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Group graded rings and smash products

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Group Graded Rings and Smash Products.

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SUNTO - Mediante una nuova caratterizzazione del Smash prodotto si dimostrano direttamente e in forma un po' più generale i teoremi di dualità di Cohen e Montgomery.

Introduction.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a k-algebra graded by a finite group G, with R_e the component corresponding to the identity element e of G (k is a commutative ring). In the paper [1] M. Cohen and S. Montgomery define the ring $R \# k[G]^*$, called the «Smash product» associated to the graded ring R. This ring may be used to obtain many properties of the graded ring R. The main tools are provided by the two Duality Theorems: Duality Theorem for Action and Duality Theorem for Coaction (see Theorem 3.2 and 3.5 of [1]). In this paper we give a new characterization of the Smash Product $R \# k[G]^*$ (Theorems 1.2 and 1.3) and we deduce directly from it and in a little more general form Cohen and Montgomery Duality Theorems (Theorems 2.2 and 2.3).

1. The rings $\mathrm{End}_{R\text{-gr}}(U)$, $\mathrm{End}_R(U)$ and their structure.

If $R = \bigoplus_{\sigma \in G} R_\sigma$ is a k-algebra graded by a finite group G, we denote by R-gr the category of all unital right graded R-modules. If $M = \bigoplus_{\sigma \in G} M_\sigma$, $N = \bigoplus_{\sigma \in G} N_\sigma$ are two objects of R-gr, then the morphisms in R-gr are R-homomorphisms $f: M \to N$ such that $f(M_\sigma) \subset f(N_\sigma)$ for all $\sigma \in G$. It is well known that R-gr is a Grothendieck category (see [2]).

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If $M = \bigoplus_{\lambda \in G} M_\lambda$ is a graded R-module and $\sigma \in G$, then $M(\sigma)$ is the graded module obtained from M by putting $M(\sigma)_\lambda = M_{\lambda\sigma}$; the graded module $M(\sigma)$ is called the *σ-suspension* of M [2]. It is well-known that the mapping $M \to M(\sigma)$ defines a functor from R-gr to R-gr which is an equivalence of categories for all $\sigma \in G$. $M \in R$-gr is said to be *G-invariant* [2] if for all $\sigma \in G$, $M \simeq M(\sigma)$ in R-gr. Consider now the graded modules M and N. A R-linear mapping $f\colon M \to N$ is said to be a graded morphism of degree τ, $\tau \in G$, if $f(M_\sigma) \subset N_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree τ build up an additive subgroup $\mathrm{HOM}_R(M, N)_\tau$ of $\mathrm{Hom}_R(M, N)$. It is clear that $\mathrm{HOM}_R(M, N) = \bigoplus_{\tau \in G} \mathrm{HOM}_R(M, N)_\tau$ is a graded abelian group of type G and $\mathrm{HOM}_R(M, N)_e = \mathrm{Hom}_{R\text{-gr}}(M, N)$. In particular, if $M = N$, then $\mathrm{HOM}_R(M, N) = \mathrm{END}_R(M)$ is a graded ring of type G. In the sequel we will denote by $U = \bigoplus_{\sigma \in G} R(\sigma)$. Since $\{R(\sigma)\}_{\sigma \in G}$ is a family of generators for R-gr [2], it follows that U is a generator for R-gr. When G is a finite group, we also remark that $\mathrm{END}_R(U) = \mathrm{End}_R(U)$ (see [2]).

PROPOSITION 1.1. [2] *If $G = \{g_1 = e, g_2, \ldots, g_n\}$ is a finite group, then the ring* $\mathrm{End}_R(U)$ *is isomorphic to the matrix ring* $M_n(R)$ *equipped with the grading*

$$M_n(R) = \bigoplus_{\lambda \in G} M_n(R)_\lambda,$$

where

$$M_n(R)_\lambda = \begin{pmatrix} R_{g_1\lambda g_1^{-1}} & R_{g_1\lambda g_2^{-1}} \cdots & R_{g_1\lambda g_n^{-1}} \\ R_{g_2\lambda g_1^{-1}} & R_{g_2\lambda g_2^{-1}} \cdots & R_{g_2\lambda g_n^{-1}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_{g_n\lambda g_1^{-1}} & R_{g_n\lambda g_2^{-1}} \cdots & R_{g_n\lambda g_n^{-1}} \end{pmatrix}.$$

In particular, the ring $\mathrm{End}_{R\text{-gr}}(U)$ *is isomorphic to the matrix ring*

$$\begin{pmatrix} R_e & R_{g_1 g_2^{-1}} \cdots & R_{g_1 g_n^{-1}} \\ R_{g_2 g_1^{-1}} & R_e \quad \cdots & R_{g_2 g_n^{-1}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ R_{g_n g_1^{-1}} & R_{g_n g_2^{-1}} \cdots & R_e \end{pmatrix}.$$

PROOF. See Corollary I.5.3 of [2].

By an *action* of a group G on a k-algebra A we mean a group morphism $\alpha: G \to \mathrm{Aut}_k(A)$; let α_g denote the image of $g \in G$ in $\mathrm{Aut}_k(A)$. We may define the *skew group ring* (or *trivial crossed product*) denoted by $A * G$, as being the free right and left A-module with basis $\{g: g \in G\}$ and with moltiplication given by $(ag)\cdot(bh) = a\alpha_g(b)gh$, where $a, b \in A$, $g, h \in G$. The ring $A * G$ is a graded ring: $A * G = \bigoplus_{\sigma \in G} (A * G)_\sigma$, where $(A * G)_\sigma = A\sigma = \{a\sigma: a \in A\}$.

THEOREM 1.2. *If $G = \{g_1 = e, g_2, \dots, g_n\}$ is a finite group, then the ring $\mathrm{End}_R(U)$ is isomorphic to the skew group ring $\mathrm{End}_{R\text{-gr}}(U) * G$.*

PROOF. By Proposition 1.1 we have that $\mathrm{End}_R(U)$ is isomorphic to the matrix ring $M_n(R)$. We consider the set

$$U_\lambda = \begin{pmatrix} 0\cdots 0 & R_e & 0\cdots 0 \\ 0\cdots R_e & 0 & 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots & R_e & \cdots 0 \end{pmatrix}$$

where on the first row R_e is on the k_1-th position, k_1 being such that $g_{k_1} = g_1\lambda$; on the second row R_e is on the k_2-th position, where k_2 is such that $g_{k_2} = g_2\lambda$; ...; on the n-th row R_n is on the k_n-th position, where $g_{k_n} = g_n\lambda$. Since G is a group it is easy to see that $\{1, 2, \dots, n\} = (k_1, k_2, \dots, k_n\}$. Moreover one may see that $U_\lambda \subset M_n(R)_\lambda$. Let $u_\lambda \in U_\lambda$,

$$u_\lambda = \begin{pmatrix} 0\cdots 0 & 1 & 0\cdots 0 \\ 0\cdots 1 & 0\cdots & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots & 1 & \cdots\cdots 0 \end{pmatrix}.$$

From the definition of U_λ it follows that u_λ has a 1 on each column and all the other entries are 0. We will show now that the system $\{u_\lambda\}_{\lambda \in G}$ has the property that $u_\lambda u_\mu = u_{\lambda\mu}$, for all $\lambda, \mu \in G$. Let

$$u_\mu = \begin{pmatrix} 0\cdots & 1 & \cdots\cdots 0 \\ 0\cdots 1 & 0 & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots & \cdots 1 & 0 \end{pmatrix}$$

where 1 in the first row is on the l_1-th column, where $g_{l_1} = g_1\mu$; 1 in the second row is on the l_2-th column, where $g_{l_2} = g_2\mu$; ...; 1 in the n-th row is on the l_n-th column, where $g_{l_n} = g_n\mu$.

There exists a unique column, say the s-th, such as its intersection with the k_1-th row has a 1 and the rest of its entries are zero. Thus we have that $g_s = g_{k_1}\mu$. Since $g_{k_1} = g_1\lambda$, then $g_s = g_1\lambda\mu$ and so in the matrix $u_\lambda u_\mu$ we have 1 on the first row in the s-th position, all the other entries of the first row being zero. Hence the first row of the matrix $u_\lambda u_\mu$ is the same as the first row of the matrix $u_{\lambda\mu}$. Using the same argument for the other rows, we deduce that $u_\lambda u_\mu = u_{\lambda\mu}$. In particular, since u_e is equal to the unit matrix, we obtain that $u_\lambda^{-1} = u_{\lambda^{-1}}$, for each $\lambda \in G$. Using now Theorem 5.3.23 of [2], we obtain that $\mathrm{End}_R(U)$ is isomorphic to the skew group ring $\mathrm{End}_{R\text{-gr}}(U) * G$. Q.E.D.

Let now $A = \bigoplus_{g\in G} A_g$ be a graded k-algebra, where k is a commutative ring and G is a finite group. By [1], the construction of the smash product $A \# k[G]^*$ is the following: $A \# k[G]^*$ is the free left and right A-module with basis $\{p_g : g \in G\}$, a set of orthogonal idempotents whose sum is 1 and with the moltiplication given by the rule: $(ap_g) \cdot (bp_h) = ab_{gh^{-1}}p_h$, $a, b \in A$.

THEOREM 1.3. *Let* $R = \bigoplus_{\sigma\in G} R_\sigma$ *be a graded k-algebra, where* $G = \{g_1 = e, g_2, \ldots, g_n\}$ *is a finite group. Then the ring* $\mathrm{End}_{R\text{-gr}}(U)$ *is isomorphic to the smash product*

$$R \# k[G]^* .$$

PROOF. We have seen that the ring $\mathrm{End}_{R\text{-gr}}(U)$ is isomorphic to the matrix ring

$$T = \begin{pmatrix} R_e & R_{g_1g_2^{-1}} \cdots R_{g_1g_n^{-1}} \\ R_{g_2g_1^{-1}} & R_e \quad \cdots R_{g_2g_n^{-1}} \\ \cdots & \cdots\cdots\cdots\cdots \\ R_{g_ng_1^{-1}} & R_{g_ng_1^{-1}} \cdots R_e \end{pmatrix}.$$

Define $\varphi : R \to T$ as follows: if $a = \sum_{g\in G} a_g$, $a_g \in R_g$ for all $g \in G$, then put

$$\varphi(a) = \begin{pmatrix} a_e & a_{g_1g_2^{-1}} \cdots a_{g_1g_n^{-1}} \\ a_{g_2g_1^{-1}} & a_e \quad \cdots a_{g_2g_n^{-1}} \\ \cdots & \cdots \\ a_{g_ng_1^{-1}} & a_{g_ng_2^{-1}} \cdots a_e \end{pmatrix}.$$

It is clear that φ is additive and injective. It is straightforward to check that if $b \in R$, then we have $\varphi(ab) = \varphi(a)\varphi(b)$. Consequently, φ is a ring morphism. Let $S = \varphi(R)$. Hence S is a subring of T. Consider the elements:

$$p_{g_k} = \begin{pmatrix} 0 \cdots\cdots\cdots 0 \\ \cdots\cdots\cdots \\ 0 \cdots 0 \;\; 1 \cdots 0 \\ 0 \cdots\cdots\cdots 0 \end{pmatrix} \leftarrow k\text{-th row}$$

$\uparrow$ k-th column

i.e. p_{g_k} is a matrix with 1 at the intersection of the k-th row with the k-th column, all its other entries being zero. Then the system of elements $\{p_{g_k}\}_{1 \leq k \leq n}$ is a system of orthogonal idempotents whose sum is 1. It is clear that

$$T = Sp_{g_1} + Sp_{g_2} + \ldots + Sp_{g_n} = p_{g_1}S + p_{g_2}S + \ldots + p_{g_n}S$$

Let us prove that $\{p_{g_k}\}$, $1 \leq k \leq n$, is a linear independent system over the ring S. Let $\sum_{k=1}^{n} s_k p_{g_k} = 0$, where $s_k \in S$. Hence

$$s_k = \begin{pmatrix} a_e^k & a^k_{g_1g_2^{-1}} \cdots a^k_{g_1g_n^{-1}} \\ a^k_{g_2g_1^{-1}} & a_e^k \quad \cdots a^k_{g_2g_n^{-1}} \\ \cdots & \cdots \\ a^k_{g_ng_1^{-1}} & a^k_{g_ng_2^{-1}} \quad a_e^k \end{pmatrix}.$$

Then

$$\sum_{k=1}^{n} s_k p_{g_k} = \begin{pmatrix} a_e^1 & 0\cdots 0 \\ a^1_{g_2 g_1^{-1}} & 0\cdots 0 \\ \cdots & \cdots \\ a_{g_n^1 g_1^{-1}} & 0\cdots 0 \end{pmatrix} + \begin{pmatrix} 0 & a^2_{g_1 g_2^{-1}} & 0\cdots 0 \\ 0 & a_e^2 & 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & a^2_{g_n g_2^{-1}} & 0\cdots 0 \end{pmatrix} + \cdots +$$

$$+ \begin{pmatrix} 0\cdots 0 & a^n_{g_1 g_n^{-1}} \\ 0\cdots 0 & a^n_{g_2 g_n^{-1}} \\ \cdots & \cdots \\ 0\cdots 0 & a_e^n \end{pmatrix}$$

and hence $a_e^k = a^k_{g_1 g_2^{-1}} = \ldots = a^k_{g_1 g_n^{-1}} = 0$, for all k, $1 \leqq k \leqq n$. Thus $s_1 = s_2 = \ldots = s_n = 0$. The ring S is a graded ring of type G with the grading $\{S_g : g \in G\}$, where

$$S_g = \begin{pmatrix} 0\cdots & R_g & 0\cdots 0 \\ 0 & R_g & \cdots & 0\cdots 0 \\ \cdots & \cdots & \cdots \\ 0\cdots\cdots\cdots & R_g\cdots \end{pmatrix}$$

and R_g is on the first row on the k_1-th position, k_1 being such that $g = g_1 g_{k_1}^{-1}$, on the second row on the k_2-th position such that $g = g_2 g_{k_2}^{-1}$, ..., on the n-th row on the k_n-th position such that $g = g_n g_{k_n}^{-1}$. Thus it is clear that $\varphi : R \to S$ becomes an isomorphism of graded rings. To finish the proof we need to show that:

$$(sp_g)(tp_h) = st_{gh^{-1}} p_h, \text{ for all } s, t \in S \, .$$

To see this let $g = g_m$, $h = g_\varrho$ and

$$t = \begin{pmatrix} b_e & b_{g_1 g_2^{-1}} \cdots b_{g_1 g_n^{-1}} \\ b_{g_2 g_1^{-1}} & b_e \quad \cdots b_{g_2 g_n^{-1}} \\ \cdots & \cdots \\ b_{g_n g_1^{-1}} & b_{g_n} \quad \cdots b_e \end{pmatrix}.$$

We have that

$$p_{g_m} t = \begin{pmatrix} 0 & 0 \cdots 0 & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & \cdots 0 \\ b_{g_m g_1^{-1}} & b_{g_m g_2^{-1}} \cdots & b_{g_m g_n^{-1}} \\ \cdots & \cdots & \cdots \\ 0 & 0 \cdots 0 & \cdots 0 \end{pmatrix} \leftarrow m\text{-th row}$$

and thus

$$p_{g_m}(tp_{g_l}) = \begin{pmatrix} 0 \cdots 0 \cdots & & 0 \\ \cdots & \cdots & \cdots \\ 0 \cdots & b_{g_m g_l^{-1}} & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 \cdots & 0 & \cdots 0 \end{pmatrix} \leftarrow m\text{-th row}.$$

$\uparrow$ l-th column

On the other hand, since $t_{g_m g_l^{-1}}$ is a matrix which has on the first row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_1-th position, for which $g_m g_l^{-1} = g_1 g_{k_1}^{-1}$, on the second row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_2-th position, where $g_m g_l^{-1} = g_2 g_{k_2}^{-1}$, ..., on the n-th row a single non zero element, $b_{g_m g_l^{-1}}$, on the k_n-th position such that $g_m g_l^{-1} = g_n g_{k_n}^{-1}$, it is clear that $k_m = 1$ and $k_s \neq 1$ for $s \neq m$, $1 \leqq s \leqq n$. We deduce that

$$t_{g_m g_l^{-1}} p_{g_l} = \begin{pmatrix} 0 \cdots & \cdots & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 \cdots & b_{g_m g_l^{-1}} & \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 \cdots & 0 & \cdots 0 \end{pmatrix} \leftarrow m\text{-th row}$$

$\uparrow$ l-th column

and thus $p_g(tp_h) = t_{gh^{-1}} p_h$, so clearly $(sp_g)(tp_h) = st_{gh^{-1}} p_h$, for all s, $t \in S$ and g, $h \in G$.

2. Cohen-Montgomery duality theorems.

The notation in this section will be the same as the one in section 1.

THEOREM 2.1. *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a graded ring of type* G, *where* G *is a finite group. If we denote* $U = \bigoplus_{\sigma \in G} R(\sigma)$, *then the functor* $M \to \to \mathrm{Hom}_{R\text{-gr}}(U, M)$ *is an equivalence from the category* R-gr *to the category* $\mathrm{End}_{R\text{-gr}}(U)$-mod.

PROOF. Since $\{R(\sigma)\}_{\sigma \in G}$ is a set of generators for R-gr [2], then U is a generator for R-gr. On the other hand, U is a finitely generated projective R-module. Hence U is a small projective generator for R-gr and therefore, after a classical result of B. Mitchel (see [3]), the functor $M \to \mathrm{Hom}_{R\text{-gr}}(U, M)$ is an equivalence between the categories R-gr and $\mathrm{End}_{R\text{-gr}}(U)$-mod. Q.E.D.

REMARKS:

1) Bearing in mind Theorem 1.2, Theorem 2.1 is nothing else that Theorem 2.2 of [1].

2) If $M = \bigoplus_{\sigma \in G} M_\sigma$, then $\mathrm{Hom}_{R\text{-gr}}(U, M) \simeq \bigoplus_{\sigma \in G} \mathrm{Hom}_{R\text{-gr}}(R(\sigma), M)$. Since $\mathrm{Hom}_{R\text{-gr}}(R(\sigma), M) = M_\sigma$, then $\mathrm{Hom}_{R\text{-gr}}(U, M) \simeq M$.

Let $R = \bigoplus_{\sigma \in G} R_\sigma$ be a graded ring. R is called a *crossed product* if for every $\sigma \in G$ there exists a homogeneous invertible element $u_\sigma \in R_\sigma$. The structure of crossed products is given in Theorem I.3.23 of [2].

THEOREM 2.2. *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a graded* k*-algebra which is a crossed product, where* G *is a finite group with* $n = |G|$. *Then*

$$R \# k[G]^* \simeq M_n(R_e) .$$

PROOF. Since R is a crossed product, then $R \simeq R(\sigma)$ in the category R-gr for any $\sigma \in G$. Therefore $U = \bigoplus_{\sigma \in G} R(\sigma) \simeq R^{(n)}$.

LEMMA. *Let* $\mathcal{C}$ *be an abelian category and* M *an object of* $\mathcal{C}$. *Then, for* $n > 0$

$$\mathrm{End}_{\mathcal{C}}(M^{(n)}) \simeq M_n(\mathrm{End}_{\mathcal{C}}(M)) .$$

PROOF. Straightforward (see [3]).

We may use the Lemma to obtain that $\mathrm{End}_{R\text{-gr}}(U) \simeq M_n(\mathrm{End}_{R\text{-gr}}(R))$. But $\mathrm{End}_{R\text{-gr}}(U) \simeq R_e$ and therefore $\mathrm{End}_{R\text{-gr}}(U) \simeq M_n(R_e)$. We apply now Theorem 1.3 and obtain that

$$R \# k[G]^* \simeq M_n(R_e). \quad \text{Q.E.D.}$$

REMARK. This result is a slight extension of Theorem 3.2 (Duality Theorem for Action) of Cohen and Montgomery [1], which is given in the case $R = S * G$ is a skew group ring.

THEOREM 2.3. (Duality for Coactions) (see [1]). *Let* $R = \bigoplus_{\sigma \in G} R_\sigma$ *be a graded k-algebra, where* G *is a finite group with* $n = |G|$. *Then*

$$(R \# k[G]^*) * G \simeq M_n(R) .$$

PROOF. By Theorem 1.2 and Theorem 1.3 we have that $\mathrm{End}_R(U) \simeq$ $\simeq (R \# k[G]^*) * G$. Since $U = \bigoplus_{\sigma \in G} R(\sigma)$, then in the category R-mod it is $U \simeq R^{(n)}$. Therefore

$$\mathrm{End}_R(U) \simeq \mathrm{End}_R(R^{(n)}) \simeq M_n(R) . \quad \text{Q.E.D.}$$

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