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# An $\Omega_+$-Estimate for the Number of Lattice Points in a Sphere.

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SUMMARY - Let $A(T)$ be the number of lattice points in the sphere $x^2 + y^2 + z^2 \leqq T$, then it is the purpose of the present paper to prove that

$$A(T) = \frac{4\pi}{3} T^{\frac{3}{2}} + \Omega_+\left(T^{\frac{1}{2}}(\log_2 T)^{\frac{1}{2}} (\log_3 T)^{-\frac{1}{2}}\right)$$

($\log_k$ denoting the $k$-fold iterated logarithm). This is done by a method due to K. S. Gangadharan [7] on the basis of an explicit formula of P. T. Bateman [1].

## 1. Introduction.

Denote by $r_3(n)$ the number of triples $(u, v, w) \in \mathbb{Z}^3$ satisfying $u^2 + v^2 + w^2 = n$, then it is the objective of the present paper to establish a result on the behaviour of the « lattice rest »

$$P(T) = \sum_{0 \leqq n \leqq T} r_3(n) - \frac{4\pi}{3} T^{\frac{3}{2}} \tag{1}$$

for $T \to \infty$. Concerning the $O$-problem, it has been proved by I. M. Vinogradov ([14] and [15], p. 29 $f$) that $P(T) = O\left(T^{\frac{3}{4}}(\log T)^c\right)$ for some absolute constant $c$. (Vinogradov gave the value $c = 6$ which

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can be readily improved at least to $c = \frac{9}{2}$; Chen J. R. [4] stated this result even with $c = 0$.)

In the other direction, G. Szegö [13] showed that

$$P(T) = \Omega_-\left(T^{\frac{1}{2}}(\log T)^{\frac{1}{2}}\right), \tag{2}$$

apparently the best result of that kind to date. On the opposite side K. Chandrasekharan and R. Narasimhan [3] proved that

$$\limsup_{T\to\infty} P(T)\, T^{-\frac{1}{2}} = +\infty .$$

In this paper we are going to establish a refinement of this last result, i.e. we prove the following estimate.

THEOREM. *For* $T \to \infty$ *we have*

$$P(T) = \Omega_+\left(T^{\frac{1}{2}}(\log_2 T)^{\frac{1}{2}}(\log_3 T)^{-\frac{1}{2}}\right) \tag{3}$$

*where* $\log_k$ *denotes the k-fold iterated logarithm.*

REMARKS. We employ the method developed by K. S. Gangadharan [7] for the divisor and the circle problem, the essential difficulty (due to the somewhat « irregular » behaviour of $r_3(n)$) being surmounted by the lemma on page 6 (the proof of which is based on the explicit formula (17) for $r_3(n)$). In fact, there are numerous contributions to the literature (see the references) which contain investigations analogous to the part of our argument leading from our lemma to (3), most of them involving generating functions (—in our case this would be Epstein's zeta-function—) which satisfy a certain functional equation due to Chandrasekharan and Narasimhan. However, apparently none of this results can be applied directly (i.e. without the necessity to get rid of some condition which is not satisfied in our case) to infer (3) from our lemma. So we prefer to give an argument as simple and self-contained as desirable which avoids the use of generating functions.

## 2. Notation and other preliminaries.

Throughout the whole paper, $n$ and $m$ are nonnegative integers, $q$ resp. $q_j$ denotes square-free integers and $p$ denotes primes. $X$ is a (large) real variable, $q_j$ are the square-free positive integers not exceeding $X$ $(1 \leqq j \leqq N = N(X))$. $\mathbb{C}^+$ is the half-plane of all complex numbers with a positive real part. Let $T(u) = \sum_\gamma a_\gamma \exp(-i\alpha_\gamma u)$ be any trigonometric polynomial and $H(s)$ an analytic function on $\mathbb{C}^+$, then we define for $\sigma > 0$

$$T \wedge H(\sigma) := \sum_\gamma a_\gamma H(\sigma + i\alpha_\gamma) .$$

We further put (for real $\alpha$)

$$K(\alpha) = 1 + \cos\alpha = 1 + \tfrac{1}{2}\exp(i\alpha) + \tfrac{1}{2}\exp(-i\alpha) \geqq 0$$

and

$$T_X(u) = \prod_{j=1}^{N(X)} K\left(2\pi u\sqrt{q_j} + \pi\right) = 1 + T_X^{(1)}(u) + \bar{T}_X^{(1)}(u) + T_X^{(2)}(u)$$

where

$$(5) \quad T_X^{(1)}(u) = -\tfrac{1}{2}\sum_{j=1}^{N}\exp(-2\pi iu\sqrt{q_j}), \qquad T_X^{(2)}(u) = \sum_\tau b_\tau \exp(-2\pi i\beta_\tau u)$$

$\bar{T}_X^{(1)}$ is the complex conjugate of $T_X^{(1)}$, $\tau$ runs through an index set of cardinality $< 3^X$, the coefficients $b_\tau$ are of modulus $\leqq 1$ and the numbers $\beta_\tau$ have the following property relevant for later purposes. If we define

$$S_X = \left\{\eta \in \mathbb{R}: \eta = \left|\sqrt{n} + \sum_{j=1}^{N} r_j\sqrt{q_j}\right|,\ n \in \mathbb{N}_0,\ r_j \in \{-1, 0, 1\},\ \sum_{j=1}^{N} r_j^2 \geqq 2\right\},$$

then $|\sqrt{n} \pm \beta_\tau| \in S_X$ for all $\tau$ and all $n \in \mathbb{N}_0$. Moreover, putting

$$q(X) := -\log\left(\min\{\eta \in S_X\}\right) ,$$

we note that Gangadharan [7] has proved that

$$(6) \qquad aX \leqq q(X) \leqq Q(X) := b^{X/\log X}$$

for sufficiently large $X$ with some positive constants a and $b (b > 2)$. Hence

$$(7) \qquad |\sqrt{n} \pm \beta_\tau| \geqq \exp(-q(X)) \geqq \exp(-Q(X))$$

for each $\tau$ and any $n \in \mathbf{N}_0$. We finally remark that it follows from the above in a straightforward manner that (for $\sigma > 0$ and $\lambda \geqq 0$)

$$T_X \wedge I_\lambda(\sigma) = \sigma^{-\lambda} + O\Big(\exp\big(\lambda q(X)\big)\, 3^X\Big) \tag{8}$$

where $I_\lambda(\sigma) = \sigma^{-\lambda}$ and the $O$-constant is an absolute one. (This will be the case for all $O$- and $\ll$-constants throughout the whole paper.)

## 3. Proof of the theorem.

3.1. For $u > 0$ we define $\Psi(u) = u^{-\theta}\big(P(u^2) - 1\big)$, then $P_\theta = \sup \Psi(u)$ (taken over all $u > 0$) is a finite positive number for $1 < \theta \leqq 3$. (If this were not true for some $\theta > 1$, a stronger estimate than our theorem would follow immediately.) We further put $r_\theta(u) := P_\theta u^\theta - P(u^2) + 1$, then $r_\theta(u) \geqq 0$, and we obtain for arbitrary $s \in \mathbf{C}^+$

$$\int_0^\infty r_\theta(u) \exp(-su)\, du = P_\theta \int_0^\infty u^\theta \exp(-su)\, du - \tag{9}$$

$$-\int_0^\infty \Big(\sum_{0 \leqq n \leqq u^2} r_3(n) - \frac{4\pi}{3} u^3\Big) \exp(-su)\, du + \int_0^\infty \exp(-su)\, du =$$

$$= P_\theta s^{-1-\theta} \Gamma(1+\theta) - f(s) s^{-1} + 8\pi s^{-4} + s^{-1} =$$

$$= P_\theta s^{-1-\theta} \Gamma(1+\theta) - g(s) + s^{-1}$$

where

$$f(s) = s \int_0^\infty \exp(-su) \sum_{0 \leqq n \leqq u^2} r_3(n)\, du = s \sum_{n=0}^\infty r_3(n) \int_{\sqrt{n}}^\infty \exp(-su)\, du = \tag{9'}$$

$$= \sum_{n=0}^\infty r_3(n) \exp(-s\sqrt{n})$$

and

$$g(s) = f(s) s^{-1} - \frac{4\pi}{3} \int_0^\infty u^3 \exp(-su)\, du = f(s) - 8\pi s^{-4}\,. \tag{9''}$$

It follows from (9) that (for $\sigma > 0$)

$$\int_0^\infty r_\theta(u) \exp(-\sigma u) T_X(u)\, du = P_\theta \Gamma(1+\theta) T_X \wedge I_{1+\theta}(\sigma) - T_X \wedge g(\sigma) + \\ + T_X \wedge I_1(\sigma) . \tag{10}$$

In this formula we now put (for a large parameter $X$)

$$\begin{cases} \sigma = \sigma(X) := \exp(-Aq(X)) , \\ \theta = \theta(X) := 1 + Q(X)^{-1} = 1 + b^{-X/\log X} \end{cases} \tag{11}$$

(with a suitable large absolute constant $A$) and proceed to establish asymptotic evaluations of the terms on the right side of (10). To this end we first infer from (9′) that

$$f(s) = 8\pi s \sum_{n=0}^{\infty} r_3(n)(s^2 + 4\pi^2 n)^{-2} \qquad (s \in \mathbf{C}^+) . \tag{12}$$

(For a proof cf. the analogue treated by G. H. Hardy [9], p. 266.) From this the following two assertions are easy consequences (see lemma 1 and 2 of Gangadharan [7] for details):

(I) For $0 < \sigma < \frac{1}{2}$, $1 \leqq m \leqq Y$, $m$ an integer, we have

$$\sigma^2 g(\sigma \pm 2\pi i \sqrt{m}) = -(2\pi)^{-1} r_3(m) m^{-1} + O(\sigma^2 Y) .$$

(II) Suppose that $s \in \mathbf{C}^+$, $|s| \leqq c$, $|s \pm 2\pi i \sqrt{n}| \geqq \omega$ for all $n \in \mathbf{N}$ (where $c \geqq 2\pi$, $0 < \omega < 1$), then

$$g(s) = O(\omega^{-2} c^3) .$$

From this we obtain an asymptotic expansion for the second term on the right side of (10):

PROPOSITION: $\sigma(X)^2 T_X \wedge g(\sigma(X)) = (2\pi)^{-1} \sum_{j=1}^{N(X)} r_3(q_j) q_j^{-1} + o(1).$

PROOF. Recalling (5), we first note that

$$\sigma(X)^2 1 \wedge g(\sigma(X)) = \sigma(X)^2 g(\sigma(X)) \ll \sigma(X)^2 = o(1) . \tag{13}$$

Applying (I) (with $Y = X$, $\sigma = \sigma(X)$, $m = q_j$ $(j = 1, \ldots, N(X))$ we obtain

$$(14)\qquad \sigma(X)^2 T_X^{(1)} \wedge g(\sigma(X)) = -\frac{1}{2}\sigma(X)^2 \sum_{j=1}^{N} g(\sigma(X) + 2\pi i\sqrt{q_j}) = = (4\pi)^{-1} \sum_{j=1}^{N} r_3(q_j) q_j^{-1} + o(1)\,,$$

and the same argument holds for $\overline{T}_X^{(1)}$. Finally we infer from (II) (with $c = 3\pi X^{\frac{3}{2}}$, $\omega = 2\pi \exp(-q(X))$, in view of (7)) that

$$(15)\qquad \sigma(X)^2 T_X^{(2)} \wedge g(\sigma(X)) = \sigma(X)^2 \sum_{\tau} b_\tau g(\sigma(X) + 2\pi i\beta_\tau) \ll \ll \exp(-2Aq(X)) X^{\frac{9}{2}} \exp(2q(X))\, 3^X = o(1)$$

(because of (6)) which completes the proof of our proposition.

Next we obtain as an immediate consequence of (8) and (11) that (for $X \to \infty$)

$$\sigma(X)^2 T_X \wedge I_{1+\theta(X)}(\sigma(X)) = \exp(Aq(X)/Q(X)) + o(1)$$

and

$$\sigma(X)^2 T_X \wedge I_1(\sigma(X)) = o(1)\,.$$

Entering this and the above proposition into (10) and noting that $\Gamma(1 + \theta(X)) = 1 + o(1)$ we get

$$(16)\qquad 0 \leqq \sigma(X)^2 \int_0^\infty r_{\theta(X)}(u) \exp(-\sigma(X)u)\, T_X(u)\, du =$$

$$= P_{\theta(X)}(1 + o(1))\big(\exp(Aq(X)/Q(X)) + o(1)\big) - (2\pi)^{-1} \sum_{j=1}^{N(X)} r_3(q_j) q_j^{-1} + o(1)$$

(since, by definition, $r_{\theta(X)}(u)$ and $T_X(u)$ are $\geqq 0$ for every $u > 0$).

3.2. We are now going to establish the essential auxiliary result of the whole argument:

LEMMA: $\displaystyle\sum_{j=1}^{N(X)} r_3(q_j) q_j^{-1} \gg X^{\frac{1}{2}} (\log X)^{-1}\,.$

PROOF. Suppose that $n \not\equiv 0 \pmod 4$ and write $n = m^2 q$ ($q$ square-free), then $m \equiv 1 \pmod 2$, hence $m^2 \equiv 1 \pmod 8$ and therefore $n \equiv q \pmod 8$. We make use of the following explicit formula for $r_3(n)$ due to P. T. Bateman [1], p. 99:

$$r_3(n) = \frac{16}{\pi}\sqrt{n}\,\chi(n)K(-4n)H(n) \tag{17}$$

where

$$K(-4n) = \sum_{k=1}^{\infty}\left(\frac{-4n}{k}\right)k^{-1} > 0\,,$$

$$H(n) = \prod_{p^2/n}\left(1 + \frac{1}{p} + \ldots + \frac{1}{p^b}\left\{1 - \left(\frac{-p^{-2b}n}{p}\right)\frac{1}{p}\right\}^{-1}\right)$$

($b$ denoting the largest integer such that $n \equiv 0 \pmod{p^{2b}}$, $(\alpha/\beta)$ being Jacobi's symbol for $(\alpha, \beta) = 1$ and 0 otherwise); $\chi(n) = 0$ if $n \equiv 7 \pmod 8$, $\chi(n) = 1$ if $n \equiv 3 \pmod 8$ and $\chi(n) = \frac{3}{2}$ otherwise.

According to E. Landau [11], p. 219, we have

$$K(-4n)/K(-4q) = \prod_{p/m}\left(1 - \left(\frac{-4q}{p}\right)\frac{1}{p}\right) \leqq \prod_{p/m}\left(1 + \frac{1}{p}\right) < \frac{m}{\varphi(m)} \tag{18}$$

(noting that Landau's proof holds also without the somewhat more restrictive concept of a «fundamental discriminant»); here $\varphi(m)$ is Euler's function.

Since obviously

$$H(n) \leqq \prod_{p/m}\left(1 + \frac{1}{p} + \ldots + \frac{1}{p^b}\left(1 - \frac{1}{p}\right)^{-1}\right) = \prod_{p/m}\left(1 - \frac{1}{p}\right)^{-1} = \frac{m}{\varphi(m)} \tag{19}$$

we conclude from (17), (18) and (19) that

$$r_3(n) \leqq r_3(q)\,m^3\varphi(m)^{-2} \qquad (n = m^2 q \not\equiv 0 \pmod 4)\,. \tag{20}$$

Summation by parts yields

$$S(X) := \sum_{1 \leqq n \leqq X} r_3(n)n^{-1} \sim 4\pi X^{\frac{1}{2}}\,,$$

hence, noting that $r_3(4k) = r_3(k)$, we infer that

$$\sum_{\substack{1 \leqq n \leqq X \\ n \not\equiv 0(4)}} r_3(n)\, n^{-1} = S(X) - \frac{1}{4} S\left(\frac{X}{4}\right) \sim \frac{7\pi}{2} X^{\frac{1}{2}} . \tag{21}$$

Therefore, by (20), we get

$$X^{\frac{1}{2}} \ll \sum_{1 \leqq m \leqq X} m\varphi(m)^{-2} \sum_{j=1}^{N(X)} r_3(q_j)\, q_j^{-1} . \tag{22}$$

Observing that

$$\sum_{1 \leqq m \leqq X} m\varphi(m)^{-2} \leqq \prod_{p \leqq X} \sum_{k=0}^{\infty} p^k \varphi(p^k)^{-2} \ll \prod_{p \leqq X} \left(1 - \frac{1}{p}\right)^{-1} \ll \log X$$

we complete the proof of the lemma.

3.3. In view of the result just obtained we infer from (16) that there exists a positive absolute constant $C$ such that, for sufficiently large $X$,

$$P_{\theta(X)} > C X^{\frac{1}{2}} (\log X)^{-1} .$$

Writing $L(X) = X^{\frac{1}{2}}(\log X)^{-1}$ for short, we conclude that, for each sufficiently large $X$, there exists a real number $u(X)$ satisfying

$$u(X)^{-\theta(X)} \big(P\big(u(X)^2\big) - 1\big) > C L(X) \tag{23}$$

and that necessarily $u(X) \to \infty$ for $X \to \infty$. We now put $v(X) = u(X)^{1/Q(X)}$ (thus $v(X) \geqq 1$) and obtain, noting that $Q(x)\, x^{-1}$ increases for large $x$,

$$2 \log u(X) = 2Q(X) \log v(X) \leqq \begin{cases} Q\big(2X \log v(X)\big) & \text{if } 2 \log v(X) \geqq 1 , \\ Q(X) & \text{if } 2 \log v(X) < 1 . \end{cases}$$

Let $Q^{-1}$ denote the inverse function and put $W_X = \max\{1, 2 \log v(X)\}$ then we infer from the above that $Q^{-1}\big(2 \log u(X)\big) \leqq X W_X$. Using this and (23) we conclude that (for large $X$)

$$
\begin{aligned}
&P\big(u(X)^2\big)u(X)^{-1}L\Big(Q^{-1}\big(2\log u(X)\big)\Big)^{-1} = \\
&\qquad = u(X)^{-\theta(X)}P\big(u(X)^2\big)v(X)L\Big(Q^{-1}\big(2\log u(X)\big)\Big)^{-1} > \\
&> CL(X)v(X)L(XW_X)^{-1} \geqq Cv(X)W_X^{-\frac{1}{2}} = C\min\big\{v(X),\, v(X)\big(2\log v(X)\big)^{-\frac{1}{2}}\big\} \\
&\qquad\qquad \geqq C > 0\,.
\end{aligned}
$$

Since (as noted earlier) $u(X) \to \infty$ we have thus proved that

$$
P(T) = \Omega_+\Big(T^{\frac{1}{2}}L\big(Q^{-1}(\log T)\big)\Big)\,. \tag{24}
$$

Infering (e.g. by de l'Hôpital's rule) from the definition of $Q(x)$ in (6) that $Q^{-1}(y) \sim (\log b)^{-1}\log y \ \log_2 y$, we obtain the assertion of our theorem.

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