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On Reduced Products of Abelian Groups.

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0. Introduction.

All groups in this paper are abelian. If $\mathfrak{x}$ is any class of groups, we call $\mathfrak{x}^{\perp} = \{A\colon \mathrm{Hom}(X, A) = 0$ for all $x \in \mathfrak{x}\}$ the torsion-free class generated by $\mathfrak{x}$ and dually ${}^{\perp}\mathfrak{x} = \{A\colon \mathrm{Hom}(A, X) = 0$ for all $x \in \mathfrak{x}\}$ the torsion class cogenerated by $\mathfrak{x}$. These classes have nice closure properties: Each torsion class $\mathcal{T}$ is closed with respect to quotients (Q), extensions (E) and direct sums $(\oplus)$ $(\mathcal{T} = \{Q, E, \oplus\}(\mathcal{T}))$ and the torsion-free class $\mathcal{F}$ is closed under subgroups (S), extensions and cartesian products (π): $\mathcal{F} = \{S, E, \pi\}(\mathcal{F})$.

Many important classes of groups are torsion (-free) classes. Some examples: $\left(\bigoplus_{n<\omega} \mathbb{Z}/n\mathbb{Z}\right)^{\perp}$ is the class of torsion-free groups, $\mathbb{Q}^{\perp}$ is the class of reduced groups, $\mathcal{C} = (\hat{\mathbb{Z}})^{\perp}$ is the class of cotorsion-free groups where $\hat{\mathbb{Z}}$ is the $\mathbb{Z}$-adic completion of $\mathbb{Z}$, $\mathcal{F}_1 = \{A\colon |A| = \aleph_0, \mathrm{Hom}(A, \mathbb{Z}) = 0\}^{\perp}$ is the class of all $\aleph_1$-free groups, ${}^{\perp}\mathbb{Z}$ is the class of all groups without free summands, etc.

The class $\mathcal{C} = (\hat{\mathbb{Z}})^{\perp}$ turns out to be important in [DG] and [GS]. This class of all cotorsion-free abelian groups can also be defined to be the class of all reduced and torsionfree abelian groups containing no copy of some p-adic integers.

We call the pair $T = ({}^{\perp}(\mathfrak{x}^{\perp}), \mathfrak{x}^{\perp})$ the torsion theory generated by $\mathfrak{x}$ and $T = ({}^{\perp}\mathfrak{x}, ({}^{\perp}\mathfrak{x})^{\perp})$ the torsion theory cogenerated by $\mathfrak{x}$. A natural problem in this context is to investigate whether the (co)generating class $\mathfrak{x}$ of a given torsion theory can be chosen to be a set or—equi-

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valently—a singleton $\mathfrak{x} = \{X\}$. We call these torsion theories singly (co)generated. Some progress along this line can be found in [DH], [DG] and [GS]. In [GS] it has been shown that the theory $({}^{\perp}\mathcal{C}, \mathcal{C})$ is *not* singly cogenerated. There does *not* exist a *set* $\mathfrak{x}$ of groups such that $\mathcal{C} = \{S, E, \pi\}(\mathfrak{x})$ is the smallest class containing $\mathfrak{x}$ and closed under subgroups, extensions and products. This fits nicely into the philosophy that the class of torsionfree groups is « huge ». The class $\mathcal{C}$ has one more closure property.

Let I be any index set, $\{A_i : i \in I\} \subseteq \mathcal{C}$ and F any $\varkappa$-complete filter on I (cf. [J p. 52, p. 56]) for some regular cardinal $\varkappa$. The *reduced product* $\prod_{i\in I} A_i/F$ is the cartesian product $\prod_{i\in I} A_i$ modulo the subgroup of all sequences $x = (x_i)_{i\in I} \in \prod_{i\in I} A_i$ such that $\{i \in I : x_i = 0\} \in F$. One can use an argument due to Los[L] to show that $\prod_{i\in I} A_i/F$ still belongs to $\mathcal{C}$ if $\varkappa > 2^{\aleph_0}$. Hence we call $\mathcal{C}$ $\pi/(2^{\aleph_0})^+$-closed.

Definition. A class $\mathcal{A}$ of groups is called $\pi/\varkappa$-closed, $\varkappa$ a regular cardinal, if each cartesian product πA_i, $A_i \in \mathcal{A}$, modulo a $\varkappa$-complete filter belongs to $\mathcal{A}$.

One more example is the class $\mathcal{F}_1$ of $\aleph_1$-free groups: $\mathcal{F}_1$ is $\pi/\aleph_1$-closed. It has been shown in [DG] that the torsion-free class $\mathcal{F}_1$ is not singly cogenerated. In contrast to this we will show in § 1 that $\mathcal{F}_1 = \{S, \pi/\aleph_1\}(\mathbb{Z})$ is the smallest class containing $\mathbb{Z}$ beeing $\pi/\aleph_1$-closed and closed under (pure) subgroups. This follows from 1.3 which also implies $\mathcal{C} = \{S, \pi/(2^{\aleph_0})^+\}(\{G : G \in \mathcal{C}, |G| \leqslant 2^{\aleph_0}\})$. This shows that the closure operator $\{S, \pi/(2^{\aleph_0})^+\}$ is much stronger than $\{S, E, \pi\}$.

S. Shelah [S] was the first to construct arbitrarily large indecomposable groups using some combinatorial machinery. Our result seems to indicate that one really has to do so.

We will prove in § 2 that the existence of a supercompact cardinal makes singly cogenerated torsion theories singly generated. Our methods are very elementary and adopted from model theory. They apply to much more general structures. Since the results seem to be most interesting in the case of abelian groups we don't care about the general situation.

1. $\pi/\varkappa$-closed classes of Abelian groups.

A class $\mathcal{A}$ of Abelian groups is closed under cartesian products (we call this π-closed) if for any set $\{A_i : i \in I\} \subseteq \mathcal{A}$ we have $\prod_{i\in I} A_i \in \mathcal{A}$.

For $\varkappa$ a regular cardinal, we call $\mathcal{A}$ $\pi/\varkappa$-closed if for any set $\{A_i: i \in I\} \subseteq \mathcal{A}$ and any $\varkappa$-complete filter F on I the reduced product $\prod_{i\in I} A_i/F$ still belongs to $\mathcal{A}$.

It is well-known that the class $\mathcal{F}_1$ of all $\aleph_1$-free Abelian groups is $\pi/\aleph_1$-closed. We need some more notations.

Let A be any set, $\varkappa$ a (regular) cardinal and $P_\varkappa(A) = \{B: B \subseteq A, |B| < \varkappa\}$. For $B \in P_\varkappa(A)$ define $\bar{B} = \{C \in P_\varkappa(A): B \subseteq C\}$ and $F_\varkappa(A) = \{X \subseteq P_\varkappa(A): \exists B \in P_\varkappa(A), \bar{B} \subseteq X\}$. Obviously $F_\varkappa(A)$ is a $\varkappa$-complete filter on $P_\varkappa(A)$.

This filter can be used to define compactness of $\varkappa$:

$\varkappa$ is a (strongly) compact cardinal if for all sets A one can extend $F_\varkappa(A)$ to a $\varkappa$-complete *ultra*filter on $P_\varkappa(A)$. Moreover, $\varkappa$ is supercompact if $F_\varkappa(A)$ can always be extended to a *normal* $\varkappa$-complete ultrafilter ([cf. [J, p. 398, p. 408]).

If A is an Abelian group and $B \in P(A)$ let $\langle B\rangle_*$ be a pure subgroup of A containing B. Observe that we can assume $|B| = |\langle B\rangle|$ if B is infinite.

We now define $\bar{A}_\varkappa = \prod_{B\in P_\varkappa(A)} \langle B\rangle_*/F_\varkappa(A)$.

If $\varkappa < |A|$ then $\{A\} \in F_\varkappa(A)$ so in this case $F_\varkappa(A)$ is a principal filter and we get $A \simeq \bar{A}_\varkappa$ in the natural way.

The following Lemma is model-theoretic in nature and very basic.

LEMMA 1.1. Let A be any Abelian group and $\varkappa$ a regular cardinal. Then one can embed A as a pure subgroup in $\bar{A}_\varkappa$.

PROOF. For each $a \in A$ let $\tilde{a} = (a_B)_{B\in P_\varkappa(A)} \in \prod_{B\in P_\varkappa(A)} \langle B\rangle_*$ where $a_B = a$ if $a \in \langle B\rangle_*$ and $a_B = 0$ if $a \notin \langle B\rangle_*$. This induces a homomorphism $\sigma = (a \to \tilde{a}/F_\varkappa(A))$ from A into $\bar{A}_\varkappa$. We only have to show that σ is mono and A^σ is pure in $\bar{A}_\varkappa$. Let $0 \neq a \in A$ be an element such that $\tilde{a}/F_\varkappa(A) = a^\sigma = 0$. Then we have $W = \{B \in P_\varkappa(A): a \notin \langle B\rangle_*\} \in F_\varkappa(A)$. Consequently, there exist $B \in P_\varkappa(A)$ such that $\bar{B} \subseteq W$. But $B \cup \{a\} \in \bar{B}$ and $B \cup \{a\} \notin W$, a contradiction. Hence σ is mono.

Let $a \in A$, $n \in \mathbb{Z}$ and $x = (X_B)_{B\in P_\varkappa(A)}$ such that $a^\sigma = n\cdot x/F_\varkappa(A)$. We derive $U = \{B \in P_\varkappa(A): a = nx_B\} \in F_\varkappa(A)$. Fix any $C \in U$ such that $\bar{C} \subseteq U$. Define $x' = (x'_B) \in \prod_{B\in P_\varkappa(A)} \langle B\rangle_*$ by $x'_B = x_C$ if $B \in \bar{C}$ and $x'_B = 0$ if $B \notin \bar{C}$. One easily checks that $a^\sigma = nx'/F_\varkappa(A)$ and $x'/F_\varkappa(A) = (x_C)^\sigma \in A^\sigma$. Hence A^σ is pure in $\bar{A}^\sigma$. $\square$

LEMMA 1.2. Let $\varkappa$ be any regular cardinal and $\mathfrak{x}$ any *set* of Abelian groups of cardinality $< \varkappa$. Then the class $\mathfrak{x}^\perp = \{A | \mathrm{Hom}(X, A) = 0$ for all $X \in \mathfrak{x}\}$ is $\pi/\varkappa$-closed.

PROOF. Let $\{A_i: i \in I\} \subseteq \mathfrak{x}^\perp$ and F any $\varkappa$-complete filter on I and $\varphi: X \to \prod_{i\in I} A_i/F$ for some $X \in \mathfrak{x}$. Since $|X| < \varkappa$ we derive from the Wald-Los-Lemma (cf. [DG, 2.5] or [W, 1.2] or [L, Theorem 2]) that $\varphi(X)$ can be embedded into $\prod_{i\in I} A_i$. Since $\mathfrak{x}^\perp$ is π-closed, we get $\varphi = 0$. Hence $\prod_{i\in I} A_i/F \in \mathfrak{x}^\perp$.

Putting together 1.1 and 1.2 we obtain.

THEOREM 1.3. Let $\mathfrak{x}$ be a set of Abelian groups of cardinality $< \varkappa$ $\varkappa$ regular and $\mathfrak{Y}$ a representing set of all groups of cardinality $< \varkappa$ in $\mathfrak{x}^\perp$. Then $\mathfrak{x}^\perp = \{S_*, \pi/\varkappa\}(\mathfrak{Y})$ is the smallest class containing $\mathfrak{Y}$ which is $\pi/\varkappa$-closed and closed under pure subgroups (S_*).

Remember that $\mathcal{F}_1 = (\oplus\{A: \mathrm{Hom}(A, \mathbb{Z}) = 0,\ |A| \leqslant \aleph_0\})^\perp$ is the class of all $\aleph_1$-free groups and $\mathcal{C} = \{\hat{\mathbb{Z}}\}^\perp$ is the class of all cotorsion-free groups where $\hat{\mathbb{Z}}$ is the $\mathbb{Z}$-adic completion of $\mathbb{Z}$. Hence we get the following.

COROLLARY 1.4. $\mathcal{F}_1 = \{S_*,\ \pi/\aleph_1\}(\mathbb{Z})$ and $\mathcal{C} = \{S_*,\ \pi/(2^{\aleph_0})^+\}(\mathfrak{Z})$ where $\mathfrak{Z}$ is the set of all cotorsion-free Abelian groups of cardinality $\leqslant 2^{\aleph_0}$.

REMARK 1.5. If a class of Abelian groups is $\pi/\varkappa$-closed for some regular cardinal then there exists a minimal one. In the case $\mathcal{F}_1$ of the class of all $\aleph_1$-free groups, $\aleph_1$ is this minimal one, since it is well-known that $\mathcal{F}_1$ is not $\pi/\aleph_0$-closed. If we assume that $2^{\aleph_0}$ is *regular*, then $\mathcal{C}$ is not $\pi/2^{\aleph_0}$-closed.

Let J_p be the additive group of p-adic integers. If $B \subseteq J_p$, $|B| < 2^{\aleph_0}$ is a subgroup of J_p, then by a result of Sasiada (cf. [FII, Proposition 94.2]) B is slender and hence cotorsion-free. But $J_p \subseteq (\overline{J_p})2^{\aleph_0}$ showing that $\mathcal{C}$ is not $\pi/2^{\aleph_0}$-closed.

REMARK 1.6. In contrast to 1.4 it has been shown by Göbel and Shelah [GS] that $\mathcal{C} \neq \{S, E, \pi\}(G)$, i.e. $\mathcal{C}$ cannot be obtained by closing a set of groups—or equivalently a single group—under subgroups (S), extensions (E) and products (π). The analogous result for $\mathcal{F}_1$ was proved in [DG]. It is somehow surprising that the closure operator $\{S_*,\ \pi/\varkappa\}$ is so much stronger than $\{S, E, \pi\}$. This may indicate that beyond $2^{\aleph_0}$ algebra has gone and combinatorial set theory is left.

Assuming that Vopenka's principle (VP) holds (cf. [J, p. 414]) one can easily see that all torsion-theories are singly generated (cf. [DH] or [GS]). Hence 1.3 implies that under $ZFC + (VP)$ all torsion-free classes are $\pi/\varkappa$-closed for some $\varkappa$. Since people have not succeeded so far to prove that (VP) is inconsistent, one has to add some extra axiom to ZFC if one wants to obtain a torsion-free class which is not $\pi/\varkappa$-closed for any $\varkappa$. For a regular $\varkappa$, let $\mathbb{Z}_\varkappa = \mathbb{Z}^\varkappa/\mathbb{Z}^{<\varkappa}$ be the reduced product of $\varkappa$-many copies of $\mathbb{Z}$ modulo the subgroup of all elements with support of size $< \varkappa$. If $\aleph_m$ is the first measurable cardinal, let $\mathfrak{C}_0 = \{\mathbb{Z}_\varkappa : \aleph_0 \leqslant \varkappa < \aleph_m,\ \varkappa \text{ regular}\}^\perp$ denote the class of all strongly cotorsion-free Abelian groups (cf. [DG]). Since $\mathbb{Z} \in \mathfrak{C}_0$—all slender groups belong to $\mathfrak{C}_0$—and $\mathbb{Z}_\varkappa \notin \mathfrak{C}_0$ we have immediately the

OBSERVATION 1.7 ($ZFC + \nexists\aleph_m$). The class $\mathfrak{C}_0$ is not $\pi/\varkappa$-closed for any cardinal $\varkappa$.

2. Singly generated radicals and compact cardinals.

If X is any Abelian group, let R_X be the subfunctor of the identity defined by $R_X(A) = \bigcap\{\operatorname{Ker} f : f \in \operatorname{Hom}(A, X)\}$. This functor is called the radical singly generated by X.

DEFINITION 2.1. The radical R_X satisfies the cardinal condition (c.c.) if there exists a cardinal $\varkappa$ such that $R_X = R_X^\varkappa$ where $R_X^\varkappa(A) = \Sigma\{R_X(B) : B \subseteq A,\ |B| < \varkappa\}$. Let $\aleph_m$ denote the first measurable cardinal, cf. [J, p. 297]. As was shown in [DG], $R_{\mathbb{Z}}$ does not satisfy the cardinal condition in $ZFC + \nexists\aleph_m$.

In contrast to this Alan Mekler pointed out in a private discussion at the 7th International Congress of Logic and Methodology of Science in Salzburg, July 83, how to use (if there is any) a supercompact cardinal and a cardinal-collapsing argument to prove that $R_{\mathbb{Z}}$ satisfies c.c. Since this kind of arguments are usually not accessible for people working in algebra we want to present here a very elementary proof only needing compactness.

THEOREM 2.2. Let X be any Abelian group and $\varkappa$ a compact cardinal $\varkappa < |x|$. Then R_X satisfies the cardinal condition: $R_X = R_X^\varkappa$.

PROOF. Let A be any Abelian group and $F_\varkappa(A)$ the filter on $P_\varkappa(A)$ introduced in § 1. Since $\varkappa$ is compact, there exists a $\varkappa$-complete ultrafilter F containing $F_\varkappa(A)$. Let $\bar{A} = \prod_{B \in P_\varkappa(A)} \langle B\rangle/F$. Similar to the proof of 1.1, A can be embedded via σ into $\bar{A}$. Since $R_X^\varkappa(A) \subseteq R_X(A)$ we may assume $a \in R_X(A) - R_X^\varkappa(A)$ for some $a \in A$. Hence for all $B \in P_\varkappa(A)$

with $a \in \langle B \rangle$ we have $a \notin R_X(\langle B \rangle)$ and therefore there exists $f_B: \langle B \rangle \to X$ such that $f_B(a) \neq 0$. Fix any $C \in P_\varkappa(A)$ such that $a \in \langle C \rangle$ and let $f = (f_B)_{B \in P_\varkappa(A)}$ where f_B is as above if $B \in \bar{C} = \{B \in P_\varkappa(A): C \subseteq B\}$ and $f_B = 0$ if $B \notin \bar{C}$. This f induces in a natural way a homomorphism from A into X: Let $(a_B)_{B \in P_\varkappa(A)}/F \in \bar{A}$. Since $|X| < \varkappa$, there exists exactly one $x \in X$ such that $\{B \in P_\varkappa(A): f_B(a_B) = x\} \in F$. Define $f((a_B)_{B \in P_\varkappa(A)}/F) = x$. It is a standard routine to verify that $f \in \in \mathrm{Hom}(\bar{A}, X)$.

Now compute $f(a^\sigma)$: We have $\bar{C} \subseteq \{B \in P_\varkappa(A): f_B(a) \neq 0\} \in F$.

Since F is an ultrafilter and by the definition of f, we have $f(a^\sigma) \neq 0$. Hence $a \notin R_X(A)$. We will continue on this line to prove the

THEOREM 2.3. Let X be any Abelian group and $\varkappa$ a supercompact cardinal $\varkappa > |X|$. Then ${}^\perp X = \{\oplus, Q\}(\oplus\{G: G \in {}^\perp X,\ |G| < \varkappa\})$.

PROOF. The assumption that $\varkappa$ is supercompact allows us to choose our ultrafilter F to be *normal*, which by definition means that *each* vector $(a_B)_{B \in P_\varkappa(A)}$ is constant on a set belonging to F. Hence $A \cong \bar{A}$ in this case. Taking any $A \in {}^\perp X$, we will show $A = \Sigma\{B \subseteq A: B \in \in {}^\perp X, |B| < \varkappa\}$. Assume not. Then there exists $a \in A$ such that $\langle B \rangle \notin {}^\perp X$ for all $B \in P_\varkappa(A)$ where $a \in \langle B \rangle$. Hence there exists $\bar{C} = \{B \in P_\varkappa(A): C \subseteq B\}$ such that $\langle B \rangle \notin {}^\perp X$ for all $B \in \bar{C}$. Consequently there exist $0 \neq f_B \in \mathrm{Hom}(\langle B \rangle, X)$ for all $B \in \bar{C}$ and an argument similar to the one given in the proof of 2.2 presents a homomorphism $0 \neq f: \bar{A} \cong A \to X$ and we derived the contradiction $A \notin {}^\perp X$.

Using the terminology of Gardner [G], Theorem 2.3 states that the radical class ${}^\perp X$ is bounded. Of course, this implies that ${}^\perp X$ is singly generated as a torsion class.

PROBLEM. Does the existence of a *compact* cardinal above $|X|$ imply that ${}^\perp X = \{\oplus, E, Q\}(H)$ is singly generated as a torsion class?

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