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Lagrange Multipliers and Geometric Measure Theory.

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What we want to present here are some theoretical and numerical remarks about solutions to variational problems with a volume constraint. The first author is responsible for the theoretical work, the second for the computations.

We shall first recall some general facts about integrable functions minimizing an elliptic integrand, then we will pass to the consideration of measurable sets minimizing an integrand depending essentially on the sets' boundaries. Finally, we will consider the case where the sets are restricted to assume a fixed measure. For this case we present some recent results of other authors, which can be seen as an extension of the classical Lagrange Multiplier Rule. The numerical experiments are related to this new Lagrange Multiplier Rule.

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1. In the late fifties, E. De Giorgi and J. Nash, independently, proved the Hölder continuity of weak solution u to the equation

$$\sum_{j,i=1}^{n} D_i\big(a_{ij}(x)\, D_j u\big) = 0, \qquad x \in \Omega \subset R^n,$$

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where the quadratic form $\sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j$ is bounded from above and below, with respect to λ, independently of x. The solution u is assumed to have first derivatives in $L^2(\Omega)$ and the coefficients $a_{ij}(x)$ are simply Lebesgue-measurable.

This result implied the Hölder continuity of first derivatives of solutions to Hilbert-Haar problems, hence their analyticity in case of analytic data (cfr. Hilbert's 19$^{\text{th}}$ problem and A. Haar's paper in Mathematische Annalen of 1927).

The problems of regularity of solutions with unbounded gradients and solutions to systems of elliptic equations were successively considered and solved with similar methods as De Giorgi-Nash's. Some remarkable contemporary successes were obtained in the study of regularity of solutions to parametric problems, say those variational problems dealing with more complicated objects than integrable functions. We must quote the results of De Giorgi and E. R. Reifenberg for the higher dimensional Plateau problem, together with their extensions to more general problems by F. J. Almgren.

The result of De Giorgi, limited to codimension one manifolds but much easier to be presented, is the following: « If the n-dimensional measure of the boundary of a set $X \subset E^{n+1}$ is minimal in the open set $\Omega \subset E^{n+1}$, then $\partial X \cap \Omega$ is an analytic n-dimensional manifold except for a closed set whose n-measure vanishes ».

Later on H. Federer proved that the codimension of the singular set is not less than 8.

It must be said that the results for elliptic systems, which we indicated above, were made possible by the techniques introduced for the parametric problems.

2. In the early seventies U. Massari extended De Giorgi-Federer's result to sets X minimizing.

$$H_n(\partial X) + \int_X A(x)\,dx$$

for a bounded $A(x)$. What comes out from Massari's paper is that De Giorgi technique works for sets X satisfying the following inequality

$$(*) \qquad H_n(\partial X) \leqslant H_n(\partial Y) + C\varrho^{n+\varepsilon},$$

for all sets Y equal to X outside a ball of radius ϱ. The constant ε must be positive. The boundaries of such sets X have $C^{2\varepsilon}$-continuous tangent plane except for a set of n-measure zero.

3. The result of Massari was successfully applied by M. Emmer and I. Tamanini to the study of the equilibrium surfaces in capillary problems. But most of the interesting capillary problems, as the ones considered by E. Gonzalez, by Gonzalez-Massari-Tamanini, and by S. Albano-Gonzalez, are problems where the sets are restricted to having a prescribed measure. To these sets neither De Giorgi's nor Massari's techniques apply directly. A rule like Lagrange multiplier would put the solutions of variational problems with a volume constraint into the range of Massari's method. The validity of a rule of this type was shown by G. Congedo-Gonzalez, who proved that the volume constraint can be forgotten if one adds to the functional a term like $K|H_{n+1}(X) - V_0|$, where K is a constant sufficiently large and V_0 is the required volume.

4. Numerical experiments were made for the one-dimensional problem

$$(**)\qquad \int_0^1 \sqrt{1 + f^2(x)}\, dx + f(0) + f(1) + K\left|\int_0^1 f(x)\, dx - V_0\right| = \min$$

in the class of non-negative real functions defined on the interval $[0, 1]$.

Theoretical considerations show that for large values of K and $V_0 \leqslant \pi/8$ the graph of the solution of Problem (**) is an arc of circle, symmetric with respect to the line $x = \frac{1}{2}$ passing through the points $(0, 0)$, $(1, 0)$ and surrounding together with the x-axis an area equal to V_0. By the same arguments one could show that for $V_0 > \pi/8$ the graph of the solution is the half circle of radius $\frac{1}{2}$, centered at $(\frac{1}{2}, 0)$ and translated upwards $V_0 - \pi/8$.

Our numerical method consisted of approximating the function $f(x)$ as a piecewise linear spline, with 21 nodes in the interval $[0, 1]$. We then minimized the functional (**) over all such splines, using the routine E04JAF from the Numerical Algorithm Group (NAG) subroutine library. The tolerance parameter for the routine above was chosen so that our results were accurate to within 10^{-3}. We restricted our runs to three specific values of V_0, namely $\pi/16$, $\pi/8$ and $\pi/4$. For each of these volumes we let the value of K range from 1.6 to 2.6 (in increments of 0.05).

Below are shown the volume of the resultant minimizing spline solution vs. the value of K. As can be seen from the results, for each volume V_0, there exists a critical value K_0, such that for $K > K_0$, the volume V of the minimizing spline satisfies the volume constant $V = V_0$.

Table 1. *Computed value of* $\int_0^1 f(x)\,dx$.

K	V_0		
	$\pi/16 \simeq .196$	$\pi/8 \simeq .393$	$\pi/4 \simeq .785$
1.6	.171	.171	.171
1.65	.180	.180	.180
1.7	.191	.191	.191
1.75	.196	.203	.202
1.8	.196	.216	.216
1.85	.196	.230	.230
1.9	.196	.248	.248
1.95	.196	.268	.268
2.0	.196	.294	.294
2.05	.196	.393	.785
2.1	.196	.393	.785
2.15	.196	.393	.785
2.2	.196	.393	.785
2.25	.196	.393	.785

5. The second problem we considered was the following:

$$\alpha H_1(\partial E) - \beta \int_E (x_1^2 + x_2^2)\,dx + K|H_2(E) - 1| = \min,$$

where E is a two dimensional set, with area $H_2(E)$ and arc length $H_1(\partial E)$. This problem was intended to approximate a rotating water drop.

By considering Problem (**) in terms of polar coordinates, we were able to approximate the boundary of E as a periodic piecewise linear spline, using 50 nodes. Once again we used the *NAG* minimization routine to produce our minimizing spline solution.

Our numerical runs consisted of letting $\alpha = 1$ and having β take the values 0, 0.5, 1.0, 2.0, and 4.0, corresponding to increasing the angular velocity of the « water drop ». Again we found critical values K_β (depending on β). For runs with $K < K_\beta$ we consistently obtained the zero function as our minimizing spline solution. On the other hand, for runs with $K > K_\beta$ we always obtained a spline solution which satisfied the volume constraint, $H_2(E) = 1.0$.

Below are tabulated our computed results for the K's close to K_β (one slightly smaller, the other slightly larger than K_β), together with $H_2(E)$, $H_1(\partial E)$ and the angular momentum $\left(\int_E (x_1^2 + x_2^2)\,dx\right)$ of the resulting spline minimizing solution.

Table 2

β	α	K	$H_2(E)$	$H_1(\partial E)$	$\int_E (x_1^2 + x_2^2)\,dx$
0	1.0	7.26	0.0	0.0	0.0
		7.28	1.0	3.547	0.955
0.5	1.0	2.62	0.0	0.0	0.0
		2.64	1.0	3.597	0.727
1.0	1.0	3.06	0.0	0.0	0.0
		3.08	1.0	3.687	0.603
4.0	1.0	4.78	0.0	0.0	0.0
		4.8	1.0	4.265	0.332

In conclusion, these computations seem to confirm that the addition of the term $K|V - V_0|$ solves the problem of transforming a conditioned variational problem into a free one. For more discussion of the numerical aspects of this method the reader is referred to the book *Practical Optimization* by Gill, Murray, and Wright.

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