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Periodic Solutions of Asymptotically Linear Systems without Symmetry.

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0. Introduction.

Consider the nonautonomous Hamiltonian system of $2n$ differential equations

$$-J\dot{z} = H_z(t, z) \tag{0.1}$$

where $H \in C^1(\mathbb{R}^{2n+1}, \mathbb{R})$, $H(t, z)$ is T-periodic in t, $z \in \mathbb{R}^{2n}$, $H_z = \partial H/\partial z$, $\cdot$ denotes d/dt and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, I being the identity matrix in $\mathbb{R}^n$.

In this paper we are concerned with the existence of T-periodic solutions of (0.1).

Many authors have studied this problem when H is superquadratic, i.e. $H(z)/|z|^2 \to +\infty$ as $|z| \to +\infty$ (cf. [7]. [9], [10], [19], [22]) or when H is subquadratic, i.e. $H(z)/|z|^2 \to 0$ as $|z| \to +\infty$ (cf. [6], [7], [10], [12]).

Here we assume that $H(t, z)$ is asymptotically quadratic, i.e. there exists a symmetric matrix $2n \times 2n$ $b_\infty(t)$ for any $t \in [0, T]$ such that

$$(H_1) \quad \begin{cases} H_z(t, z) = b_\infty(t)z + g(t, z) \\ \text{where} \\ g(t, z)/|z| \to 0 \quad \text{as } |z| \to +\infty \quad \text{uniformly in } t \in \mathbb{R}. \end{cases}$$

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Denote by L_∞ the «linearized operator at infinity» i.e. $L_\infty z = -J\dot{z} - b_\infty(t)z$ (for a more precise definition see section 1). Then we say that (0.1) is not resonant if

$$(H_2) \qquad\qquad 0 \notin \sigma(L_\infty),$$

$\sigma(L_\infty)$ being the spectrum of L_∞.

On the contrary we shall say that (0.1) is resonant if

$$(H_3) \qquad\qquad 0 \in \sigma(L_\infty).$$

We recall that asymptotically linear autonomous Hamiltonian systems have been studied in [2], [3], [8].

Nonautonomous and asymptotically linear Hamiltonian systems have been studied in [1], [2], [3], [15], [16] under the non resonance assumption H_2).

Here we study the existence of T-periodic solutions of (0.1) with the assumption (H_1) both in the non resonant case and in the resonant case (cf. th. 1.1, 1.2 and 1.3).

The proof of theorems is based on an abstract critical point theorem contained in [12] (cf. th. 1.4 in section 1).

In the second section we shall look for T-periodic solutions of the nonlinear wave equation:

$$(0.2) \qquad \begin{cases} u_{tt} - u_{xx} = f(x, t, u) & \\ u(x, t) = u(x, t+T) & \text{for any } t \in \mathbf{R},\ x \in [0, \pi] \\ u(0, t) = u(\pi, t) & \text{for any } t \in \mathbf{R} \end{cases}$$

where

$$(W_1) \qquad \begin{cases} T \text{ is a rational multiple of } \pi,\ f \in C^1([0, \pi] \times \mathbf{R} \times \mathbf{R}, \mathbf{R}) \\ \text{and } f \text{ is } T\text{-periodic in } t. \end{cases}$$

(0.2) has been studied in the case where $f(x, t, \cdot)$ is monotonic (cf. [1], [2], [9], [11], [21]) and without monotonicity assumption of f (cf. [5], [17], [20], [24]). Here we study (0.2) in the asymptotically linear case, i.e. we suppose that there exists a real continuous function

$a_\infty(x, t)$ defined on $[0, \pi]\times\mathbf{R}$ such that

$$(W_2)\qquad \begin{cases} f(x, t, u) = a_\infty(x, t)u + g(x, t, u) \\ \text{where} \\ g(x, t, u)/|u| \to 0 \quad \text{as } |u| \to +\infty \quad \text{uniformly in} \\ (x, t) \in [0, \pi]\times\mathbf{R}\,. \end{cases}$$

Using a trick contained in [17] and theorem 1.4 we prove, under suitable assumptions, the existence of T-periodic solutions of (0.2), without assuming that f is monotonic (cf. th. 2.5 and 2.9).

1. Consider the sistem (0.1). First of all we shall prove the following theorem:

THEOREM 1.1. *If H_1), H_2) hold, then* (0.1) *has at least one T-periodic solution.*

The solution we find in this theorem can be constant. If we suppose that 0 is an equilibrium point of the Hamiltonian vector field, it is interesting to find T-periodic and nontrivial solutions.

Precisely, we shall require that

$$(H_4)\qquad H(t, 0) = H_z(t, 0) = 0\,, \quad H \text{ is } C^2 \quad \text{in } z = 0\,.$$

In this case H_z can be written

$$H_z(t, z) = b_0(t)z + o(|z|) \quad \text{as } |z| \to 0$$

where

$$b_0(t) = b_\infty(t) + g_z(t, 0)\,.$$

We set

$$G(t, z) = H(t, z) - \tfrac{1}{2}\big(b_\infty(t)z, z\big)_{\mathbf{R}^{2n}}\,.$$

We shall denote by λ_1^∞ (resp. λ_{-1}^∞) the smallest positive (resp. the greatest negative) eigenvalue of L_∞ in L^2 and by $\tilde\lambda_1^\infty$ and $\tilde\lambda_{-1}^\infty$ the analogous in $W^{\frac12}$ (cf. the following for definition of $W^{\frac12}$). The following theorem holds:

THEOREM 1.2. *Under the assumptions* H_1), H_2), H_4) *and*

$$(H_5) \qquad G(t, z) \leqslant 0\,, \qquad \forall t \in \mathbb{R},\ z \in \mathbb{R}^{2n}$$

$\big($*resp.* (H_5') $G(t, z) \geqslant 0\big)$

$$(H_6) \qquad \bar{\lambda} = \max_{0 \leqslant t \leqslant T} \left[\max \sigma\big(g_z(t, 0)\big)\right] < \lambda_{-1}^{\infty}$$

$$\left(resp.\ (H_6')\ \lambda = \min_{0 \leqslant t \leqslant T} \left[\min \sigma\big(g_z(t, 0)\big)\right] > \lambda_1^{\infty}\right)$$

there exists at least one T-periodic nontrivial solution of (0.1).

Analogous results as in theorems 1.1 and 1.2 have been obtained in [3] under the assumption that b_0 and b_∞ do not depend on t; moreover in [3] the hamiltonian function $H(t, z)$ is C^2 and the Hessian H_{zz} is uniformly bounded. On the other hand in theorem 1.2 we need an additional condition on the signe of G. (H_6) establishes the connection between $b_0(t)$ and $b_\infty(t)$ which guarantees that the solution we find is nontrivial. (H_6) corresponds to the assumption of theorem 2 in [3]

$$(*) \qquad i\left(b_0,\, b_\infty,\, \frac{2\pi}{T}\right) > 0\,;$$

infact in the special case of two harmonic oscillators with frequencies α_0 and α_∞, one can easily verify that (H_6) and (*) are equivalent.

Now we suppose that the problem has *a* « strong resonance » at infinity, i.e. H_3) holds and

$$(H_7) \qquad \begin{cases} G(t, z) \to 0 & \text{as } |z| \to +\infty \text{ uniformly in } t \in \mathbb{R} \\ \big(g(t, z), z\big)_{\mathbb{R}^{2n}} \to 0 & \text{as } |z| \to +\infty \text{ uniformly in } t \in \mathbb{R}\,. \end{cases}$$

Autonomous Hamiltonian systems with a strong resonance at infinity have been studied in [14] under assumptions of symmetry. For time-dependent Hamiltonian systems we shall prove the following theorem:

THEOREM 1.3. *If* $H(t, z)$ *satisfies* (H_1), (H_3), (H_7), *then* (0.1) *has at least one T-periodic solution. Moreover, if* (H_4), (H_5), (H_6) $\big($*resp.* (H_4), (H_5') *and* $(H_6')\big)$ *hold too, then there exists at least one T-periodic nontrivial solution.*

Proof of the theorems.

In order to prove theorems 1.1, 1.2 and 1.3 we need an abstract critical point theorem proved in [12]. For completeness we shall state here this result.

THEOREM 1.4. *Given an Hilbert space E and α, β real constants, $\alpha < \beta$, let f be a functional satisfying the following assumptions:*

(f_0) $f \in C^1(E, \mathbb{R})$

(f_1) $f(z) = \frac{1}{2}(Lz, z) - \psi(z)$, where

i) *L is a continuous self-adjoint operator on E*

ii) *$\psi \in C^1(E, \mathbb{R})$ and ψ' is a compact operator*

(f_2) *0 does not belong to the essential spectrum of L*

(f_3) *given $c \in]\alpha, +\infty[$, every sequence $\{u_n\}$, for which $\{f(u_n)\} \to c$ and $\|f'(u_n)\|\,\|u_n\| \to 0$, possesses a bounded subsequence.*

Moreover given a constant $R > 0$ and two closed L-invariant subspaces E^1 and E^2 such that $E = E^1 \oplus E^2$, we set $Q = B_R \cap E^1$; $S = q + E^2$ (with $q \in Q$, $\|q\| < R$) and suppose that

(f_4) $f(u) \geq \beta$ *on S*

(f_5) $f(u) \leq \alpha$ *on ∂Q*

(f_6) $\sup_Q f(u) = c_\infty$ where $c_\infty < +\infty$

then f possesses at least a critical value $c \in [\beta, c_\infty]$.

Now some notations are needed. We set $L^2 = L^2([0, T], \mathbb{R}^{2n})$ and

$$W^{\frac{1}{2}} = W^{\frac{1}{2}}([0, T], \mathbb{R}^{2n}) = \{u \in L^2 | \sum_{j,k} (1 + |j|^2)^{\frac{1}{2}} |u_{jk}|^2 < +\infty\},$$

where $u_{jk}(j \in Z,\ k = 1, 2, \ldots, 2n)$ are the Fourier components of u with respect to the basis in L^2

$$\psi_{jk} = \exp[jtJ/\omega]\Phi_k = \cos\left(\frac{jt}{\omega}\right)\Phi_k + J \operatorname{sen}\left(\frac{jt}{\omega}\right)\Phi_k$$

where $\omega = T/2\pi$ and $\{\Phi_k\}$ $(k = 1, \dots, 2n)$ is the standard basis in $\mathbf{R}^{2n}$.

$W^{\frac{1}{2}}$ equipped with the inner product

$$(u, v)_{W^{\frac{1}{2}}} = \sum_{j,k} (1 + |j|^2)^{\frac{1}{2}} u_{jk} v_{jk}$$

is an Hilbert space.

We denote by $(\cdot,\cdot)$ and $((\cdot,\cdot))$ the inner products in L^2 and in $W^{\frac{1}{2}}$, and by $|\ |$ and $\|\cdot\|$ the corresponding norms. We consider the functional

$$f(z) = \tfrac{1}{2}((L_\infty z, z)) - \int_0^T G(t, z)\, dt \tag{1.5}$$

where $L_\infty \colon W^{\frac{1}{2}} \to W^{\frac{1}{2}}$ is the self-adjoint, continuous operator defined by

$$((L_\infty u, v)) = \sum_{j,k} j u_{jk} v_{jk} - (b_\infty(t) u, v) \qquad u,\ v \in W^{\frac{1}{2}}\,.$$

It is known that f is continuously Fréchet-differentiable and that its critical points are the solutions of (0.1).

We are proving that f satisfies the assumptions of the theorem (1.4). Standard arguments show that (f_0), (f_1), (f_2) hold. It is known that the non resonance assumption (H_2) implies (f_3) (cf. e.g. [18]). Now we show that also (f_4)-(f_6) hold. Set

$$Q = H_\infty^- \cap B_R \qquad S = H_\infty^+$$

with H_∞^- (resp. H_∞^+) the subspace of $W^{\frac{1}{2}}$ where L_∞ is negative (resp. positive) definite and B_R the sphere of center 0 and radius R in $W^{\frac{1}{2}}$, R large enough.

In the following we shall denote by c_i a positive constant. By (H_1), there exists a positive constant M such that, fixed ε positif, it results

$$|G(t, z)| \leqslant c_1 |z|_{R^{2n}} + \varepsilon/2 |z|^2_{R^{2n}} \qquad \forall z \in \mathbf{R}^{2n},\ |z|_{R^{2n}} \geqslant M \tag{1.6}$$

(cf. [8] for a detailed proof). Obviously there are real constants

β, c_∞ s.t.

$$f(z) = \tfrac{1}{2}((L_\infty z, z)) - \int_0^T G(t, z)\,dt \geqslant \beta \qquad z \in S \tag{1.7}$$

and

$$f(z) \leqslant c_\infty \qquad z \in Q . \tag{1.8}$$

Moreover there exists $\alpha \in \mathbf{R}$ s.t.

$$f(z) < \alpha \qquad z \in \partial Q \tag{1.9}$$

In fact, by (1.6) there exists c_2 depending on ε and α real numbers, s.t.

$$f(z) \leqslant \tfrac{1}{2}\tilde{\lambda}_{-1}^{\infty}\|z\|^2 + \frac{\varepsilon}{2}|z|^2 + c_1|z| + c_2 \leqslant \tfrac{1}{2}(\tilde{\lambda}_{-1}^{\infty} + \varepsilon)\|z\|^2 + c_3\|z\| + c_2 < \alpha .$$

We can choose R large enough such that $\alpha < \beta$. ■

Theorem 1.4 assures that f has at least one critical value $c \geqslant \beta$. Clearly, we can not exclude the trivial solution.

We prove now theorem 1.2. Let L_0 be the selfadjoint realization of $-J\dot{z} - b_0(t)z$ in $W^{\frac{1}{2}}$. It follows that

$$L_\infty - L_0 = b_0(t) - b_\infty(t) = g_z(t, 0) . \tag{1.10}$$

Let H_0^+ (resp. H_0^-) be the subspace of $W^{\frac{1}{2}}$ where L_0 is positive (resp. negative) definite. The following lemma holds:

LEMMA (1.11). *Under the assumptions of theorem* 1.2 *it results*

$$\text{(i) } H_\infty^- \cap H_0^+ \neq \{0\} \qquad \text{(ii) } H_\infty^+ \subset H_0^+ .$$

PROOF. Let q be an eigenvector corresponding to the eigenvalue λ_{-1}^{∞} in L^2. Then it is known that $q \in W^{\frac{1}{2}}$ and q is an eigenvector corresponding to $\tilde{\lambda}_{-1}^{\infty}$ in $W^{\frac{1}{2}}$. Moreover

$$((L_0 q, q)) = ((L_\infty q, q)) - (g_z(t, 0)q, q) \geqslant \lambda_{-1}^{\infty}|q|^2 - \big(\max \sigma(g_z(t, 0))\,q, q\big) \geqslant$$
$$\geqslant \lambda_{-1}^{\infty}|q|^2 - \bar{\lambda}|q|^2 \geqslant (\lambda_{-1}^{\infty} - \bar{\lambda})\,|q|^2 > 0 .$$

Obviously $q \in H_0^+$ and i) follows. Proving (ii), we observe that

$$((L_0 z, z)) = ((L_\infty z, z)) - (g_z(t, 0) z, z) > 0 \qquad z \in H_\infty^+$$

and the inclusion (ii) is strict because of property (i). ■

Let us prove theorem 1.2. It is obvious that the functional (1.5) verifies the hypotheses (f_1)-(f_3) of the abstract theorem 1.4; (f_4)-(f_6) hold as before setting

$$Q = H_\infty^- \cap B_R \qquad S = q + H_\infty^+$$

where q is an eigenvector of L_∞ corresponding to λ_{-1}^∞ (and to $\tilde{\lambda}_{-1}^\infty$) with $\|q\| < R$. We shall show that, in this case, the functional f is bounded from below on S by a strictly positive constant β. In fact, taken $z \in S$, $z = q + z_+$, $z_+ \in H_\infty^+$, we have

$$f(z) = \tfrac{1}{2}((L_\infty z, z)) - \int_0^T G(t, z)\, dt \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2) - \\ - \int_0^T G(t, z)\, dt \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2) .$$

Fixed ε small, we distinguish two cases:

a) $\|z_+\|^2 < -\tilde{\lambda}_{-1}^\infty/\tilde{\lambda}_1^\infty \|q\|^2 + \varepsilon$

b) $\|z_+\|^2 \geq -\tilde{\lambda}_{-1}^\infty/\tilde{\lambda}_1^\infty \|q\|^2 + \varepsilon$.

In the first case, if we choose $\|q\|$ small enough, $\|z\|$ is small and it turns out that

$$f(z) = \tfrac{1}{2}((L_0 z, z)) + o(\|z\|^2) \geq \tilde{\beta} > 0$$

because $z \in H_0^+$ and $\|z\| \geq \|q\| > 0$.

In the second case, it results

$$f(z) \geq \tfrac{1}{2}(\tilde{\lambda}_1^\infty \|z_+\|^2 + \tilde{\lambda}_{-1}^\infty \|q\|^2) \geq \varepsilon \cdot \tilde{\lambda}_1^\infty > 0 .$$

Setting $\beta = \min(\tilde{\beta}, \tilde{\lambda}_1^\infty \cdot \varepsilon)$, we conclude that

$$f(z) \geqslant \beta > 0 \quad \text{for every } z \in S .$$

Thus, there exists a critical value $c \geqslant \beta > 0$ and therefore, being $f(0) = 0$, there exists at least one critical nontrivial point. ■

REMARK. If (H_1), (H_2), (H_4), (H_5') and (H_6') hold, we can prove that $H_\infty^+ \cap H_0^- \neq \{0\}$ and the functional $-f$ satisfies the assumptions of the theorem 1.4 setting $Q = H_\infty^+ \cap B_\varrho$, $S = q + H_\infty^-$, $q \in H_\infty^+ \cap H_0^-$, $\|q\|$ small.

Lastly we prove theorem 1.3. Following the same argument as in [14] it can be proved that (f_0)-(f_3) hold. In order to prove (f_4)-(f_6) we have to make a different choise of Q and S since $\operatorname{Ker} L_\infty \neq \{0\}$.

We set now

$$Q = (\operatorname{Ker} L_\infty \oplus H_\infty^-) \cap B_R \qquad S = q + H_\infty^+ ,$$

q being an eigenvector corresponding to $\tilde{\lambda}_{-1}^\infty$ as above, $\|q\|$ small enough and R the constant which will be determined in the following.

As usual, we have

$$f(z) \geqslant \beta > 0 \quad \text{for every } z \in S .$$

Let $M = 2\pi \sup\{|G(t, z)|, z \in \mathbb{R}^{2n}, 0 \leqslant t \leqslant T\}$ and ϱ a positive constant such that

$$\tfrac{1}{2}\tilde{\lambda}_{-1}^\infty \varrho^2 + M < 0 . \tag{1.12}$$

Then (cf. lemma (3.2) of [8]) there exists $R > 0$ large enough such that for $z \in \operatorname{Ker} L_\infty \oplus H_\infty^-$, $\|z\| = R$, $z = z_0 + z_-$, $z_- \in B_\varrho$, we have

$$\int_0^T |G(t, z)|\, dt < \beta_{/2} . \tag{1.13}$$

Taking $z \in \partial Q$, there are two possibilities

$$\text{a) } z_- \in B_\varrho \qquad b) \ z_- \notin B_\varrho .$$

In the first case, by (1.13) we have

$$f(z) = \tfrac{1}{2}((L_\infty z, z)) - \int_0^T G(t, z)\,dt < \beta_{/2} < \beta\,.$$

In the second case, by (1.12) it follows that

$$f(z) \leqslant \tfrac{1}{2}\bar{\lambda}_{-1}^{\infty}\,\|z_-\|^2 + M \leqslant \tfrac{1}{2}\bar{\lambda}_{-1}^{\infty}\,\varrho^2 + M < 0 < \beta_{/2} < \beta\,.$$

Then (f_4) holds with $\alpha = \beta_{/2}$; on the other hand it is obvious that f is bounded from above on Q, So by theorem 1.4, the conclusion of theorem 1.3 follows.. ■

2. Now we study (0.2). Obviously the T-periodic solutions of (0.2) are the 2π-periodic solutions of

$$(2.1)\qquad \begin{cases} u_{tt} - \omega^2 u_{xx} = \omega^2 f(x, \omega t, u) \\ u(x, t + 2\pi) = u(x, t) & x \in [0, \pi],\ t \in \mathbb{R} \\ u(0, t) = u(\pi, t) = 0 & t \in \mathbb{R} \end{cases}$$

where $\omega = T/2\pi$.

Let us denote by L_∞ (resp. L) the selfadjoint realization of u_{tt}-$\omega^2 u_{xx} - \omega^2 a_\infty(x, t)u$ (resp. u_{tt}-$\omega^2 u_{xx}$) in a suitable Hilbert space E (cf. [5] for definition of E).

We recall that the eigenvalues of L in $L^2 = L^2([0, \pi) \times [0, 2\pi])$ are

$$\lambda_{jk} = j^2\omega^2 - k^2, \quad j \in N, \quad k \in Z\,.$$

At first we shall assume a non resonance condition at infinity, i.e.

$$(W_3)\qquad 0 \notin \sigma(L_\infty)\,.$$

As in section 1, we assume that f is linear at $u = 0$. More precisely, we suppose that

$$(W_4)\qquad f(x, t, 0) = 0 \quad \text{for every } (x, t) \in [0, \pi] \times \mathbb{R}\,.$$

By (W_2) and (W_4) it follows that

$$f(x, t, u) = \frac{\partial f}{\partial u}(x, t, 0)u + \varepsilon(x, t, u) \tag{2.2}$$

where

$$\varepsilon(x, t, u)/u \to 0 \quad \text{as } |u| \to 0 \quad \text{uniformly in } (x, t)$$

and

$$\frac{\partial f}{\partial u}(x, t, 0) = a_\infty(x, t) + \frac{\partial g}{\partial u}(x, t, 0)\,. \tag{2.3}$$

It is known that the periodic solutions of (2.1) correspond to the critical points of the functional

$$I(u) = \tfrac{1}{2}((L_\infty u, u)) - \omega^2 \int_Q G(x, \omega t, u)\, dx\, dt$$

$((\cdot, \cdot))$ being the inner product in E, $Q = [0, \pi] \times [0, 2\pi]$ and $G(x, t, u) = \int_0^u g(x, t, \xi)\, d\xi$.

We observe that we do not apply directly theorem (1.4) to the functional I, because E is not compactly imbedded in L^2 and therefore the non linear term $\int_Q g(x, t\omega, u)\, dx\, dt$ is non compact. In order to overcome this difficulty, we restrict, as in [17], I to a suitable closed subspace $\hat{E}$ of E such that

i) $\hat{E} \underset{c}{\hookrightarrow} L^2$;

ii) $L(\hat{E}) \subset \hat{E}$;

iii) $\hat{E} \cap \ker L = \{0\}$;

iv) $f(\hat{E}) \subset \hat{E}$.

Conditions ii)-iv) assure that the critical points of $I_{|\hat{E}}$ are still critcal points of I; hence they are classical solutions of (2.1).

Following Coron, we define

$$\hat{E} = \{u \in E : u(x, t) = u(x, t + \pi)\,, \quad u(\pi - x, t) = u(x, t)\}\,.$$

We assume now

$$(W_5) \qquad f(x,t,u) = f\left(x, t+\frac{T}{2}, u\right) \qquad f(x,t,u) = f(\pi - x, t, u)\,.$$

Then the following lemma holds:

LEMMA 2.4. *If $T = 2\pi b/a$, b odd and f satisfies (W_5), the subspace $\hat{E}$ verifies* (i)-(iv). *Moreover*

$$u \in \hat{E} \Leftrightarrow u_{jk} = 0 \quad \textit{for } j \textit{ even or } k \textit{ odd}$$

$$u \in \hat{E}^{\perp} \Leftrightarrow u_{jk} = 0 \quad \textit{for } j \textit{ odd and } k \textit{ even}\,,$$

u_{jk} being the components of u belonging to eigenspaces corresponding to eigenvalues λ_{jk} of L in L^2.

Let us denote the restrictions of I and L_∞ to $\hat{E}$ by $\hat{I}$ and $\hat{L}_\infty$. If we denote by λ_{-1}^∞ (resp. λ_1^∞) the first negative (resp. positive) eigenvalue of $\hat{L}_\infty$ in L^2 and by $\tilde{\lambda}_{-1}^\infty$ and $\tilde{\lambda}_1^\infty$ the analogous in $\hat{E}$, the following theorem holds:

THEOREM 2.5. *Under the assumptions (W_1), (W_2), (W_3) and (W_5), if $T = 2\pi b/a$, b odd,* (0.1) *has at least one T-periodic solution. Moreover if (W_4),*

$$(W_6) \qquad G(x,t,u) \leqslant 0 \qquad (\textit{resp.}\ (W_6')\ G(x,t,u) \geqslant 0)$$

$$(W_7)\ \max_Q \partial g/\partial u(x,t,0) < \lambda_{-1}^\infty \qquad (\textit{resp.}\ (W_7')\ \min \partial g/\partial u(x,t,0) > \lambda_1^\infty)$$

are satisfied, then the solution we find is nontrivial.

In the case where $f'(\infty) = a_\infty(x,t)$ and $f'(0) = \partial f/\partial u\,(x,t,0)$ are constant, it follows that $L_\infty = L - \omega^2 f'(\infty) I$. Therefore the eigenvalues of $\hat{L}_\infty$ in L^2 are

$$j^2\omega^2 - k^2 - \omega^2 f'(\infty)\,, \quad j \in N,\ j \text{ odd, and } k \in Z,\ k \text{ even}\,.$$

Thus (W_7) (resp. (W_7')) becomes

$$(2.6) \qquad \begin{cases} \text{there exists } \lambda_{jk} \text{ eigenvalue of } L \text{ in } L^2,\ j \text{ odd and } k \text{ even, s.t.} \\ f'(0) < \lambda_{jk} < f'(\infty) \quad (\text{resp. } f'(\infty) < \lambda_{jk} < f'(0))\,. \end{cases}$$

We recall that Amann and Zendher have studied (0.2) in the case where there exist $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, such that $\alpha > 0$ or $\beta < 0$ and

$$\alpha \leqslant (\partial f/\partial u)\,(x, t, \eta) \leqslant \beta \qquad x, t, \eta \in [0, \pi] \times \mathbb{R} \times \mathbb{R}\,. \tag{2.7}$$

Moreover they assume that

$$f'(0) < \bar{\lambda} < f'(\infty) < \hat{\lambda} \tag{2.8}$$

$\bar{\lambda}$ and $\hat{\lambda}$ being two consecutive eigenvalues of L belonging to $]\alpha, \beta[$.

In theorem (2.5) we have dropped assumption (2.7), which implies the monotonicity of f in u, but we have to add the new assumptions (W_5)-(W_6) and to replace (2.8) by (2.6). We find a T-periodic solution of (0.2) if b is odd.

If b is even, we should study (2.1) replacing (W_5) by the assumption

$$f\left(x, t + \frac{T}{2}, u\right) = f(x, t, u) \qquad \text{and } f \text{ odd in } u \tag{W_5'}$$

and choosing

$$\hat{E} = \{u \in E : u(x, t + \pi) = -u(x, t)\}\,.$$

We recall that Amann and Zendher consider only the case where $f'(0)$ and $f'(\infty)$ are constant and problem (2.1) is non resonant. On the contrary, we consider the strong resonance case too, i.e. we assume

$$\begin{cases} 0 \in \sigma(L_\infty) \\ g(x, t, u)u \to 0 \quad \text{as } u \to \infty \text{ uniformly in } (x, t) \\ G(x, t, u) \to 0 \quad \text{as } u \to \infty \text{ uniformly in } (x, t)\,. \end{cases} \tag{W_7}$$

The following theorem holds:

THEOREM 2.9. *The conclusion of theorem* (2.5) *still holds if we replace* (W_3) *by* (W_7).

REMARK. The proof of theorems (2.5) and (2.9) follows as in section 1.

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