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On the Existence of Projective and Quasi-Projective Covers.

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1. Introduction.

It is worth noting when, for a module theoretical property P, the conditions

(fgP): Every finitely generated R-module has property P;

and

($1gP$): Every 1-generated (i.e. cyclic) R-module has property P;

are, in fact, equivalent. As is well known, the property P_1 of possessing a projective cover is an example of this occurance.

Hyman Bass [2] introduced the semi-perfect rings as the class of rings satisfying ($1gP_1$). He proved the equivalence of ($1gP_1$) with (fgP_1) and other ring-theoretical properties (cf. [1, p. 304, 27.6]). Further characterizations of the class of semi-perfect rings of the form (fgP) were obtained by Jonathan Golan [6, 7] and Anne Koehler [10]. We collect their results.

THEOREM 1.1 (Bass [2], Golan [6, 7], Koehler [10]). The following properties of the ring R are equivalent.

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(i) Every finitely generated R-module has a projective cover.

(ii) Every finitely generated R-module has a quasi-projective cover.

(iii) Every finitely generated quasi-projective R-module is quasi-perfect.

(iv) Every finitely generated R-module has a quasi-perfect cover.

(v) R is semi-perfect.

In this note we show that the replacement of « finitely generated » by « 1-generated », well known to be valid in (i), is also valid in (iii) and (iv), and that « finitely generated » may be replaced by « 2-generated » in (ii). We will prove

THEOREM 1.2. The following properties of the ring R are equivalent.

(i) R is semi-perfect.

(ii) Every 2-generated R-module has a quasi-projective cover.

(iii) Every 1-generated quasi-projective R-module is quasi-perfect.

(iv) Every 1-generated R-module has a quasi-perfect cover.

Koehler showed that a ring R is semi-simple artinian if and only if every finitely generated R-module is quasi-projective. Along the same lines we prove

THEOREM 1.3. The ring R is semi-simple artinian if and only if every 2-generated R-module is quasi-projective.

Neither in Theorem 1.3 nor in (ii) of Theorem 1.2 is it possible to replace « 2-generated » by « 1-generated ». This prompts us to consider, in Section 3, the class of *qpc*-rings (rings whose cyclic modules have quasi-projective covers), and Koehler's class of q^*-rings (rings all of whose cyclic modules are quasi-projective). We give a characterization of those submodules K of a ring R for which R/K is quasi-projective. This leads to a new description, in terms of the ring arithmetic, of the classes of q^*-rings and *qpc*-rings which may prove useful in the future.

An example is given to show that the class of *qpc*-rings properly contains the class of q^*-rings.

2. Proof of the reduction theorems.

Throughout, R is a ring with identity and all modules are unital left R-modules. A module is *n-generated* if, as R-module, it is generated by a subset of cardinality at most n. Thus, 1-generated equals cyclic. Mappings are written on the right.

For the definition of quasi-projective and quasi-perfect modules see [13] and [7]. We use Rangaswamy and Vanaja's definition of quasi-projective covers [12]. Since it is only the existence of quasi-projective covers that we are concerned with, the difference in terminology between [13, 6, 4] and [12] can be ignored.

Proof of Theorem 1.2. It suffices to derive (i) from each of the other conditions. To show that (ii) implies (i), let M be a cyclic R-module. Then $R \oplus M$ is 2-generated and thus, has a quasi-projective cover. By Golan [8, p. 339, 2.2(2)] M has a projective cover. This being true for every cyclic R-module proves R to be semi-perfect by definition. Assume (iii). Then R is supplemented, and a result of Friedrich Kasch and Erika Mares states that every epimorphic image of R has a projective cover [9, p. 526, SATZ]. Hence R is semi-perfect. Likewise, if (iv) is valid, there exists a quasi-perfect cover $\pi: Q \twoheadrightarrow R$ of R. Then $R \simeq Q/\ker \pi$ being projective with $\ker \pi$ small in Q implies $\ker \pi = 0$ and $R \simeq Q$. Since Q is supplemented, so is R, and the proof is again completed by [9, p. 526, SATZ].

Proof of Theorem 1.3. Let S be a submodule of R and assume that every 2-generated R-module is quasi-projective. Then $R \oplus R/S$ is quasi-projective and [8, p. 339, 2.2(1)] implies that R/S is projective. Hence S is a direct summand. This being true for every submodule S of R shows R to be semi-simple [1, p. 115, 13.9]. The proof is completed by Koehler [11, p. 656, 2.1].

3. Quasi-projective cyclic modules.

Since we regard R as a left R-module, the R-endomorphisms of R are precisely the right multiplications with elements of R. Consequently, for two left ideals K and L of R with $K \subseteq L$,

$$\varphi: R/K \to R/L$$

is an R-module homomorphism if and only if there exists $r \in R$ such that $Kr \subseteq L$ and $(x + K)\varphi = xr + L$ for all $x \in R$. Hence $r \in (L:K)$ where we define

$$(L:K) = \{r \in R | Kr \subseteq L\} .$$

Since $K \subseteq L$, we have $L + (K:K) \subseteq (L:K)$. These remarks essentially prove the following characterization of submodules K of R with R/K quasi-projective.

PROPOSITION 3.1. Let K be a submodule of R. Then R/K is quasi-projective if and only if, for all submodules L of R, whenever $K \subseteq L$, then $(L:K) = L + (K:K)$.

Immediate consequences are the following facts. The first one is well known (cf. Fuchs and Rangaswamy [5], Golan [7]).

COROLLARY 3.2. Let S be a submodule of R. In each of the following cases R/S is quasi-projective.

(i) S is a two-sided ideal of R.

(ii) S is a maximal submodule of R.

(iii) R/S has order pq where p and q are (not necessarily distinct) primes.

Another illustration of the usefulness of 3.1 is a short and transparent proof of a result by Wu and Jans [13].

PROPOSITION 3.3. If K is a small submodule of R and R/K is quasi-projective, then K is a two-sided ideal of R.

PROOF. Let $r \in R$. By 3.1 we have

$$\begin{aligned} r \in \big((Kr + K):K\big) &= Kr + K + (K:K) \\ &= Kr + (K:K) \end{aligned}$$

so that $r = kr + x$ with $k \in K$ and $Kx \subseteq K$. Hence

$$(1 - k)(r - x) = r - kr - x + kx = kx ,$$

and $1 - k$ is a unit in R since K is small. It follows that

$$R(r - x) = R(1 - k)(r - x) = Rkx \subseteq RKx = Kx \subseteq K$$

which implies

$$Kr \subseteq K(r-x) + Kx \subseteq R(r-x) + Kx \subseteq K + K = K .$$

We obtain the following characterization of q^*-rings.

THEOREM 3.4. The ring R is a q^*-ring if and only if, for each pair of submodules K and L of R, whenever $K \subseteq L$, then $(L:K) = L + (K:K)$.

Combining 3.2 and 3.3, it is immediate that a local ring (cf. [3, p. 35]) is a q^*-ring if and only if every submodule contained in the radical is a two-sided ideal (cf. Koehler [10, p. 312, 2.1]).

We turn to cyclic modules with quasi-projective covers. If

$$\varphi: Q \twoheadrightarrow R/S$$

is a quasi-projective cover then it is easy to see that there is a submodule K of R with $K \subseteq S$ such that

$$\pi: R/K \twoheadrightarrow R/S$$

is a quasi-projective cover where π is the natural map. Thus we have

THEOREM 3.5. The ring R is a qpc-ring if and only if, for each submodule S of R, there exists a submodule K of R such that 1) $K \subseteq S$, 2) S/K is small in R/K, and 3) for every submodule L of R, whenever $K \subseteq L$, then $(L:K) = L + (K:K)$.

To demonstrate that not every qpc-ring is a q^*-ring we give the following

EXAMPLE (Koehler [10]). For n an integer, let $\hat{n} \in Z_4$ and $\bar{n} \in Z_2$ be the congruence classes containing n modulo 4 and modulo 2, respectively. Let

$$R = \left\{ \begin{bmatrix} \hat{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix} \middle| \, a, b, c \in Z \right\}$$

with addition defined componentwise and multiplication given by

$$\begin{bmatrix} \hat{a} & \bar{b} \\ 0 & \bar{c} \end{bmatrix} \begin{bmatrix} \hat{d} & \bar{e} \\ 0 & \bar{f} \end{bmatrix} = \begin{bmatrix} \widehat{ad} & \overline{ae + bf} \\ 0 & \overline{cf} \end{bmatrix}.$$

Koehler has shown that R is not a (left) q^*-ring. The radical of R is

$$N = \left\{ \begin{bmatrix} \widehat{2a} & \bar{b} \\ 0 & 0 \end{bmatrix} \middle| \, a, b \in Z \right\}.$$

In order to show that R is a *qpc*-ring, let S be a submodule of R such that R/S is not quasi-projective. By 3.2, S has order 2. One verifies that then $S \subseteq N$. Hence S is small and $R \overset{\pi}{\twoheadrightarrow} R/S$ is a projective cover of R/S.

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