangle} \nleqslant C_{\Omega_1(\bar{G})}(\langle h_1\rangle)$. Set $\langle x_1\rangle = \overline{\langle x\rangle}$. Then

$$(12) \qquad [h_1, x_1] = f_1\,.$$

If we let $\langle b\rangle = \Omega_n(\langle a\rangle) = \langle a^l\rangle$ (say) in (11), it follows from the action of a on $\Omega_1(G)$ that

(13) $m = 2^l + 1$ and $\Omega_1(G) \cap Z(\Omega_n(G)) = \langle e_0, e_1, \dots, e_{m-1} \rangle$.

The 2^l elements x^{a_j}, $0 \leqslant j \leqslant 2^l - 1$, form a basis of $\langle e_1, \dots, e_{m-1} \rangle$. Therefore

(14) $$C_{\langle a \rangle}(\langle x \rangle^{\langle x, a^2 \rangle}) = 1 .$$

Moreover, by Lemma 2.2 (iv) and (vi), $\Omega_1(\langle za^2 \rangle) = \langle e_0 \rangle$ for all $z \in H$. Hence $\Omega_1(\langle x, a^2 \rangle) = \Omega_1(\langle x, za^2 \rangle) = \langle e_0 \rangle \times \langle x \rangle^{\langle x, a^2 \rangle}$. In particular

$$\Omega_1(\langle x_1, a_1^2 \rangle) = \Omega_1(\langle x_1, h_1 a_1^2 \rangle)$$

and so, by (12),

$$f_1 = [h_1, x_1] = [h_1, x_1]^{a_1^2} = [h_1 a_1^2, x_1]\,[a_1^2, x_1] \in \Omega_1(\langle x_1, a_1^2 \rangle) .$$

But then $e_1 \in \Omega_1(\langle x, a^2 \rangle) \cap H = \langle x \rangle^{\langle x, a^2 \rangle}$, contradicting (14). This contradiction completes the proof of (a).

In order to show (b) observe that, by (a), every element $y \in \Omega_1(\bar{G})$ induces a power automorphism on $\Omega_n(\bar{G})$. Then, from the fact that $\Omega_n(\bar{G})$ contains two cyclic subgroups of order 2^n intersecting trivially, using Lemma 2.2 (iv), it follows that the power automorphism induced by y on $\Omega_n(\bar{G})$ is universal, and it is either the identity or the power $1 + 2^{n-1}$. Hence $|\Omega_1(\bar{G}) : \Omega_1(\bar{G}) \cap Z(\Omega_n(\bar{G}))| \leqslant 2$ and so, since $\Omega_1(\bar{G}) \cap \cap Z(\Omega_n(\bar{G})) \lhd \bar{G}$, we get

$$\Omega_1(\bar{G}) \cap Z(\Omega_n(\bar{G})) \geqslant \langle f_0, \dots, f_{m-1} \rangle .$$

where $\{f_0, \dots, f_m\}$ and $\{e_0, \dots, e_m\}$ are the usual bases of $\Omega_1(\bar{G})$ and $\Omega_1(G)$, according to Lemma 2.2 (*ii*). Assume, by way of contradiction, that f_m induces the power $1 + 2^{n-1}$ on $\Omega_n(\bar{G})$. Set $\langle a_1^{2^l} \rangle = \Omega_n(\langle a_1 \rangle)$. Then $[f_m, a_1^{2^l}] = f_0$ implies that $m = 2^l$. The 2^l elements $e_m^{a_j}$ $(0 \leqslant j \leqslant 2^l - 1)$ form a basis of $\Omega_1(H)$. Therefore

(15) $$C_{\langle a \rangle}(\langle e_m \rangle^{\langle e_m, a^2 \rangle}) = 1 .$$

Moreover, by Lemma 2.2 (iv) and (vi), $\Omega_1(\langle a^2 z \rangle) = \langle e_0 \rangle$ for all $z \in H$.

In particular $\langle a^2z\rangle \cap H = 1$. Therefore

$$\langle e_m\rangle^{\langle e_m, a^2\rangle} = \langle e_m\rangle^{\langle e_m, a^2 z\rangle} = \langle e_m, a^2\rangle \cap H = \langle e_m, a^2 z\rangle \cap H \,,$$

and

$$\Omega_1(\langle e_m, a^2\rangle) = \langle e_0\rangle \times \langle e_m\rangle^{\langle e_m, a^2\rangle} = \Omega_1(\langle e_m, a^2 z\rangle) \,.$$

Then

$$\Omega_1(\langle f_m, a_1^2\rangle) = \Omega_1(\langle f_m, a_1^2 z_1\rangle) \tag{16}$$

for all $z_1 \in \bar{H}$. As we have seen in proving (a), there exists $h_1 \in \bar{H}$ such that $h_1^{2^{n-1}} = f_1$. Thus

$$f_1 = [h_1, f_m] = [h_1, f_m]^{a_1^2} = [h_1 a_1^2, f_m]\,[a_1^2, f_m] \in \Omega_1(\langle f_m, a_1^2\rangle)$$

by (16). Therefore $e_1 \in \langle e_m, a^2\rangle \cap H = \langle e_m\rangle^{\langle e_m, a^2\rangle}$, contradicting (15). This completes the proof of Proposition 5.1. □

We finally point out that, using Lemma 2.1, we can improve a result in [3] (Theorem) as follows:

THEOREM 5.2. *Let G be a group and $\pi\colon G \to \bar{G}$ a projectivity. If $H \lhd G$, then $H/H_{\bar{G}}$ and $\bar{H}/\bar{H}_{\bar{G}}$ are soluble groups of derived length at most* 3.

(In [3] the upper bound for the derived length of $\bar{H}/\bar{H}_G$ was 4).

PROOF. Looking at the proof of [3], Theorem, it is immediately seen that the critical case is when $G = H\langle a\rangle$, $\bar{H}$ is core-free in $\bar{G}$ and G, $\bar{G}$ are finite 2-groups. In this case, applying Theorem 3.1 (b), it follows that there exists r such that $\bar{H}/\Omega_r(\bar{H})$ is metacyclic and $\Omega_r(\bar{H})$ is a non-Hamiltonian modular 2-group such that $\Omega_r(\bar{H})/\Omega_{r-1}(\bar{H})$ is non-cyclic. Then, as a result of Lemma 2.1, $\bar{H}^{(3)} = 1$, as required. □

REFERENCES

[1] G. BIRKHOFF, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications, **25**, New York, 1949.

[2] G. BUSETTO, *Sottogruppi normali e proiettività*, Rend. Sem. Mat. Univ. Padova, **67**, pp. 105-110.

[3] G. Busetto, *Normal subgroups and projectivities*, J. Algebra (to appear).

[4] F. Gross, *p-subgroups of core-free quasi-normal subgroups*, Rocky Mountains J. Math., **1** (1971), pp. 541-550.

[5] F. Menegazzo, *Normal subgroups and projectivities of finite groups*, Rend. Sem. Mat. Univ. Padova, **59** (1978), pp. 11-15.

[6] R. Schmidt, *Modulare Untergruppen endlicher Gruppen*, Illinois J. Math., **13** (1969), pp. 358-377.

[7] M. Suzuki, *On the lattice of subgroups of finite groups*, Trans. Amer. Math. Soc., **70** (1951), pp. 345-371.

[8] M. Suzuki, *Structure of a group and the structure of its lattice of subgrvups*, Erg. der Math., **10**, Springer-Verlag, Berlin (1956).

[9] B. V. Yakovlev, *Lattice isomorphisms ofs olvable groups*, Alg. i Log., **9** (1970), pp. 349-369.

[10] G. Zacher, *Una caratterizzazione reticolare della finitezza dell'indice di un sottogruppo in un gruppo*, Rend. Accad. Naz. Lincei, **69** (1980), pp. 317-323.

[11] G. Zappa, *Sulla risolubilità dei gruppi finiti in isomorfismo reticolare con un gruppo risolubile*, Giornale di Mat. Battaglini, **80** (1951), pp. 213-255.

Manoscritto pervenuto in redazione l'11 maggio 1984.