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# Some Cardinal Invariants for Valuation Domains.

LUIGI SALCE - PAOLO ZANARDO

## Introduction.

The condition on a valuation domain $F$ of being maximal, which goes back to Krull [8], and the very close condition of being almost maximal, due to Kaplansky [7], were extensively investigated by many authors, on account of their importance for the consequences deriving for the ring structure of $R$ and for many classes of $R$-modules.

On the contrary, the problem of measuring in some way the failure of the maximality did not receive too much attention up to now; the only contributions in this direction known by the authors are by Brandal [1], and by Facchini and Vamos [2].

As any valuation domain $R$ is a subring of a maximal valuation domain $S$, which is an immediate extension of it, it is natural to try to measure the failure of the maximality for $R$ by looking for cardinal invariants which measure, roughly speaking, how large is $S$ over $R$.

In this paper, given any ideal $I$ of $R$, we will introduce two cardinal invariants associated with $I$: the *completion defect at* $I$, denoted by $c_R(I)$, and the *total defect at* $I$, denoted by $d_R(I)$; their definition, which seems very technical, raised up naturally in the investigation of indecomposable finitely generated modules in [13].

The completion defect $c_R(I)$ measures how large is $(R/I)^\wedge$, the com-

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pletion of $R/I$ in the ideal topology, and the total defect $d_R(I)$ measures how large is $S/IS$ over $R/I$; recall that both $R/I$ and $(R/I)^\wedge$ are contained in $S/IS$ up to canonical isomorphisms.

It is noteworthy that the invariants that we are introducing do not depend on the ring structure of $S$, which is not unique up to isomorphism, but only on the $R$-module structure of $S$, which is a pure-injective hull of $R$ (see [7] and [12]).

In the first section we introduce the *breadth ideal* (*of non maximality*) of the valuation domain $R$, a concept originally due to Brandal [1], and the *breadth ideal o a unit* of $S$, a concept defined in a slightly different way by Ostrowsky [11], Kaplansky [7] and Nishi [10].

The breadth ideals of the units of $S$ are used in the second section to define the completion defect and the total defect at an ideal $I$ of $R$. The main result in this section is an inequality which relates the total defect at an ideal $I$ with the completion defects at the ideals containing $I$. This inequality however is in general strict, as is shown, for a special class of discrete valuation domains, by Facchini and the second author in [3].

In section 3 we compare the total defect at an ideal $I$ with the Goldie dimension $g_R(I)$ of $S/IS$ as an $R/I$-module; they turn out to be equal if $I$ is a prime ideal, while in the non-prime case the total defect becomes generally larger.

We remark that the invariants of the valuation domain $R$ that we investigate here play a relevant role in the study of many classes of $R$-modules: besides finitely generated modules (see [13]), the $R$-modules $JS/IS$, where $I > J$ are fractional ideals of $R$(see [4]); indecomposable injective $R$-modules (see [2]) and torsion-free $R$-modules of finire rank (see [5]).

## 1. The breadth.

$R$ will always denote a valuation domain, $P$ its maximal ideal and $Q$ its field of quotients. Recall that $R$ is maximal if it is linearly compact (in the discrete topology); $R$ is almost maximal if every proper factor of it is linearly compact.

A valuation domain $S$ containing $R$ as a subring is an immediate extension of $R$ if

(i) every ideal of $S$ is of the form $IS$, where $I$ is an ideal of $R$, and $IS \cap R = I$;

(ii) $S/PS$ is naturally isomorphic to $R/P$ or, equivalently, $S = PS + R$.

An immediate extension $S$ of $R$ is maximal if, given any valuation domain $S'$ containing $S$ as a proper subring, either (i) or (ii) fails for $S'$.

It is well known (see [7] or [12]) that every valuation domain $R$ is contained in a maximal immediate extension $S$, which is a maximal valuation domain. However $S$ is uniquely determined up to $R$-isomorphism only, and not as a ring, unless $R$ is almost maximal, in which case $S$ is the completion of $R$ (in the valuation tolopogy). $R$ coincides with $S$ if and only if it is maximal.

Brandal considered in [1] the family of ideals of $R$

$$\mathcal{F} = \{I \leqslant R : R/I \text{ is not linearly compact}\}$$

and he showed that either $\mathcal{F} = \emptyset$, or there exists a prime ideal $L$ of $R$ such that

$$\mathcal{F} = \{I : I \leqslant L\} \quad \text{or} \quad \mathcal{F} = \{I : I < L\}.$$

Fixed a maximal immediate extension $S$ of $R$, we reformulate this result by introducing the following subset of $R$, called the *breadth of* $R$ (with a more meaningful term we could call it the *breadth of non maximality of* $R$):

$$B(R) = \{a \in R : S > R + aS\}.$$

Notice that $B(R) = \emptyset$ whenever $S = R + aS$ for all $a \in R$; this happens exactly if $S = R$, i.e. if $R$ is maximal; thus from now on we shall assume that $R$ is a valuation domain not maximal, so $B(R)$ is an ideal of $R$.

PROPOSITION 1.1. Let $R$ be a valuation domain not maximal. Then its breadth $B(R)$ is a prime ideal of $R$ such that:

$$\begin{aligned} B(R) &= \cap \{I : R/I \text{ is linearly compact}\} \\ &= \cup \{I : R/I \text{ is not linearly compact}\}. \end{aligned}$$

PROOF. Assume that $a, b \in R \setminus B(R)$. Then $S = R + aS = R + bS$, and $bS = b(R + aS)$ implies $S = R + b(R + aS) = R + abS$. Therefore $ab \notin B(R)$, so that $B(R)$ is prime.

If $R/I$ is linearly compact, then $R/I \simeq S/IS$ in a natural way, therefore $S = R + IS$; thus $I \geqslant B(R)$. It follows that $B(R)$ is contained in $\cap\{I: R/I$ is linearly compact$\}$. Conversely, if $I > B(R)$, then $S = R + IS$, thus $R/I \simeq S/IS$ is linearly compact. Being $B(R)$ prime, either $B(R) = P$, in which case the first equality is trivial, or $B(R)$ is the intersection of the ideals properly containing it, thus the first equality is obvious. The second equality can be proved in a similar way. ///

From Proposition 1.1 it follows that $B(R)$ does not depend on the choice of $S$, and that it coincides with the ideal $L$ quoted in the Brandal's result. Notice that $R$ is almost maximal (and not maximal) if and only if $B(R) = 0$.

The valuation domain $R/B(R)$ is always almost maximal; Brandal gives examples in [1] showing that $R/B(R)$ can be maximal or not.

Let us denote by $U(S)$ and $(UR)$ respectively the multiplicative groups of the units of $S$ and $R$. Every $0 \neq x \in S$ can be written in a unique way, up to units of $E$, in the form $x = \varepsilon r$, with $\varepsilon \in U(S)$ and $r \in R$; by this reason we will confine ourselves to consider units of $S$ in the following discussion.

Given any $\varepsilon \in U(S)\setminus R$, consider the ideal of $R$, called the *breadth of* $\varepsilon$

$$B(\varepsilon) = \{a \in R: \varepsilon \notin R + aS\}.$$

REMARK. Our definition of breadth of a unit of $S$ is essentially the same as the one given by Nishi [10], which is a slight modification of the original definition of breadth of a pseudoconvergent set of elements of $R$ given by Kaplansky [7], and originally due to Ostrowsky [11]. The definition of breadth (of non maximality) of $R$ is originated by the two above definitions.

From the definitions of $B(R)$ and $B(\varepsilon)$ it trivially follows that $B(\varepsilon) \leqslant B(R)$. Conversely, let $a \in B(R)$; then $S > R + aS$, thus there exists $\varepsilon \in U(S)$ such that $\varepsilon \notin R + aS$, therefore $a \in B(\varepsilon)$; we have proved

PROPOSITION 1.2. Let $R$ be a valuation domain not maximal. Then $B(R) = \cup\{B(\varepsilon): \varepsilon \in U(S)\setminus R\}$.

The following result will be useful in the next section; it is similar to [10, Prop. 6].

LEMMA 1.3. Let $R$ be a valuation domain not maximal and $\varepsilon \in U(S)\setminus R$. If $u \in U(R)$ and $0 \neq r \in P$, then $B(u + r\varepsilon) = rB(\varepsilon)$.

PROOF. $\varepsilon \notin R + aS (a \in R)$ if and only if $r\varepsilon \notin R + raS$, and this obviously is equivalent to $u + r\varepsilon \notin R + raS$. ///

We introduce the following notation: given $I \leqslant R$, let $f_I\colon S \to S/IS$ be the canonical surjection; then the image $f_I R$ of $R$ is a subring of $S/IS$ isomorphic to $R/I$; its completion, whenever $R/I$ is Hausdorff, is denoted by $(f_I R)^\wedge$. Notice that, being $f_I R$ pure in $S/IS$ and $S/IS$ complete, we have the following inclusions:

$$f_I R \leqslant (f_I R)^\wedge \leqslant S/IS\,. \tag{1}$$

The topology considered above, as in the following proposition, on the factor ring $R/I$ is the «ideal topology», which has as a basis of neighborhoods of 0 the ideals $(aR)/I$, $a \in R\setminus I$.

PROPOSITION 1.4. Let $R$ be a valuation domain, and $I \leqslant R$. Then $R/I$ is Hausdorff and non complete if and only if $I = B(\varepsilon)$ for some $\varepsilon \in U(S)\setminus R$.

PROOF. In order to show that $R/B(\varepsilon)$ is Hausdorff, it is enough to prove that $a \notin B(\varepsilon)$ implies $pa \notin B(\varepsilon)$ for some $p \in P$. So let $\varepsilon \in R + aS$; then $\varepsilon = r + as (r \in R,\ s \in S)$. But $S = R + PS$ implies that $s = t + ps'$, for some $t \in R$, $p \in P$ and $s' \in S$; therefore we get: $\varepsilon = r + at + aps' \in R + paS$, as we want. Clearly $\varepsilon + B(\varepsilon)S \notin f_{B(\varepsilon)}R$, but it is the limit of a Cauchy net of elements of $f_{B(\varepsilon)}R$: for, given $r \notin B(\varepsilon)$, $\varepsilon \in R + rS$ implies that there exists $u_r \in U(R)$ such that $\varepsilon - u_r \in rS$; thus $\varepsilon + B(\varepsilon)S$ is the limit of the Cauchy net $\{u_r + B(\varepsilon)S : r \notin B(\varepsilon)\}$. So we have proved that $R/B(\varepsilon) \cong f_{B(\varepsilon)}R$ is not complete.

Conversely, assuming that $R/I$ is Hausdorff and not complete, from the inclusions (1) we get an element $\varepsilon \in U(S)\setminus R$ such that $\varepsilon + IS$ is the limit of a Cauchy net $\{u_r + IS : r \notin I\}$ in $f_I R$. So $\varepsilon \in R + rS$ if and only if $r \notin I$, therefore $I = B(\varepsilon)$. ///

From the proof of the preceding proposition we deduce the following

COROLLARY 1.5. Let $\varepsilon \in U(S)\setminus R$, and $I \leqslant R$. Then $\varepsilon + IS \in f_I R$ if and only if $B(\varepsilon) < I$; $\varepsilon + IS \in (f_I R)^\wedge \setminus (f_I R)$ if and only if $B(\varepsilon) = I$. ///

A particular case is when $I = 0$; then the elements of the completions $\hat{R}$ of $R$ are exactly those $x = r\varepsilon \in S(0 \neq r \in R,\ \varepsilon \in U(S))$ such that $B(\varepsilon) = 0$. It was shown by Nishi [10] that $\hat{R}$ is the center $Z(A)$ of the ring $A = \mathrm{End}_R E(R/P)$, which is isomorphic to $\mathrm{End}_R(S)$; so we have the inclusions:

$$R \leqslant \hat{R} = Z(A) \leqslant S \leqslant A = \mathrm{End}_R E(R/P) \cong \mathrm{End}_R S .$$

## 2. The completion defect and the total defect.

We introduce now a new concept, which first appeared in [13]. Let $R$ be a valuation domain not maximal, and $S$ a maximal immediate extension of $R$; let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in U(S)$ and $I \leqslant P$; we say that the $\varepsilon_i$'s are *u-independent over I* if

$$(2) \qquad a_0 + \sum_1^n a_i \varepsilon_i \in IS \qquad (a_i \in R,\ 0 \leqslant i \leqslant n)$$

implies $a_i \in P$ for all $i$. Conversely, if (2) holds for some $a_i \in U(R)$ the $\varepsilon_i$'s are said *u-dependent over I*.

LEMMA 2.1. (i) If $\varepsilon \in U(S) \setminus R$, then $\varepsilon$ is $u$-independent over $B(\varepsilon)$. (ii) If $\varepsilon_1, \ldots, \varepsilon_n \in U(S)$ are $u$-independent over $I \leqslant P$, then

$$\varepsilon_i \notin R \text{ and } B(\varepsilon_i) \geqslant I \text{ for all } i .$$

PROOF. (i) If $a_0 + a_1\varepsilon \in B(\varepsilon)S$, then $a_0 \notin P$ if and only if $a_1 \notin P$ and, in this case, $\varepsilon \in B(\varepsilon)S + R$, which is absurd.

(ii) If, for some $j$, $\varepsilon_j \in R$, then (2) holds with $a_0 = \varepsilon_j$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. Assume now that $B(\varepsilon_j) < I$ for some $j$. Then $\varepsilon_j + IS = u + IS$ for some $u \in U(R)$, so (2) holds with $a_0 = u$, $a_j = -1$ and $a_i = 0$ for $0 \neq i \neq j$. ///

We say that a family $\{\varepsilon_\lambda \colon \lambda \in \Lambda\}$ of units of $S$ not in $R$ is $u$-independent over an ideal $I \leqslant P$, if any finite subset of it is $u$-indipendent over $I$; so the $u$-independence is a property of finite character, and maximal families of units with this property do exist.

Having fixed the ideal $I \leqslant P$, we consider all the families $\{\varepsilon_\lambda\}_{\lambda \in \Lambda}$ of units of $S$ which are $u$-independent over $I$, such that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$. Let $c_R(I)$ be the minimal cardinal such that $c_R(I) \geqslant |\Lambda| + I$ for all these families.

We call $c_R(I)$ the *completion defect of $R$ at $I$*; clearly $c_R(I)$ is an invariant of $R$ not depending on the choice of $S$, being the $u$-independence defined by linearity.

Obviously $R$ is almost maximal if and only if $c_R(I) = 1$ for all nonzero ideals $I$.

The following result compares the completion defects at isomorphic ideals.

PROPOSITION 2.2. Let $I \simeq J$ be isomorphic ideals of $R$ contained in $P$. Then $c_R(I) = c_R(J)$.

PROOF. It is enough to show that, given a family $\{\varepsilon_\lambda : \lambda \in \Lambda\} \subseteq S$ which is $u$-independent over $I$, where $I = B(\varepsilon_\lambda)$ for all $\lambda \in \Lambda$, there exists a family $\{\eta_\lambda : \lambda \in \Lambda\}$ which is $u$-independent over $J$, where $J = B(\eta_\lambda)$ for all $\lambda \in \Lambda$. Being $I \simeq J$, there exists $a \in R$ such that either $J = aI$ or $aJ = I$; we can assume $a \in P$, otherwise $J = I$ and the claim is trivial. If $J = aI$, let $\eta_\lambda = 1 + a\varepsilon_\lambda$ for all $\lambda \in \Lambda$. Then $B(\eta_\lambda) = J$ for all $\lambda \in \Lambda$ follows from Lemma 1.3. Assume now that

$$a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \ (a_i \in R,\ 0 \leqslant i \leqslant n);$$

then $a_0 + \sum_1^n a_i(1 + a\varepsilon_{\lambda_i}) \in aIS$ implies that

$$a^{-1}\left(a_0 + \sum_1^n a_i\right) + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS$$

thus, by the $u$-independence of the $\varepsilon_\lambda$'s, we deduce that $a_1, \dots, a_n \in P$, and $a^{-1}\left(a_0 + \sum_1^n a_i\right) \in P$; it follows that $a_0 \in P$ too.

Conversely, assume that $aJ = I (a \in P)$. Notice that $a \notin I$, because $J \leqslant P$. Being $B(\varepsilon_\lambda) = I$, there exists an $u_\lambda^a \in U(R)$ such that $\varepsilon_\lambda = u_\lambda^a + a\eta_\lambda$ for some $\eta_\lambda \in S$. Without loss of generality, we can assume that $\eta_\lambda \in U(S)$: for, if $\eta_\lambda \in PS$, substitute $u_\lambda^a$ and $\eta_\lambda$ respectively by $u_\lambda^a - a \in U(R)$ and $1 + \eta_\lambda \in U(S)$. From $aJ = I = B(\varepsilon_\lambda)$ and from Lemma 1.3, we deduce that $aJ = aB(\eta_\lambda)$, so $B(\eta_\lambda) = J$. Assume now that

$$a0 + \sum_1^n a_i \eta_{\lambda_i} \in JS \qquad (a_i \in R,\ 0 \leqslant i \leqslant n).$$

Then

$$\left(aa_0 - \sum_1^n a_i u^a_{\lambda_i}\right) + \sum_1^n a_i(u_{\lambda_i} + a\eta_{\lambda_i}) \in aJS = IS;$$

recalling that $u^a_{\lambda_i} + a\eta_{\lambda_i} = \varepsilon_{\lambda_i}$ $(1 \leqslant i \leqslant n)$, and that the $\varepsilon_{\lambda_i}$'s are $u$-independent over $I$, it follows that $a_1, \ldots, a_n \in P$; then $a_0 + \sum_1^n a_i \eta_{\lambda_i} \in JS$ implies $a_0 \in P$ too. ///

Given an ideal $I \leqslant P$, we introduce now another invariant; we consider all the families $\{\varepsilon_\lambda : \lambda \in \Lambda\}$ of units of $S$ as in the definition of $c_R(I)$, but we assume only that the $\varepsilon_\lambda$'s are $u$-independent over $I$, without assuming that $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$; thus, by Lemma 2.1, we only know that $B(\varepsilon_\lambda) \geqslant I$ for all $\lambda \in \Lambda$. Let $d_R(I)$ be the minimal cardinal such that $d_R(I) \geqslant |\Lambda| + 1$ for all these families. We call $d_R(I)$ the *total defect of $R$ at $I$*; here too we notice that $d_R(I)$ is an inavriant of $R$ not depending on the choice of $S$.

The following result is an immediate consequence of the definition.

LEMMA 2.3. (i) If $I \leqslant J \leqslant P$, then $d_R(I) \geqslant d_R(J)$.

(ii) $d_R(I) = 1$ if and only if $I > B(R)$ or $I = B(R)$ and $R/B(R)$ is complete.

(iii) $R$ is almost maximal if and only if $d_R(I) = 1$ for every $I \neq 0$. ///

Given an $R$-module $M$ and an ideal $I \leqslant P$, we say that the elements $x_1, \ldots, x_n \in M$ are *linearly independent over $I$* if $x_1 + IM, \ldots, x_n + IM$ are linearly independent elements of the $R/I$-module $M/IM$, i.e. if $\sum_1^n a_i x_i \in IM (a_i \in R)$ implies that $a_i \in I$ for all $i$. Obviously one can extend this definition to a family of elements of $M$.

Recall that, if $M$ is torsion-free, then the rank $rk_R\, M$ of $M$ is the dimension of the $Q$-vector space $M \otimes_R Q$, where $Q$ is the field of quotients of $R$, or, equivalently, the cardinality of a maximal system of linearly independent elements of $M$.

PROPOSITION 2.4. Let $R$ be a valuation domain and $I$ a prime ideal of $R$. Then $c_R(I) = rk_{R/I}(R/I)^\wedge$ and $d_R(I) = rk_{R/I} S/IS$.

PROOF. Given a family of elements $\{x_\lambda: \lambda \in \Lambda\}$ of $(R/I)^\wedge$ which are linearly independent over $I$, one can assume, without loss of generality, that $x_\lambda \in U(S)$ for all $\lambda \in \Lambda$, and that one of them, say $x_{\bar{\lambda}}$, is 1. It follows trivially from the definition that $\{x_\lambda: \lambda \neq \bar{\lambda}\}$ is a family of elements $u$-independent over $I$, and $B(x_\lambda) = I$ by Corollary 1.5; therefore $rk_{R/I}(R/I)^\wedge \leqslant c_R(I)$. In a similar way one can see that $rk_{R/I} S/IS \leqslant d_R(I)$.

Conversely, to prove that $c_R(I) \leqslant rk_{R/I}(R/I)^\wedge$ (respectively $d_R(I) \leqslant rk_{R/I} S/IS$) it is enough to show that, given a family $\{\varepsilon_\lambda: \lambda \in \Lambda\}$ of units of $S$ with $B(\varepsilon_\lambda) = I$ for all $\lambda \in \Lambda$ (resp. with $B(\varepsilon_\lambda) \geqslant I$), which are $u$-independent over $I$, then $\{1, \varepsilon_\lambda: \lambda \in \Lambda\}$ are linearly independent over $I$. Assume that

$$a_0 + \sum_1^n a_i \varepsilon_{\lambda_i} \in IS \qquad (a_1 \in R);$$

if some $a_i \notin I$, let $a_j$ be one of the coefficients not in $I$ with minimal value. By multiplying by $a_j^{-1}$, we get

$$a_j^{-1}\left(a_0 + \sum_1^n a_i \varepsilon_{\lambda_i}\right) \in a_j^{-1} IS = IS ,$$

because $a_j I = I$; the last relation is absurd, because the coefficient of $\varepsilon_{\lambda_i}$ is equal to 1, which contradicts the $u$-independence of $\{\varepsilon_\lambda: \lambda \in \Lambda\}$ over $I$. ///

We wish to compare now the total defect $d_R(I)$ at the ideal $I$ with the completion defects $c_R(J)$ at the ideals $J \geqslant I$.

LEMMA 2.5. For every $i = 1, \dots, n$, let $E_i$ be a family of units of $S$ $u$-independent over the ideals $J_i$, such that $J_i = B(\varepsilon)$ for all $\varepsilon \in E_i$. If $J_1 > J_2 > \dots > J_n$, and the $J_i$'s are pairwise non isomorphic, then $\cup\{E_i: 1 \leqslant i \leqslant n\}$ is $u$-independent over $J_n$.

PROOF. We induct on $n$, the claim being trivial for $n = 1$. So, assume that $n > 1$ and that $\cup\{E_i: 1 \leqslant i \leqslant t\}$ is $u$-independent over $J_t$, for $1 \leqslant t \leqslant n-1$. Let

$$a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h \eta_h \in J_n S \tag{3}$$

where $a_j,\ b_h \in R\setminus\{0\}$ $(0 \leqslant j \leqslant k)$, $1 \leqslant h \leqslant m$; $\varepsilon_j \in \cup\{E_i : 1 \leqslant i \leqslant n-1\}$ for $1 \leqslant j \leqslant k$ and $\eta_h \in E_n$ for $1 \leqslant h \leqslant m$. First, notice that $a_1, \ldots, a_k \in P$: for, let $r \in J_n \setminus J_{n-1}$; then, for all $h \leqslant m$ there exists $v_h^r \in U(R)$ such that $\eta_h - v_h^r \in rS$. It follows that

$$a_0 + \sum_1^k a_j \varepsilon_j + \sum_1^m b_h v_h^r \in J_{n-1} S$$

and the $u$-independence of the $\varepsilon_j$'s implies that $a_1, \ldots, a_k \in P$. Recall now that $B(\varepsilon_j)$ is one of the $J_i$'s, for $1 \leqslant i \leqslant n-1$, for all $j$; we will show that

$$a_j B(\varepsilon_j) = B(1 + a_j \varepsilon_j) < J_n \text{ for all } j\ .$$

Being $J_n$ not isomorphic to $J_1, \ldots, J_{n-1}$, $a_j B(\varepsilon_j) \neq J_n$ for all $j$; let

$$(4) \qquad A_1 = \{j : a_j B(\varepsilon_j) < J_n\}; \qquad A_2 = \{j : a_j B(\varepsilon_j) > J_n\}\ .$$

Let $t \in J_n \setminus \cup\{a_j B(\varepsilon_j) : j \in A_1\}$; for all $j \in A_1$ we can choose an element $w_j^t \in U(R)$ such that

$$(1 + a_j \varepsilon_j) - w_j^t \in tS < J_n S\ ;$$

substituting in (3), we get:

$$(5) \qquad a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h \eta_h \in J_n S$$

where the sums with indexes in $A_1$, $A_2$ ate intended to be 0 whenever $A_1$ or $A_2$ is void.

Assume now that $A_2 \neq \emptyset$. Let $c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j) \setminus J_n$, and choose for all $h \leqslant m$, $v_h^c \in U(R)$ such that $\eta_h - v_h^c \in cS$; substituting in (5) we get:

$$(6) \qquad a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_{j \in A_2} (1 + a_j \varepsilon_j) + \sum_1^m b_h v_h^c \in \bigcap_{j \in A_2} a_j B(\varepsilon_j)\, S\ ;$$

we will show that (6) is absurd. Let $j_0 \in A_2$ be such that $a_{j_0}$ has minimal value among the $a_j$'s with $j \in A_2$; then

$$(7) \qquad a_{j_0}^{-1}(a_0 - k1 + \sum_{j \in A_1} w_j^t + \sum_1^m b_h v_h^c + |A_2| \cdot 1) + \\ + \sum_{j \in A_2} a_{j_0}^{-1} a_j \varepsilon_j \in \bigcap_{j \in A_2} a_{j_0}^{-1} a_j B(\varepsilon_j) S ;$$

notice that in (7) $a_{j_0}^{-1} a_j \in R$ for all $j \in A_2$, and that

$$B(\varepsilon_{j_0}) \geqslant a_{j_0}^{-1} \bigcap_{j \in A_2} a_j B(\varepsilon_j)$$

therefore also the first summand in (7) is in $R$. But then (7) is absurd, because the coefficient of $\varepsilon_{j_0}$ is 1 and by the inductive hypothesis. Thus we have proved that $A_2 = \emptyset$, therefore (5) becomes simply:

$$(8) \qquad (a_0 - k1 + \sum_{j \in A_1} w_j^t) + \sum_1^m b_h \eta_h \in J_n S ;$$

by the $u$-independence of the $\eta_h$'s over $J_n$, (8) gives that $b_1, \ldots, b_m \in P$; being $a_1, \ldots, a_k \in P$, from (3) it follows that $a_0 \in P$. ///

Given an ideal $J \leqslant R$, let $[J]$ denote the isomorphism class of $J$; if $I \leqslant R$ is another ideal, then $[J] \geqslant I$ means that there exists $J' \in [J]$ such that $J' \geqslant I$. By Proposition 2.2, we can define $c_R[J]$ as the common value $c_R(J')$, where $J'$ ranges over $[J]$. We can easily obtain from the preceding Lemma 2.5 the following

PROPOSITION 2.6. $d_R(I) \geqslant \sum \{c_R[J] - 1 : [J] \geqslant I\} + 1$. ///

The inequality in Proposition 2.6 is in general strict, as is shown by Facchini and the second author in [3]; actually, they prove a multiplicative formula relating $d_R(I)$ and the $c_R[J]$'s, $[J] \geqslant I$, for $I$ a prime ideal of a discrete valuation domain $R$ with Spec $R$ well ordered by the opposite inclusion; they also give a realization theorem for these domains with preassigned completion defects, using an idea of Nagata [9].

## 3. – Total defect and Goldie dimension.

Let I be an ideal of the valuation domain $R$, and let $g_R(I)$ denote the Goldie dimension of $S/IS$ as an $R/I$-module. If $I$ is a prime ideal, then $g_R(I) = rk_{R/I} S/IS$. It follows from the definitions that, for an arbitrary ideal $I$, $g_R(I) \leqslant d_R(I)$, and Proposition 2.4 shows that this inequality becomes an equality if $I$ is a prime ideal.

Recall that, if $0 \neq I \leqslant P$, then the subset of $R$

$$I^{\#} = \{r \in R : rI < I\}$$

is a prime ideal, which is the union of all the ideals $< R$ isomorphic to $I$ (see [10] and [4]). If $I = 0$, we set $I^{\#} = 0$.

LEMMA 3.1. Given any $I < R$, $g_R(I) = g_R(I^{\#})$.

PROOF. It is enough to show that, given $\varepsilon_1, \ldots, \varepsilon_n \in U(S)$, they are linearly independent over $I^{\#}$ if and only if they are linearly independent over $I$. So, assume that they are linearly independent over $I^{\#}$ and let $\sum_1^n a_i \varepsilon_i \in IS$. If some $a_i \notin I$, let $a \in R$ be such that $v(a) = \min\{v(a_i) : 1 \leqslant i \leqslant n\}$. Then

$$a^{-1} \sum_1^n a_i \varepsilon_i \in a^{-1} IS \leqslant I^{\#} S$$

because $a^{-1}I$ is an ideal isomorphic to $I$, hence $a^{-1}I \leqslant I^{\#}$. But this is a contradiction, because some $a^{-1}a_i$ is a unit.

Conversely, let $\varepsilon_1, \ldots, \varepsilon_n$ be linearly independent over $I$ and let $x = \sum_1^n a_i \varepsilon_i \in I^{\#} S$. If $x \in IS$, then $a_i \in I$ for all $i$, hence $a_i \in I^{\#}$ for all $i$. If $x \notin IS$, then $x = r\eta$, where $\eta \in U(S)$ and $r \in I^{\#} \setminus I$. Being $I^{\#}$ the union of the ideals isomorphic to $I$, there exists an ideal $J \leqslant I^{\#}$ such that $tJ = I$ for some $t \in P$ and $r \in J$. Then $tr \in I$, therefore $t \sum_1^n a_i \varepsilon_i \in IS$; the independence of the $\varepsilon_i$'s over $I$ implies that $ta_i \in I = tJ$, hence $a_i \in J \leqslant I^{\#}$ for all $i$. ///

Recall that an ideal $I < R$ is archimedean if $I^{\#} = P$. As an immediate consequence of the preceding lemma we get

COROLLARY 3.2. Given two ideals $I \simeq J$, then $g_R(I) = g_R(J)$; moreover $g_R(I) = 1$ if $I$ is archimedean.

PROOF. The first claim follows from the equality $I^{\#} = J^{\#}$; the second equality follows from the isomorphism $S/PS \simeq R/P$. ///

Lemma 3.1 and Proposition 2.4 give the following

COROLLARY 3.4. Given any ideal $I < R$, $g_R(I) = d_R(I^\#)$. ///

COROLLARY 3.5. Given any ideal $I < R$, then $g_R(I) = 1$ if and only if either $I^\# > B(R)$, or $I^\# = B(R)$ and $R/B(R)$ is complete. ///

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