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Measurable products of modules


<http://www.numdam.org/item?id=RSMUP_1985__73__261_0>
Measurable Products of Modules.

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SUMMARY - In this paper all groups are abelian, rings are associative with identity, and modules are left unitary. We are interested in modules of the form \( \prod G_i \), a direct product of submodules \( G_i \) over an index set \( I \).

Many theorems about such modules require that \(|I|\) be non-measurable. Here we let \(|I|\) be arbitrary, put mild restrictions on the \( G_i \)'s, and obtain new results. In Section 1 we establish some decomposition theorems. We then apply them to homomorphisms of the form \( f: \prod G_i \to A \) where: in Section 2 \( A \) is a slender module and in Section 3 \( A \) is an infinite direct sum of submodules.

0. Preliminaries.

Let \( I \) be a set and \( P(I) \) its power set. Here (as in [3]) \(|I|\) is measurable if there is a \( 0,1 \) countably additive function \( \mu \) on \( P(I) \) such that \( \mu(I) = 1 \) and \( \mu(\{i\}) = 0 \) for each \( i \in I \). If no such function exists, \(|I|\) is non-measurable. If \( \beta \) is the least measurable cardinal, it is a regular limit cardinal such that \( \alpha < \beta \) implies \( 2^\alpha < \beta \). If all sets are constructible (\( V = L \)), measurable cardinals do not exist. A good discussion of these matters may be found in [5].

If \((S, +, \cdot)\) is a Boolean ring, an ideal \( K \) is here called a \( \gamma \)-ideal if, whenever \( \{s_1, s_2, \ldots\} \) is a countable set of orthogonal elements in \( S \), \( \sum_{n \geq k} s_n \in K \) for some \( k \) in \( N \), the natural numbers,

Let \( R \) be a ring and \( A \) a \( R \)-module. A filter \( F \) is a set of principal

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right ideals in $R$ such that, for each $aR$, $bR$ in $F$, there is a $cR$ in $F$ contained in $aR$ and $bR$. $A$ is torsion-free if, for $r \in R$ and $x \in A$, $rx = 0$ implies $r = 0$ or $x = 0$. $A$ is divisible if $rA = A$ for all non-zero $r$ in $R$. $D(A)$ is the maximal divisible submodule of $A$ and $A$ is reduced if $D(A) = 0$.

In general our terminology agrees with that in Fuchs [3].

1. Decomposition theorems.

We begin with a set-theoretic lemma. We omit its proof since its statements are well-known or easily proved (e.g. see page 161 in II of [3] or pages 342-356 in [5]).

**Lemma 1.1.** Let $I$ be a set, $S = P(I)$, and $K$ a proper $\gamma$-ideal in the Boolean ring $(S, +, \cdot)$.

(a) $S/K$ is finite and there are orthogonal elements, say $u_1, \ldots, u_s$, in $S$ which map onto the atoms of $S/K$.

(b) If $J$, is a set of orthogonal elements in $S$, almost all $s_j$'s are in $K$. If $|J|$ is non-measurable, then $\sum_{J'} s_j \in K$ for some cofinite subset $J'$ of $J$.

(c) If $|I|$ is non-measurable, then $K = P(I')$ for some cofinite subset $I'$ of $I$.

**Remark.** If $K$ is an ideal of finite index in a complete Boolean ring, it need not be a $\gamma$-ideal (consider $S = 2^I$ and let $K$ be a maximal ideal containing $2^{(\kappa)}$). However, if $I$ is a set, then $|I|$ is measurable if and only if $S = P(I)$ has a proper $\gamma$-ideal $K$ containing the atoms of $S$.

We now use Lemma 1.1 to obtain decompositions of specific modules. In the next two theorems the ring $R$ is arbitrary.

**Theorem 1.2.** Let $X = \prod_{i} G_i$ be a $R$-module where the $G_i$'s are pairwise isomorphic of non-measurable cardinality. If $K$ is a proper $\gamma$-ideal in $S = P(I)$ and $H = \langle \prod_{s} G_i : s \in K \rangle$, then $X = L \oplus H$ where $L \cong \bigoplus_{E} G_i$ for some finite subset $E$ of $I$. If $I$ is infinite, then $X \cong H$.

**Proof.** Let $\pi_i : G_i \to G$ be an isomorphism for some group $G$ and each $i$. Let $u_1, \ldots, u_s$ be as in Lemma 1.1. For each $u_n$ let $A_n =$
Clearly each $A_n \cong G$ and we set $L = \bigoplus_{i=1}^{e} A_n$.
We claim $X = L \oplus H$. Let $x = \sum_{i} x_i$, $x_i \in G_i$, be an element in $X$.
If $s \in S$, we define $x_s = \sum_{i \in s} x_i$. Hence $x = \sum_{s} x_{s*}$ where $i \in s_e$ exactly
if $\pi_i(x_i) = g \in G$. Since the $s_e$’s partition $I$ and $|G|$ is non-measurable,
$\sum_{G'} s_e \in K$ for some cofinite subset $G'$ of $G$ by Lemma 1.1 and so $\sum_{G'} x_{s*}$
is in $H$. Now $x$ will be in $L + H$ if we can show $x_{s*}$ is in it for any
fixed $g$. But $s_e = \sum a_n u_n + v$ where each $a_n = 0$ or 1 and $v \in K$.
So $x_{s*} = \sum a_n x_{u_n} + x_v - 2 \sum a_n x_{u_nv}$, which is in $L + H$ ($u_nv$ is in $K$).
Suppose now $y_1 + \ldots + y_s + z = 0$ with $y_n$ in $A_n$ and $z$ in $H$. For
any fixed $n$ and some $i$ in $u_n$ the $i$th component of $z$ is 0 since $u_n$ is not
in $K$. Hence the $i$th component of $y_n$ is 0 and $y_n = 0$. Therefore
$z = 0$ and $X = L \oplus H$, as desired. If $E$ consists of one element
from each $u_n$, then $L \cong \bigoplus_{i \in \bar{E}} G_i$. Since $X = L \oplus \prod_{I \setminus E} G_i$, we have $H \cong$
\[ \prod_{I \setminus \hat{E}} G_i \cong X \text{ if } I \text{ is infinite.} \]

**Theorem 1.3.** Let $X = \prod_{I} G_i$ be a $R$-module where each $G_i$ and
the set of their isomorphism classes have non-measurable cardinality.
If $K$ is a $\gamma$-ideal in $S = P(I)$ and $H = \langle \prod_{s} G_i : s \in K \rangle$, then $X = L \oplus H$ where, for some finite subset $E$, $L \cong \bigoplus_{i \in \bar{E}} G_i$ and $H \cong \prod_{I \setminus E} G_i$.

**Proof.** Write $I = \bigcup_{J} s_j$ where $i, i'$ are in the same $s_j$ exactly if
$G_i \cong G_i'$. Then $\{s_j\}$ partitions $I$, $|J|$ is non-measurable, and $\prod_{I} G_i = \prod_{J} G_{s_j}$ where $G_{s_j} = \prod_{i \in J} G_i$. By Lemma 1.1 there is a $J'$ cofinite
in $J$ such that $\sum_{J'} s_j \in K$ and $\prod_{J'} G_{s_j} \in H$. For fixed $j G_{s_j}$ is a product
of isomorphic groups and, if $S_j = P(s_j)$, then $K_j = S_j \cap K$ is a $\gamma$-ideal
in $S_j$. By the last theorem $G_{s_j} = L_j \oplus H_j$ where $H_j = \langle \prod_{s} G_i : s \in K_j \rangle$
and, for a finite subset $t_j$ of $s_j$, $L_j \cong \bigoplus_{i \in t_j} G_i$ and $H_j \cong \prod_{s_j \setminus t_j} G_i$. So
$X = \bigoplus_{J'} L_j \oplus \left[ \bigoplus_{J' \setminus J''} H_j \oplus \prod_{J''} G_{s_j} \right]$ Let $L$ be the left sum and let $H'$ be
the module in the bracket. If $E = \bigcup_{J'} t_j$, we have $L \cong \bigoplus_{i \in E} G_i$ and
$H' \cong \prod_{I \setminus E} G_i$. Since $H' \subset H$ and $H \cap L = 0$, $H = H'$ and the proof
is complete.

Our next proposition will prove useful later.
PROPOSITION 1.4. Let \( \{G_i\}, i \in I \), be a set of \( R \)-modules and let \( V \) be the set of their isomorphism classes.

(a) If \(|R|\) and each \(|G_i|\) is \( < \alpha \), a fixed non-measurable cardinal, then \(|V|\) is non-measurable.

(b) If each \(|G_i|\) and \(|V|\) are non-measurable, then \(|G_i| < \alpha\) for some non-measurable \( \alpha \) and all \( i \).

PROOF. (a) Let \( Y \) be a free \( R \)-module of rank \( \alpha \). Then \( G_i \cong Y/H_i \) for some \( H_i \) and each \( i \). The number of distinct submodules of \( Y \) is \( \leq 2^{|Y|} < 2^\alpha \) which is non-measurable since \( \alpha \) is. So \(|V|\) is non-measurable since \( \alpha \) is. So \(|V|\) is non-measurable. (b) Let \( \beta \) be the least non-measurable cardinal (it's an ordinal). Let \( J \) be a subset of \( I \) such that the ordinals \(|G_j|, j \in J\), are distinct with least upper bound \( \alpha < \beta \). Since \( \beta \) is a regular cardinal if \( \alpha = \beta \), we have \( \beta = |J| < |V| < \beta \), a contradiction. So \( \alpha < \beta \).

2. Slender modules.

In this section we apply the theorems of the last section to mappings of the form \( f: \prod_{i} G_i \to A \) where \( A \) is a slender \( R \)-module. In the literature there appear various definitions of a «slender» module (or group). Actually these definitions are essentially the same as we shall show. Therefore, we define: a \( R \)-module \( A \) is slender if it satisfies any (hence all) of the four conditions of Proposition 2.1.

PROPOSITION 2.1. Let \( A \) be a \( R \)-module. The following are equivalent.

1. If \( f: R^N \to A \) is a homomorphism, almost all components in \( R^N \) map to 0.

2. If \( f: R^N \to A \) is a homomorphism, there is a cofinite subset \( C \) in \( N \) such that \( f(R^C) = 0 \).

3. If \( \{G_n\}, n \in N \), is a countable set of \( R \)-modules and \( f: \prod_{n} G_n \to A \) is a homomorphism, then \( f(\prod_{n=k}^{N} G_n) = 0 \) for some \( k \) in \( N \).

4. If \( \{G_n\}, n \in N \), is a countable set of \( R \)-modules and \( f: \prod_{n} G_n \to A \) is a homomorphism, then \( f(G_n) = 0 \) for almost all \( n \) in \( N \).
Proof. (1) \(\Rightarrow\) (2). We write \(R^n = \prod_N Re_n\) where \(e_n\) is a \(N\)-tuple with 1 in the \(n\)th position and 0 elsewhere. It suffices to assume \(f(e_n) = 0\) for all \(n\) but \(f(x) \neq 0\) for some \(x \in R^n\) and to derive a contradiction. Write \(x = \sum r_n e_n\), \(r_n \in R\), and, for each \(n\) in \(N\), set \(a_n = x - (r_1 e_1 + \ldots + r_n e_n)\). For each \(k\) the \(k\)th component in \(R^n\) of \(a_n\) is 0 for almost all \(n\). Hence, if \(B = \prod Ra_n\), there is a natural imbedding \(\varphi: B \to R^n\) with \(\varphi(a_n) = a_n\) for each \(n\). Consider the map \(f: B \to A\). Since \(f\varphi(a_n) = f(x) \neq 0\) for each \(n\), we have a contradiction of (1) with respect to \(f\varphi\). Therefore (1) \(\Rightarrow\) (2).

(2) \(\Rightarrow\) (3). Suppose (3) is false. For each \(k\) in \(N\) choose \(x_k \in \bigcap_i G_{n \geq k}\) so that \(f(x_k) \neq 0\). There is a natural map \(\varphi: R^n \to \prod G_n\) carrying \(e_n\) to \(x_n\). Then \(f\varphi: R^n \to A\) is a homomorphism and by (2) \(f\varphi(\prod_{k \geq m} Rx_k) = 0\) for some \(m\). So \(0 = f\varphi(x_k) = f(x_k)\) for \(k \geq m\), a contradiction. Thus (2) \(\Rightarrow\) (3). Clearly (3) \(\Rightarrow\) (4) \(\Rightarrow\) (1) and the proposition is true.

Remark. (1) is the definition of slender used by Fuchs [3, vol. II, pg. 159] in the case \(R = Z\) and \(A\) is torsion-free (a slender \(R\)-module is torsion-free if \(R = Z\) but not in general. See Example 3 on p. 399 of [4]). (2) and (3) are the definitions of slender used in [2] and [4].

We now apply Theorem 1.3 to a map from a direct product of modules to a slender module.

Theorem 2.2. Let \(X\) be a module where each \(|G_i|\) and the set of their isomorphism classes are non-measurable. If \(f: X \to A\) is a homomorphism and \(A\) is slender, then \(X = L \oplus H\) where \(f(H) = 0\) and, for some finite subset \(E\), \(L \cong \bigoplus_{i \in E} G_i\) and \(H \cong \prod_{i \notin E} G_i\).

Proof. For \(S = P(I)\) let \(K = \{s \in S: f(\prod_i G_i) = 0\}\). Clearly \(K\) is an ideal and it is a \(\gamma\)-ideal by (3) of Proposition 2.1. Theorem 1.3 completes the proof.

Corollary 2.3. Let \(R\) be a commutative integral domain not a field and let \(A\) be a countable torsion-free reduced \(R\)-module. If \(R\)-module \(X\) equals \(\prod G_i\) where \(|G_i| < \alpha\) for some non-measurable \(\alpha\) and all \(i\) and if \(f: X \to A\) is a homomorphism, then \(X = L \oplus H\) where \(f(H) = 0\) and, for some finite subset \(E\), \(L \cong \bigoplus_{i \in E} G_i\) and \(H \cong \prod_{i \notin E} G_i\).
Proof. We assume $A$ is non-zero and hence $|E|$ is countable. Thus the set of isomorphism classes of the $G_i$'s is non-measurable by Proposition 1.4. The result now follows from Theorem 2.2.

Corollary 2.4. Let $A$ be a torsion-free abelian group which does not contain a copy of $Q$, $Z^n$, or the $p$-adic integers for a prime $p$. Let $X = \prod_{i} G_i$ be an abelian group where $|G_i| < \alpha$ for some non-measurable $\alpha$ and all $i$. If $f: X \to A$ is a homomorphism, then $X = L \oplus H$ where $f(H) = 0$ and, for some finite $E$, $L \cong \bigoplus_{E} G_i$ and $H \cong \prod_{I \setminus E} G_i$.

Proof. $A$ is a slender $Z$-module by Theorem 95.3 in [3]. Proposition 1.4 and Theorem 2.2 complete the proof.

Note. If $\{G_i\}$ is a set of indecomposable groups of non-measurable cardinality, the least upper bound of $|G_i|$ may not be non-measurable. There exists arbitrarily large indecomposable groups (Theorem 2.1 in [7]).

3. Direct products and sums.

Suppose $X = \prod_{i} G_i$, $A = \bigoplus_{j} A_j$ are modules and $f: X \to A$ is a homomorphism. Most known theorems dealing with this situation require that $|I|$ be non-measurable (see [6] for references). In this section we let $I$ be arbitrary, put some restrictions on the $G_i$'s and obtain new results. Other results with $I$ arbitrary may be found in part 2 of [6]. By $f_i$ we will mean the map $f$ followed by the projection to $A_j$.

Theorem 3.1. Let $X = \prod_{i} G_i$, $A = \bigoplus_{j} A_j$ be two $R$-modules and let $f: X \to A$ be a homomorphism. Suppose each $G_i$ and the set of their isomorphism classes have non-measurable cardinality. If (a) $F$ is a filter of non-zero principal right ideals in $R$ or (b) $R$ is a commutative integral domain, then $X = L \oplus H$ where $L \cong \bigoplus_{E} G_i$ and $H \cong \prod_{I \setminus E} G_i$ for some finite $E$ such that, for some non-zero $b$ in $R$, $bf_j(H)$ is contained in (a) $\bigcap_{j} rA$ or (b) $D(A)$ for almost all $j$ in $J$. 
PROOF. Let $S = P(I)$. (a) Let $K = \{ s \in S : \text{for some } r, R \text{ in } F \text{ we have } r_i f_i(\prod_{i \in I} G_i) \subseteq \bigcap_{r \in R} rA \text{ for almost all } j \in J \}$. It is easy to see that $K$ is an ideal in $S$ and it is a $\gamma$-ideal by Chase’s Theorem (Theorem 2.1 in [1] or Theorem 1.1 in [6]). Conclusion (a) follows immediately from Theorem 1.3 above and from Theorem 1.3 in [6]. (b) Let $K = \{ s \in S : \text{for some } r, R \neq 0 \text{ in } R \text{ we have } r_i f_i(\prod_{i \in I} G_i) \subseteq D(A) \text{ for almost all } j \}$. Then $K$ is a $\gamma$-ideal in $S$ by Theorem 1.5 in [6] and, from that theorem and Theorem 1.3 above, we have conclusion (b).

We next apply Theorem 3.1 to the case where $f$ is the identity map. For best results we let $A$ be torsion free.

THEOREM 3.2. Let $A$ be a torsion-free $R$-module with decompositions $A = \prod_i G_i = \bigoplus_j A_j$ where $|G_i| \leq \alpha$ for some non-measurable $\alpha$ and all $i$. If $F$ is a filter of non-zero principal right ideals in $R$ such that $\bigcap_{r \in F} rA = 0$ or if $R$ is a commutative integral domain and $D(A) = 0$, then there are finite subsets $I_1$ in $I$ and $J_1$ in $J$ such that $\bigoplus G_i \cong B \bigoplus \bigoplus_{i \in I_1} A_i$ with $B \subseteq \bigoplus_{j \in J_1} A_j$.

PROOF. Since $A$ is torsion-free, we may assume $|R| \leq \alpha$. By Proposition 1.4 the set of isomorphism classes of the $G_i$’s has non-measurable cardinality. Let $f$ be the identity map in Theorem 3.1. By that theorem and the fact that $A$ is torsion-free we have, for some finite subsets $I_1$ in $I$ and $J_1$ in $J$, $A = L \bigoplus H$ with $L \cong \bigoplus_{i \in I_1} G_i$ and $H \subseteq \bigoplus_{j \in J_1} A_j$. The conclusion of the theorem now follows.

If $R = \mathbb{Z}$ and $A$ in the last theorem is just an abelian group, torsion-freeness is not required to obtain a meaningful decomposition theorem.

THEOREM 3.3. Let $A = \prod_i G_i = \bigoplus_j A_j$ be a reduced abelian group where $|G_i| \leq \alpha$ for some non-measurable $\alpha$ and all $i$. There are decompositions $G_i = B_i \bigoplus C_i$, $A_j = T_j \bigoplus U_j$ and finite subsets $I_1$ in $I$, $J_1$ in $J$ such that

(a) $\prod_i B_i \cong \bigoplus_j T_j$ and is bounded

(b) $\prod_i C_i \cong \bigoplus_j U_j$
PROOF. By Proposition 1.4 the set of isomorphism classes to which the \( G_i \)'s belong has non-measurable cardinality. By Theorem 3.1 (with \( f \) the identity map and \( D(A) = 0 \)) there is a decomposition \( A = L \oplus H \) and finite subsets \( I_i \) and \( J_i \) such that \( L \cong \bigoplus_{I_i} G_i \), \( H \cong \bigoplus_{J_i} G_i \) \( \cong \coprod_{I_i \cap J_i} G_i \), and \( nH \subseteq \bigoplus_{J_i} A_i \) for some \( n \) in \( N \). Since our goals are isomorphisms not identities, we may assume \( n(A_i) \subseteq \bigoplus_{J_i} A_i \) for \( n \) in \( N \). The rest of the proof is exactly like that of Corollary 1.9 in [6].

COROLLARY 3.4. Suppose \( A = \prod G_i \) is a reduced abelian group, \( |G_i| < \kappa \) for some non-measurable \( \kappa \) and all \( i \), and let \( \beta \) be an infinite cardinal. Then \( A \) is a direct sum of \( \beta \) non-zero subgroups if and only if (a) some finite sum of \( G_i \)'s is or (b), for some \( n \) in \( N \), each \( G_i \) has a \( n \)-bounded direct summand \( B_i \) such that \( |\prod B_i| > \beta \).

PROOF. Sufficiency is clear. To show necessity set \( A = \bigoplus_{J} A_J \), \( A_J \neq 0 \) and \( |J| = \beta \), and apply Theorem 3.3. Assume the decompositions there have been made and \( |\prod B_i| < \beta \). By (c) then \( \bigoplus_{I_i} C_i \) is a direct sum of \( \beta \) non-zero subgroups.

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Manoscritto pervenuto in redazione il 11 maggio 1984