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Rings S-Radical Over PI-Subrings.

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1. A ring R is said to be radical over a subring A if, for every $x \in R$, there exists an integer $n(x) \geq 1$ such that $x^{n(x)} \in A$. One of the results concerning the structure of radical extensions is a result due to Herstein and Rowen. In [5] they proved: if R is a ring with no nil right ideals, radical over a subring A and A satisfies a polynomial identity, then R satisfies the same multilinear identities. In [6] Zelmanov showed that the conclusion still holds if we merely assume that R is without nil ideals.

In this paper we shall be concerned with the same problem of lifting polynomial identities in the setting of rings with involution. If R is a ring with involution and S the set of symmetric elements of R, we say that R is S-radical over a subring A if, given $s \in S$, then $s^{n(s)} \in A$ for some integer $n(s) \geq 1$.

S-radical extensions were studied in [1] where it was shown that if R is a division ring S-radical over a proper subring A then, for all $x \in R$, xx^* is central in R and so, R is at most 4-dimensional over its center.

Here we shall prove the following: let R be a prime ring with no nil right ideals and char $R \neq 2, 3$. If R is S-radical over a subring A and A satisfies a polynomial identity of degree d, then R satisfies a polynomial identity (PI) and PI-deg $(R) \leq d$.

We remark that if every element in S is nilpotent then R contains

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a nonzero nil right ideal; however it is not known if R contains a nonzero nil ideal (this is tied in with a conjecture due to McCrimmon [4]).

Throughout this paper R will denote a ring with involution $*$, Z its center and $S = \{x \in R : x = x^*\}$ and $K = \{x \in R : x = -x^*\}$, the set of symmetric and skew elements respectively. Finally $N = \{xx^* : x \in R\}$ will denote the set of norms of R.

If R is a prime ring satisfying a polynomial identity, then its ring of central quotients, Q, is a central simple algebra of dimension n^2 over its center and we define PI-deg $(R) = n$.

2. We first prove a result of independent interest which will be very useful in proving the main theorem, namely:

THEOREM 1. *Let R be a ring with no nonzero nil right ideals. If R is S-radical over a division ring A, $A \neq R$ then either*

1) *R is a direct sum of a division ring and its opposite with the exchange involution or*

2) *R is simple, $N \subseteq Z$ and $\dim_Z R \leqslant 4$.*

PROOF. Since R is also S-radical over $A \cap A^*$, we may assume $A = A^*$. Let $U = U^*$ be a proper $*$-ideal of R. Since U is proper and A is a division ring, $U \cap A = 0$. Thus $U \cap S$ consists of nilpotent elements. Let $s \in U \cap S$ be such that $s^2 = 0$. If $r \in R$, $sr + r^*s \in U \cap S$, so, for a suitable n, $0 = (sr + r^*s)^n = (sr)^n + (r^*s)^n + sys$ for some $y \in R$. Hence $(sr)^n s = 0$. This shows that sR is nil and so, $sR = 0$ consequently $s = 0$. Therefore we get $U \cap S = 0$. Let $x \in U$, then $x + x^* = 0$ implies $x = -x^* \in K$ and so $x^2 \in U \cap S = 0$. Thus every element in U is nilpotent of index 2. It follows that $U = 0$.

We have proved that R is $*$-simple. Since $J(R)$, the Jacobson radical of R, is a $*$-ideal and $J(A) = 0$, we immediately get $J(R) = 0$ that is R is semisimple. Now each $s \in S$ is either nilpotent or invertible so by ([4], Theorem 2. 3. 4) R is one of the following types:

(i) a division ring,

(ii) a direct sum of a division ring and its opposite with the exchange involution,

(iii) the 2×2 matrices over a field F, or

(iv) a commutative ring with trivial involution.

If the first case occurs, by the result of Chacron and Herstein [1] we are done. In case (ii) or (iii) we are obviously done. In case (iv) R is radical over a division ring and so by ([2], Theorem 1.1) R is a field. This completes the proof of the theorem.

We now state our main theorem.

THEOREM 2. Let R be a prime ring with involution of characteristic $\neq 2, 3$ which is S-radical over a subring A. If R has no nonzero nil right ideals and A satisfies a polynomial identity of degree d, then R satisfies a polynomial identity and PI-deg $(R) \leqslant d$.

The proof of theorem 2 requires several lemmas; we first make a few preliminary remarks and then state and prove the required lemmas.

In what follows $A \subset R$ *will be rings satisfying the hypotheses of the theorem* and $f(X_1, \ldots, X_d)$ will be a multilinear polynomial identity of degree d satisfied by A. Moreover we assume, as we may, that $A = A^*$.

We remark that, by a theorem of Giambruno [3], either $S \subseteq Z(R)$ or $Z(A) \subseteq Z(R)$. In the former case R satisfies the standard identity of degree 4 and there is nothing to show. Hence, we shall always assume that $Z(A) \subseteq Z(R)$. In particular since R is prime, every nonzero element in $Z(A)$ is regular in R.

We begin with

LEMMA 1. If A is a domain then R is PI.

PROOF. By ([4], Theorem 1.4.2) we have that $Z(A) \neq 0$. If we localize A and R at $Z(A)$ we get rings with induced involution A_1, R_1 respectively. Then R_1 has no non-zero nil right ideals and is S-radical over A_1. Moreover, since A is a domain, by ([4], Theorem 1.3.4), A_1 is a division algebra. From theorem 1 we get that either $A_1 = R_1$ or $S = S(R_1) \subseteq Z(R_1)$. In any case R_1, and so R, is PI.

LEMMA 2. If R is PI then PI-deg $(R) \leqslant d$.

PROOF. By ([4], Theorem 1.4.2), $Z(R) \neq 0$. Hence, since $Z(R)$ is S-radical over $Z(A)$, $Z(A) \neq 0$. If we localize R at $Z(R)$ and A at $Z(A) \subseteq Z(R)$, we get rings R_1, A_1 respectively. Then, by ([4], Theorem 1.4.3), R_1 is a finite dimensional central simple algebra with induced involution which is S-radical over A_1. Moreover, A_1 satisfies the polynomial identity $f(X_1, \ldots, X_d)$. Thus, in order to complete the proof of the lemma, we may assume that R is a finite dimensional central simple algebra. Therefore, $R = D_n$, the ring of $n \times n$ matrices over a division ring D, and the involution * is either symplectic or of transpose type.

Suppose first that * is symplectic. Then D is a field, moreover, since $S \not\subset Z(R)$, $n > 2$. Let e_{ij} be the usual matrix units in R. For $\alpha \in D$ and $i > 1$ odd, the elements

$$e_{11} + e_{22}\,,$$

$$e_{11} + e_{22} + \alpha(e_{1i} + e_{i+1,2})\,,$$

and

$$e_{11} + e_{22} + \alpha(e_{i1} + e_{2,i+1})$$

lie in A since they are symmetric idempotents. Hence $\alpha(e_{1i} + e_{i+1,2})$, $\alpha(e_{i1} + e_{2,i+1}) \in A$ and multiplying these elements first from the left and then from the right by $e_{11} + e_{22}$ we conclude that

$$De_{1i} + De_{i+1,2} + De_{i1} + De_{2,i+1} \subseteq A \qquad (i > 1 \text{ odd})\,. \tag{1}$$

Similarly, since for $i > 2$ even the elements

$$e_{11} + e_{22} + \alpha(e_{1i} - e_{i-1,2})$$

$$e_{11} + e_{22} + \alpha(e_{i1} - e_{2,i-1})$$

are symmetric indempotents, we obtain

$$De_{1i} + De_{i-1,2} + De_{i1} + De_{2,i-1} \subseteq A\,, \qquad (1 > 2 \text{ even})\,. \tag{2}$$

From (1) and (2) since $e_{11} + e_{22} \in A$, it follows that $De_{ij} \subseteq A$ for all i, j. Thus $A = R$ and we are done.

Suppose now that * is of transpose type, that is, there exists an invertible diagonal matrix $C = \operatorname{diag}\{c_1, \ldots, c_n\} \in D_n$ with $c_i = c_i^* \in D$ such that $(x_{ij})^* = C(x_{ji}^*)C^{-1}$ for all $(x_{ij}) \in D_n$. In this case e_{ii} $(i = 1, \ldots, n)$ is a symmetric idempotent and so lies in A.

We claim that for every e_{ij} there exists $0 \neq \alpha = \alpha_{ij} \in Z$, the center of D, such that $\alpha \cdot e_{ij} \in A$. Since A is a subring and $e_{ii} \in A$ $(i = 1, \ldots, n)$, it is enough to show that this holds for $e_{i,i+1}$ and $e_{i+1,i}$ $(i = 1, \ldots, n-1)$.

Moreover, since * restricted to the diagonal 2×2 block $De_{ii} + De_{i,i+1} + De_{i+1,i} + De_{i+1,i+1}$ is still an involution of transpose type, in order to prove the claim, we may assume that $R = D_2$.

Now, since D is S-radical over $A \cap D$, it follows by [1] that either $S(D) \subseteq Z$ or $D \subseteq A$. Moreover, by [3], since $e_{11} \notin Z$, there exists $s \in S$

such that, for some k, $e_{11}s^k \neq s^k e_{11}$ and $s^k \in A$. In particular $s^k = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not a diagonal matrix, say $b \neq 0$.

If $S(D) \subseteq Z$, then $C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \in Z_2$ and $*$ induces an involution on Z_2. Thus, in this case we may assume that $s \in Z_2$. Hence $e_{11}s^k e_{22} = be_{12} \in A$ and $(be_{12})^* = b' e_{21} \in A$ with b, $b' \in Z$.

On the other hand, if $D \subseteq A$, $be_{12} = e_{11}s^k e_{22} \in A$ and $e_{12} \in A$. Hence $c_2 c_1^{-1} e_{21} = e_{12}^* \in A$ and e_{21} lies also in A. Thus the claim is established; in other words, there exist $0 \neq \alpha_{ij} \in Z$ such that $\alpha_{ij} e_{ij} \in A (i, j = 1, \ldots, n)$.

Now, if $D \subseteq A$, then clearly $D_n = A$ and there is nothing to prove. Therefore we may assume that $S(D) \subseteq Z$ and so PI-deg $(D_n) \leqslant 2n$.

Let f be the multilinear identity for A of degree d. If $d < 2n$, then

$$f(\alpha_{11}e_{11}, \alpha_{12}e_{12}, \alpha_{22}e_{22}, \ldots) \neq 0 ,$$

a contradiction. Hence $d \geqslant 2n \geqslant$ PI-deg (D_n) and the lemma is proved.

LEMMA 3. *If R satisfies a generalized polynomial identity (GPI), then R is PI and PI-deg $(R) \leqslant d$.*

PROOF. Suppose that R is not a PI ring. Then, by a theorem of Montgomery ([4], Corollary to Theorem 2.5.1), for every positive integer n, R contains a $*$-subring $R^{(n)}$ which is a prime PI ring with PI-deg $(R^{(n)}) \geqslant n$. But $R^{(n)}$ is S-radical over $R^{(n)} \cap A$ and $R^{(n)} \cap A$ satisfies the polynomial identity $f(X_1, \ldots, X_d)$ of degree d. By Lemma 2, $d \geqslant$ PI-deg $(R^{(n)}) \geqslant n$, for every positive integer n, a contradiction. Thus R is PI and by Lemma 2, PI-deg $(R) \leqslant d$.

We are finally able to prove our main theorem.

PROOF OF THEOREM 2. Since, by assumption, $S \not\subseteq Z(R)$, by ([4], Theorem 2.2.1), either S contains non-zero nilpotent elements or the involution is positive definite, that is $xx^* = 0$ in R forces $x = 0$.

Suppose first that there exists $s \neq 0$ in S with $s^2 = 0$. If $x \in R$, let $n(x, s) \geqslant 1$ be such that $(sx + x^*s)^{n(x,s)} \in A$ and let A_1 be the subring of R generated by all $(sx)^{n(x,s)}$, $x \in R$. Then $R_1 = s\ R$ is radical over A_1. Now, if $b \in A_1$, say

$$b = \sum (sx_{i_1})^{n_{i_1}} (sx_{i_2})^{n_{i_2}} \ldots (sx_{i_k})^{n_{i_k}}$$

then, since $s^2 = 0$,

$$bs = \sum (sx_{i_1} + x_{i_1}^* s)^{n_{i_1}} (sx_{i_2} + x_{i_2}^* s)^{n_{i_2}} \ldots (sx_{i_k} + x_{i_k}^* s)^{n_{i_k}} \cdot s = as$$

where $a \in A$. From this it easily follows that if $b_1, \ldots, b_d \in A_1$ then $(b_1 \ldots b_d)s = (a_1 \ldots a_d)s$ where $a_1, \ldots, a_d \in A$. Hence,

$$\begin{aligned} f(b_1, \ldots, b_d)s &= \sum \alpha_\sigma b_{\sigma(1)} \ldots b_{\sigma(d)} s \\ &= \sum \alpha_\sigma a_{\sigma(1)} \ldots a_{\sigma(d)} s \\ &= f(a_1, \ldots, a_d)s = 0 . \end{aligned}$$

In other words A_1 satisfies the polynomial identity $f(X_1, \ldots, X_d)X_{d+1}$.

Let $R_2 = R_1/N(R_1)$ where $N(R_1)$ is the nil radical of R_1. Since R has no non-zero nil right ideals, neither does R_2. Moreover, R_2 is radical over A_2, the image of A_1 in R_2. Since A_1, and so A_2, satisfies $f(X_1, \ldots, X_d)X_{d+1}$ by [5], R_2 also satisfies $f(X_1, \ldots, X_d)X_{d+1}$. Therefore R satisfies a GPI and by Lemma 3 the result follows.

Suppose now that * is positive definite. We proceed by induction on the degree of the multilinear polynomial identity $f(X_1, \ldots, X_d)$ satisfied by A.

Since * is positive definite, A is semiprime. Moreover, since the center of a prime ring is a domain, $Z(A) \subseteq Z(R)$ is also a domain. But in a semiprime PI-ring, every ideal hits the center non trivially ([4], Corollary to Theorem 1.4.2), therefore A is prime.

If A has no non-zero nilpotent elements, then A is a domain and we are done by Lemma 1. Hence we may assume that there exists $a \neq 0$ in A with $a^2 = 0$.

Let $R' = aRa^*$; then R' is a *-subring of R, S-radical over $A' = aRa^* \cap A$, and, since * is positive definite, R' is a prime ring.

Let

$$f(X_1, \ldots, X_d) = X_d h(X_1, \ldots, X_{d-1}) + g(X_1, \ldots, X_d)$$

where X_d never appears as first variable in any monomial of g. Since $a^2 = 0$, if $x_1, \ldots, x_{d-1} \in A'$ and $x_d \in A$, we have

$$0 = af(x_1, \ldots, x_{d-1}, x_d) = ax_d h(x_1, \ldots, x_{d-1})$$

Hence $aAh(x_1, \ldots, x_{d-1}) = 0$ and, since $a \neq 0$, the primeness of A forces $h(x_1, \ldots, x_{d-1}) = 0$. In other words A' satisfies $h(x_1, \ldots, x_{d-1})$. By our induction hypothesis, R' is PI. From this we get that R satisfies a GPI. By Lemma 3, the result follows.

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